STOCHASTIC RESONANCE IN SYMMETRIC DOUBLE WELL: HIGHER HARMONICS

A.V. Zhiglo

Institute for Theoretical Physics,
National Science Center"Kharkov Institute of Physics and Technology", Kharkov, Ukraine

The work deals with the phenomenon of Stochastic Resonance in its genuine model, proposed by B. McNamara and K. Wiesenfeld for explanation of long-term climatic changes on Earth. It is shown that in two state model the higher harmonics behave in a non-monotonous way with increase of the noise level, possessing one or more maxima. Explicit formulae for third and fifth harmonic amplitudes and corresponding SNR are obtained. Studied by other authors peculiarities, like dips and sharp peaks in output signal do not occur in two state model, thus they only exist in systems with continuous configuration space.

PACS: 05.40. +j

1. INTRODUCTION

Stochastic resonance (SR) has been studied for about 20 years. The principal manifestation of the phenomenon is a strong reaction of different output characteristics of the system (like component with initial frequency in residual-time distribution, signal/noise ratio etc.) to a weak periodic signal. This reaction grows with increasing of the noise level up to certain extent. The motivations for study of the phenomenon as well as theoretical model are rather naturally presented in [1,2], early works describing SR (initially put forward for explanation of the correlation between glacial periods on Earth with the periodic changing of the Earth orbit eccentricity). Now SR constitutes an important subfield of non-linear physics. According to usual understanding of stochastic resonance phenomenon as non-monotonous dependence of the output as a function of noise intensity, the majority of investigations (both theoretical and experimental) in the case of monochromatic input analyze the component of output with initial frequency.

There are several works where higher harmonics are investigated as well as the first one — using different models in continuous configuration space (most common approaches are linear-response theory, numerical integration of Fokker-Planck equation, matrix continued fraction technique) they study the problems of optimal generation or suppressing of higher harmonics [3,4,5]; the strengths of higher harmonics show various peculiarities such as extremely sharp peaks and resonance-absorption like dips at certain noise intensities. The dependence of non-zero intensity of even harmonics on the potential asymmetry is studied in [6]. Different methods used for study of SR-like phenomena and further numerous references can be found in [7].

We will demonstrate some qualitative aspects of the problem in two state model, intending to find out what of studied features of higher harmonics can be observed in it. We follow the notation from the work [1]; we will propose some useful representations of higher harmonics intensities — absolute and in relation to noise power densities at corresponding frequencies.

Let us consider a Brownian particle in an external smooth potential $U(x)$ with a barrier between two wells, subjected to a strong friction. If the noise intensity D (we consider a white noise that provides a term $\sqrt{2D}\,\xi(t)$ in the Langevin equation (3); $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$) is much smaller than the height of the barrier between wells ΔU, the average reciprocal transition time through the barrier is given by Kramers formula:

$$r = \frac{1}{2\pi}\sqrt{|U''(0)|U''(x_m)}\exp\left(-\frac{\Delta U}{D}\right);$$

0 and x_m are positions of potential maximum and minimum.

We consider a symmetric potential (with dimensionless x)

$$U(x) = -\frac{x^2}{2} + \frac{x^4}{4} \qquad (1)$$

thus

$$r = \frac{1}{\sqrt{2}\,\pi}\exp\left(-\frac{1}{4D}\right). \qquad (2)$$

Let us assume that beyond the chaotic force $\sqrt{2}D\xi(t)$, a friction and $-\partial U/\partial x$ the particle is subjected to an external periodic force $A\cos\omega t$. For overdamped system the dependence $x(t)$ is to be found from

$$\dot{x} = x - x^3 + A\cos\omega t + \sqrt{2D}\xi(t). \qquad (3)$$

Under these conditions $\langle x(t)\rangle$ is a periodic function having period $T = 2\pi/\omega$ (providing $t - t_0 \gg 1/r$); the point is to study the behavior of its different harmonics as functions of D.

We mainly study the power spectrum, being obtained as a Fourier transform of autocorrelation function $\langle x(t)x(t+\tau)\rangle$. $SNR_k(D)$, defined as a ratio of signal power in δ-spike at $\Omega = k\omega$ to noise power density at this frequency is also investigated; non-monotonous $SNR_1(D)$ dependence is a usual manifestation of SR.

Two state model, in which there are only two states for the system to occupy ($x = \pm x_m$), may be considered a limit of described above dynamical system in which the transition time is much greater than the relaxation time within a well. It is natural to assume minima of $U(x)$ as $\pm x_m$; for the potential (1) $x_m = 1$.

In such an approximation the distribution $p(x,t)$ reduces to $n_\pm(t)$ — probabilities of location near $\pm x_m$ (within right or left well) $(n_+ + n_- = 1)$:

$$p(x,t) = n_+(t)\delta(x - x_m) + n_-(t)\delta(x + x_m)\,. \tag{4}$$

Their evolution is given by the rate equation:

$$\frac{dn_\pm}{dt} = -n_\pm W_\mp(t) + n_\mp W_\pm(t)\,, \tag{5}$$

where $W_\pm(t)$ are normalized probabilities of transition into $\pm$ state.

In the work [1] the following expression was proposed:

$$W_\pm(t) = r\exp\left(\pm\frac{Ax_m}{D}\cos\omega t\right) \tag{6}$$

with r, A, D introduced above.
The solution of (5) (for sign +)

$$\frac{dn_+}{dt} = -2n_+ r\cosh\left(\frac{Ax_m}{D}\cos\omega t\right) + r\exp\left(\frac{Ax_m}{D}\cos\omega t\right) \equiv -P(t)n_+ + Q(t) \tag{7}$$

is

$$n_+(t) = \exp\left(-\int_{-\infty}^{t} P\,dt\right)\left(C + \int_{-\infty}^{t} Q(t)\exp\left(\int_{-\infty}^{t} P(\tau)\,d\tau\right)dt\right) \tag{8}$$

The integrals in (8) cannot be calculated in terms of known functions. In [1] integrals were evaluated with accuracy $\sim \varepsilon = Ax_m/D$, considered to be a small parameter. As a result $n_+(t)$ only contained the first harmonic that was $\sim\varepsilon$ (comparing with the constant component of distribution at $t = t_0+1/r$).

Using more precise expansion in small ε it is possible to account for higher harmonics and to calculate their amplitudes with desired accuracy. However, the problem is rather complicated especially if the aim is to obtain SNR_k – because in such a way it is inevitable to take into account terms with different frequencies (not only $k\omega$). In the work [8] the authors obtained results for higher harmonics in similar model considering hopping between the wells as process discrete in time (i.e. not only configuration space, but also the time scale was treated as discrete). They also studied interesting features connected with modulation of equilibrium positions $x_m(t)$; this resulted in (rather weak) peaks at even harmonics in power density.

We will obtain sum representation of amplitudes in power spectrum; these sums:

1) have sense at all ε
2) can be used for evaluating the largest terms (those are $\sim\varepsilon^{2k}$ for SNR_k) at small ε.

2. POWER SPECTRUM

Using $\exp(\varepsilon\cos\omega t) = \sum_{n\in Z} \mathrm{I}_n(\varepsilon)\cos n\omega t$, where I_n are modified Bessel functions we can obtain

$$\int_0^t P\,dt = 2r_0 t + \sum_{n=1}^{\infty} \varepsilon_{2n}\sin 2n\omega t\,, \tag{9}$$

where

$$r_0 = r\,\mathrm{I}_0(\varepsilon),\quad \varepsilon_{2n} = \frac{2r\,\mathrm{I}_{2n}(\varepsilon)}{n\omega}\,. \tag{10}$$

So, using (8)

$$n_+(t) = e^{-\sum \varepsilon_{2n}\sin 2n\omega t}\left(Ce^{-2r_0 t} + e^{-2r_0 t}\int_{-\infty}^{t} dt\, r\, e^{2r_0 t}e^{\varepsilon\cos\omega t}e^{\sum\varepsilon_{2n}\sin 2n\omega t}\right) \equiv \pi(t)\left(Ce^{-2r_0 t} + V(t)\right). \tag{11}$$

Here $V(t+\pi/\omega) = V(t)$.

For the conditional probability $n_+(t\,|\,x_0,t_0)$:

($n_+(t_0\,|\,x_0,t_0) = \delta_{x_0 x_m}$)

$$n_+(t) = \pi(t)\left[V(t) + e^{-2r_0(t-t_0)}\left(-V(t_0) + \delta_{x_0 x_m}\pi^{-1}(t_0)\right)\right]. \tag{12}$$

The amplitudes of different harmonics in output power are connected with the coefficients G_k in

$$H(t) = \lim_{t_0\to-\infty}\frac{\langle x(t)|x_0,t_0\rangle}{x_m} = 2\pi(t)V(t) - 1 \equiv G_0 + \sum_{k=0}^{\infty} G_k\cos(k\omega t + \psi_k)\,. \tag{13}$$

The periodicity of $H(t)$ is obvious; the autocorrelation function $\langle x(t)x(t+\tau)\rangle = K(t,\tau\,|\,x_0,t_0)$

$$\langle x(t)x(t+\tau)|x_0,t_0\rangle = \iint xy\,dx\,dy\,p(x,t+\tau\,|\,y,t)p(y,t\,|\,x_0,t_0) \tag{14}$$

is periodic as a function of t in the limit $t_0\to-\infty$. On averaging over random initial phase in external force (or, equivalently, over the period of t, taking $t - t_0$ and τ constant)

$$\lim_{t_0\to-\infty} x_m^{-2}\langle x(t)x(t+\tau)|x_0,t_0\rangle = H(t+\tau)H(t) + e^{-2r_0\tau}\left(1 - H^2(t)\right)\frac{\pi(t+\tau)}{\pi(t)}\,. \tag{15}$$

The first term (see (13)) gives (after averaging)

$$\langle H(t+\tau)H(t)\rangle_t = G_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} G_k^2\cos k\omega\tau\,. \tag{16}$$

Fourier transform of (16) only contains δ-spikes at $\Omega = k\omega$:

$$-2\pi G_0^2\,\delta(\Omega)+K_S(\Omega)=\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}e^{-i\Omega t}\frac{G_k^2}{2}\cos k\omega t\,dt$$

$$=\sum_{k=1}^{\infty}\frac{\pi}{2}G_k^2\left[\delta(\Omega-k\omega)+\delta(\Omega+k\omega)\right]. \qquad (17)$$

Using the same procedure one can see that the 2nd term in (15) corresponds to noise component of output power since

$$\left\langle e^{-2r_0\tau}\frac{\pi(t+\tau)}{\pi(\tau)}\left(1-H^2(t)\right)\right\rangle_t=e^{-2r_0\tau}U(t). \qquad (18)$$

with $U(t+\pi/\omega)=U(t)$. Fourier transform $K_N(\Omega)$ of (18) (i.e. the second term in (15)) is a regular function with Gaussian maxima near $\Omega=2n\omega$ ($n\in\mathbb{N}$ – positive integers). So, the useful signal is given by (17); in part, if some $G_k=0$ the power spectrum is regular at $\Omega=k\omega$ (because of the symmetry of the potential all even harmonics vanish: $G_{2l}=0\ \forall l\in\mathbb{N}$). According to the definition

$$SNR_k=\frac{G_k^2}{4K_N(k\Omega)}. \qquad (19)$$

3. HARMONICS AMPLITUDES EVALUATION

Let us represent $V(t)$ (see(11)) as a sum:

$$V(t)e^{2r_0t}=\int_{-\infty}^{t}\exp(\varepsilon\cos\omega t)e^{2r_0t}\,dt\,\exp\sum_{k=1}^{\infty}\varepsilon_{2k}\sin 2k\omega t$$

$$=\sum_{n,n_k}I_n(\varepsilon)\prod_k I_{n_k}(\varepsilon_{2k})\times i^{-\sum n_k}\frac{e^{2r_0t}e^{i\omega t\left(n+2\sum kn_k\right)}}{2r_0+i\omega\left(n+2\sum kn_k\right)}. \qquad (20)$$

Here $\sum_{n_k}$ means a sum over all $\{n_k\}_{k=1}^{\infty}\in\mathbb{Z}^{\infty}$ – consequences of integers (with finite number of non-zero elements). Let us obtain representation of $H(t)$ as a Fourier sum:

$$H(t)=2\pi(t)V(t)-1=\sum_{s=-\infty}^{\infty}e^{i\omega st}\sum_{n_k,l_k}I_n(\varepsilon)$$

$$\times\prod_k\left(I_{n_k}(\varepsilon_{2k})I_{l_k}(\varepsilon_{2k})\right)i^{-\sum_k(n_k-l_k)}\times\frac{2r}{2r_0+im\omega}-1$$

$$\equiv\sum H_s e^{i\omega st}. \qquad (21)$$

In the last equation we have introduced $s\in\mathbb{Z}$, $s=n+2\sum_{k=1}^{\infty}k(n_k+l_k)$, thus n is to be calculated from here; $m=s-2\sum_{k\in\mathbb{N}}kl_k$.

H_s are simply connected with G_k:

$$G_k\cos(k\omega t+\psi_k)=H_ke^{i\omega kt}+H_{-k}e^{-i\omega kt},\ G_k=2\left|H_k\right|$$

For $\varepsilon\ll 1$ (21) enables us to obtain approximations for H_s: below

$$z=r/\omega,\ B\equiv B(\{n_k\},\{l_k\})=\frac{i^{\sum(l_k-n_k)}}{2r_0+im\omega},$$

$N!$ means $|N|!$

$$\frac{H_s}{2r}=\sum_{n_k,l_k}\frac{(\varepsilon/2)^{|n|}}{n!}\prod_{k=1}^{\infty}\left(\frac{z(\varepsilon/2)^{2k}}{k(2k)!}\right)^{|n_k|+|l_k|}\frac{B}{n_k!\,l_k!}\left(1+O(\varepsilon^2)\right)$$

$$\approx\sum_{n_k,l_k}\left(\frac{\varepsilon}{2}\right)^{|n|+2\sum k\left(|n_k|+|l_k|\right)}\prod_{k=1}^{\infty}\left(\frac{z}{k(2k)!}\right)^{|n_k|+|l_k|}\frac{B}{n_k!\,l_k!}. \qquad (22)$$

It is obvious that the terms containing the lowest power of ε correspond to those with equal-signed n, n_k and l_k. There is a finite number of such terms. Using only them we get for $s>0$

$$\frac{H_s}{2r}=\sum_{\substack{n,n_k,l_k\geq 0\\ n+2\sum k(l_k+n_k)=s}}\left(\frac{\varepsilon}{2}\right)^{s}\frac{i^{\sum(l_k-n_k)}}{2r_0+im\omega}$$

$$\times\prod_{k=1}^{\infty}\left(\frac{z}{k(2k)!}\right)^{n_k+l_k}\frac{1}{n_k!\,l_k!}. \qquad (23)$$

The calculations give for first non-zero amplitudes G_k:

$$G_1=2z\varepsilon\sqrt{\frac{1+\varepsilon^2/4}{4z^2+1+2z^2\varepsilon^2}}+O\left(\varepsilon^5\right)=\frac{2z}{\sqrt{4z^2+1}}+O\left(\varepsilon^3\right)$$

$$G_3=\frac{z\varepsilon^3}{3}\sqrt{\frac{z^2+1/16}{(4z^2+1)(4z^2+9)}} \qquad (24)$$

$$G_5=\frac{z\varepsilon^5}{320}\sqrt{\frac{\left(64/3\,z^2-1\right)^2+\left(14z\right)^2}{\left(4z^2+1\right)\left(4z^2+9\right)\left(4z^2+25\right)}}.$$

Using the terms with the lowest power of ε in K_N one can obtain from (19) expressions for SNR_k applicable for small ε:

$$SNR_k=\frac{\pi G_k^2}{8r_0}\left(4r_0^2+k^2\omega^2\right)\left(1+O(\varepsilon^4)\right). \qquad (25)$$

Thus

$$SNR_1=\frac{\pi}{2}\omega z\varepsilon^2\left(1+\frac{2z^2\varepsilon^2}{4z^2+1}\right)+O\left(\varepsilon^6\right)$$

$$SNR_3=\frac{\pi}{72}\omega z\varepsilon^6\frac{z^2+1/16}{4z^2+1} \qquad (26)$$

$$SNR_5=\frac{\pi\omega z\varepsilon^{10}}{10^2\times 2^{13}}\frac{\left(64/3\,z^2-1\right)^2+\left(14z\right)^2}{\left(4z^2+1\right)\left(4z^2+9\right)}.$$

The dependence $SNR_k(\omega)$ has non-monotonous character. Unless A or ε is large SNR_k attains its maximum at $\varepsilon<1$ where our approximation is valid. So, maximum positions can be calculated from the above expressions. From expansion of $I_n(\varepsilon)$ one can derive the following structure of the main term of SNR:

$$SNR_k=\omega z\left(\varepsilon/2\right)^{2k}Q_k(z^2),$$

where Q_k is some rational function, $Q_{2k}(0)\neq 0$. Therefore, for small z (i.e. large ω or low noise, though the last alternative is restricted by the requirement $Ax_m/D\equiv\varepsilon<1$) one can evaluate the position of principal maximum of SNR_k:

$$\left.\frac{dQ_k(z^2)}{d\varepsilon}\right|_{z=0} = 0 \Rightarrow \left[\frac{d\,SNR_k}{d\varepsilon} = 0 \Leftrightarrow D_{\max} = \frac{\Delta U}{2k}\right].$$

Corresponding graphics of SNR_k are depicted in Fig. 1–3. Maxima in Fig. 1 are located at predicted values. Figs. 2,3 show deviation from description based on neglecting of intricate z-containing factor behavior – they correspond to low ω, when maxima are situated in the region of D, where $\varepsilon \geq 1$. Fig. 3 shows anomalous relative heights and position of multiple maxima of the first three non-zero harmonics.

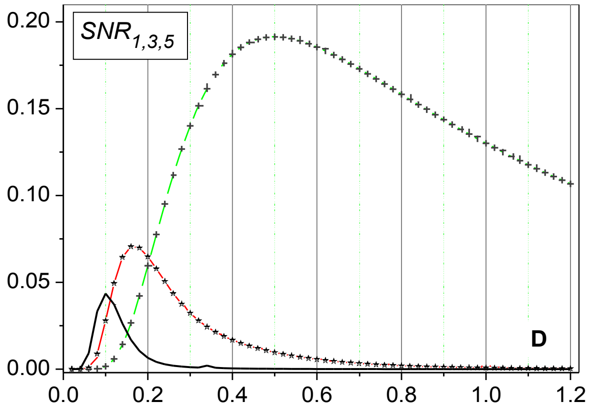

Fig. 1. *$SNR_{1,3,5}|_{\omega=0.1}$ – regular situation. Crosses correspond to SNR_1, the light curve to SNR_3, and the dark one to SNR_5*

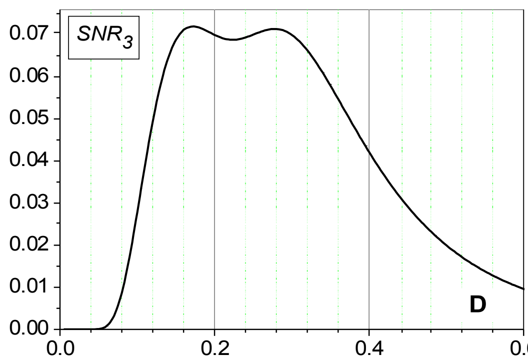

Fig. 2. *$SNR_3|_{\omega=0.018}$ – low frequency, 2 maxima with equal heights*

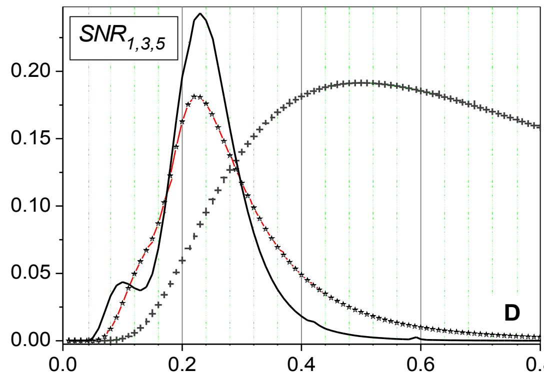

Fig. 3. *$SNR_{1,3,5}|_{\omega=0.003}$*

The main difference from corresponding characteristics of the first harmonic is due to more complicated role of z in the last expressions. This makes possible multiple maxima, which are really observed at such ω that the "resonant" D corresponds to $z \gg 1$ (or $r \gg \omega$). In this case the behavior of $z(D)$ is necessary to be accounted for. Due to the different locations of maxima of different factors containing z in G_k and SNR_k one can find special values of ω that provide several maxima of comparable heights for these functions, though universally these functions possess only one maximum in the region of lower and higher frequencies.

For $\varepsilon \geq 1$ one should use greater number of terms in (21). The number of required terms increases rapidly when ε becomes greater than 1. So, in order to find out whether there exist other peculiarities in output characteristic for considered system, it is more convenient to use numerical calculation of different harmonics of $H(t)$. Such numerical integration shows that the corrections to (24) in the vicinities of maxima are not significant. These precise results do not change the non-monotonous character of both curves $G_k(\varepsilon)$ and $SNR_k(\varepsilon)$ (or, equivalently $G_k(D)$ and $SNR_k(D)$). The only difference I would like to mention is less rapid vanishing of both functions at $\varepsilon \to \infty$ $(\Leftrightarrow D \to 0)$ – these vanish exponentially according to (24,26) and less rapidly following the numerical results.

I am glad to acknowledge the participation of Yu.L. Bolotin in both the putting of the problem and further elaboration of questions needed consideration as well as for his support in search for relevant literature and inspiring discussions.

REFERENCES

1.B. McNamara, K. Wiesenfeld. Theory of Stochastic Resonance // *Phys. Rev. A.* 1989, v. 39, p. 4854-4869.

2.R. Benzi, A. Sutera, A. Vulpiani. The mechanism of stochastic resonance // *J. Phys. A: Math.Gen.* 1982, v. 34, p. 10-16.

3.L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni. Stochastic Resonance // *Rev. Mod. Phys.* 1998, v. 70, p. 223-287.

4.P. Jung, P. Hänggi. Stochastic Nonlinear Dynamics Modulated by External Periodic Forces // *Europhys. Lett.* 1989, v. 8 (6), p. 505-510.

5.P. Bartussek, P. Hänggi, P. Jung. Stochastic Resonance in Optical Bistable Systems // *Phys. Rev. E.* 1994, v. 49, №5, p. 3930-3939.

6.P. Jung, P. Talkner. Suppression of Higher Harmonics at Noise Induced Resonances // *Phys. Rev. E.* 1995, v. 51, №3, p. 2640-2644.

7.A.R. Bulsara, M.E. Inchiosa, L. Gammaitoni. Noise-Controlled Resonance Behavior in Nonlinear Dynamical Systems with Broken Symmetry // *Phys. Rev. Lett.* 1996, v. 77, №11, p. 2162-2165.

8.V.A. Shneidman, P. Jung, P. Hänggi. Power Spectrum of a Driven Bistable System // *Europhys. Lett.* 1994, v. 26, №8, p. 571-576.