STOCHASTIC REFINEMENT SYSTEMS AND NONSEPARABLE RANDOM GEOMETRIC STRUCTURES

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Probabilistic study method of fractally disodered media is proposed. It is suggested on the base of introducing of the space refinement concept.

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1. PROBABILISTIC DESCRIPTION PROBLEM OF FRACTALLY STRUCTURED MEDIA

Study of disordered solid-state media during last decades has resulted in the concept that fractality is the typical property of their structure [1]. We mean the fractality as a strong nonuniformity of medium components space distribution with a wide spectrum of length scales characterizing that distribution (f.e., from 10^{-1}sm to 10^{-6} sm). Simplest example of such a situation is medium with a large volume fraction of hollows in comparison with volume fraction occupied properly by substance. Besides, it is impossible to characterize such a structure by definite number of space scales when averaging on space domains which are rather small and contain large number of particles. Most likely, one can observe, in this case, such a particle distribution which is transformed to a structure being locally similar to itself when the scale of averaging is changed in several times (conventionally, on the order). Thus, the fractality is structural recurrence on essential different scales of observation. It is developed on background of general disorder (of stochasticity).

Traditional apparatus of theoretical physics is found inadequate to description of the above described physical situation and some similar ones. Firstly, it is connected with the fact that theoretical models were constructed of those objects which are locally very simple from topological point of view, i.e. they have definite topological dimensions (see, f.e., [2]). Mostly, opposite geometric concepts in the sense of Euclidean topology are used. Models were built either of isolated mass points (zero dimension) or on the base of the concept of continuous medium (Euclidean space dimension).

In accordance with the above-pointed out approach, following probabilistic models named *random point fields* were usually used when modelling disordered structures in statistical physics. They are some stochastic analogies of nonrandom geometric objects with definite topological dimension. If dimension equal to zero, corresponding point fields are called the *ordinary* ones. Their realizations consist of isolated points with the probability one. Many models of statistical mechanics are built on the base of point fields of similar type (see f.e. [3]). In the opposite case, when modelling heterogeneous disordered media (see f.e. [4]), random point fields must have another qualitative properties. They represent, with probability one, random domains in Euclidean space which either consist of disconnected components with different forms and are randomly situated or formate random connected labyrinths penetrated whole system space. In last case, ones say about presence of the *percolation* in stochastic geometry structure. It is possible mixed case also. From the qualitative point of view, random point fields formed by the above-described way are called *separable* [5–7]. Last term means that each their realization, with the probability one, is uniquely reconstructed by enumeration of all its points which coincide with points of a standard countable set □ being dense everywhere in space.

Obviously, constructing of probability distribution for both above-described opposite qualitative types of random point fields is possible on the base of countable collection of distribution functions which depend on coordinates of space points. In first case, it is connected with the fact that each realization contains only countable set of points and, therefore, one can realize corresponding distribution functions as depending ones on coordinates of sequentially increasing finite subsets of realization points [3]. In second case, distribution functions depend on coordinates of sequentially increasing finite subsets of □. In this time, probability distributions of random sets X are built by Kolmogorov's method (see [7]) on the base of separable random field $\Theta(\mathbf{x} \mid □ X)$ such that their realizations are indicator functions of □X.

The above-marked countability of distribution functions collection that is sufficient for probabilistic description of random point fields under consideration, is their essential property, since each probability distribution being built on the base of an uncountable set of generating random events has some unsuitable properties that makes it to be unavailable for using in physical problems. In particular, it does not allow to "measure" physical quantities whose possible values depend on system state in an uncountable space point set [8]. For example, when modelling stochastic geometry structures on the base of point random fields, such a quantity is the fractal dimension.

Fractal is somewhat intermediate structure between the model of isolated mass points and the continuity medium. It contains the continuum of points which, however, are distributed rather rare. That is why, a stochastic fractal is impossible to describe in probabilistic sense neither on the base of the ordinary random field scheme or on the base of the separable field notion. It takes place because of its realizations contain a point continuum and are not defined by each countable subset among them. At this, in connection with the above-mentioned circumstance, i.e. due to necessity of introduction of a function collection being no more than countable and defining the situation of stochastic fractal points, each probability distribution for corresponding structure is impossible to realize analytically on the base of functions depending only on point coordinates.

Undevelopment of mathematical models for description of fractal media taking place to date may be explained by the above-shown contradiction. Just it is in spite of long history of the fractal theory beginning from mathematical work [9] and from pioneer attempts of physical applications in [10].

2. PROBABILITY DISTRIBUTIONS ON SPACE REFINEMENTS

Overcoming of the contradiction appearing when trying to determine probability distribution for stochastic fractally distributed structures is suggested if coordinate description of their points is interchanged on description on the base of imbedding space *refinements*. For this aim, concept of the space basis is extended. Together with unit vectors $\mathbf{e}_1,..., \mathbf{e}_d$ of coordinate axes and the origin, it is included additional element in the basis, i.e. the integer N>1 that we shall call *subdivision parameter*. Let, for simplicity, the imbedding space be the cube $\Lambda= [0, L]^d \subset \mathbb{R}^d$, d=2,3. One of its vertices is the origin and corresponding ages form the basic frame. We shall call collection of disjoint identical cubes $\mathrm{A}_{\mathbf{x}}^{(m)}$, m=0,1,2,.. having the age size L/N^m . This collection we shall name the m -order *cellular subdivision* of space Λ. Each vector **x** serves as a mark for each of these cubes. Vectors **x** refer to vertices which are similar to the origin. Let us denote $\{□\}_m$ the set of all possible vectors **x** at fixed m. Then each set Z⊂Λ is defined by sequence of its *roughening* $K_m(Z)$, m=0,1,2,...on corresponding subdivisions. Roughening operators are defined by formula

$$\mathrm{K}_m(Z) = \bigcup_{\mathbf{x}:\mathrm{A}_{\mathbf{x}}^{(m)} \cap Z \neq \varnothing} \mathrm{A}_{\mathbf{x}}^{(m)} .$$

The sequence $□=\{□_m$, m=0,1,2,...} of all subdivisions is named the *refinement* of imbedding space. At fixed value m, the collection of all subsets $H\subset K_m$ is finite and equal to $2^{|□_m|}$, $|□_m|=N^{dm}$. Further, the sign | · | denotes number of elements in the set that must take place of the dot. Therefore, the collection of all $H\subset□_m$ for possible values m is countable. Thus, we get opportunity of probabilistic description of each stochastic geometric structure with realizations X by means of probabilities

$$P_m(H) = \{X : K_m (X) = H\}. \quad (1)$$

At this, naturally, it is necessary to exclude H≠∅, i.e. $P_m(\varnothing)=0$, and, due to obvious inclusions $K_{m+1}(X) \subset K_m(X)$, probabilities $P_m(H)$ must satisfy the following consistency condition

$$P_m(H) = \sum_{G:K_m(G)=H} P_{m+1}(G), \; H \subset□_m \quad (2)$$

It is found that probabilities $P_m(H)$ for all $□_m$, m=0,1,2,...of space refinement such that Eqs. (2) are satisfied, determine a probability distribution on so-called σ-algebra generated by random events $\{X: K_m(X) = H\}$, $H\subset□_m$, m=0,1,2,.... However, a peculiar proof of this fact must be done, since these events do not form neither Borel's system [8] traditionally used in the random processes theory nor Dynkin's system [11]. In other words, it is necessary to prove the special theorem about the measure continuation for case under consideration.

After solution of basic problem about statistical description of stochastic fractals, it is necessary to be learned how suitable phenomenological models may be synthesized using some simple probabilistic conjectures concerns their geometric structure. In this case, many possibilities appear and it is necessary to be guided by definite physical arguments when constructing model in order to describe somewhat physical reality.

3. PRINCIPLES OF MODEL CONSTRUCTION OF FRACTALLY STRUCTURED MEDIA

First of all, we must notice that the mathematical method described in previous item is universal. It permits to define probability distribution, in principle, for any random set in Λ, in particular, well-known geometric models used in the stochastic fractals theory are included, i.e. fractal lines and fractal surfaces which are graphs of separable random fields. However, at first, description of these structures by the way pointed out is found to be inconvenient, and, at second, they are not objects of our consideration. Since each fractal medium is a thermodynamic system, so, in particular, it must be fill out whole imbedding space in average. Therefore, for selecting those random point fields which describe fractal media, we must require reliability of the translation invariance in stochastic sense (at least, on sufficiently large scales of averaging). In ideal case, this property is expressed mathematically in the form

$$\mathbf{P}\{\Gamma + \mathbf{a}\} = \mathbf{P}\{\Gamma\},$$

where Γ is an arbitrary random event, Γ={X} that has definite probability,

$$\Gamma+ \mathbf{a} = \{X+ \mathbf{a}\}, \mathbf{a}\in \mathbb{R}^d , \|\mathbf{a}\| <<L .$$

Further, geometric structures having modelled must be topologically and metrically uniform. The most important topological characterization of each fractal that dictates it properties in a great many, is the so-called fractal dimension D (see [12]). Here, we determine it on the space refinement by the following way

$$D = \inf\{\delta:\ \lim_{m\to\infty} \frac{K_m(X)}{N^{md\delta}} = 0\}. \qquad (3)$$

It follows from this formula that D, in general case, is a functional of random realization X, so it is a random value. Therefore, the requirement of the topological uniformity is reduced to selecting such models of stochastic fractals that value D is constant with the probability one. Metrical value closely connected with the dimension D is the so-called fractal measure $\mu(\cdot)$. It is a function of X̧B, where B is any domain in Λ. From the physical point of view, this function is the volume of matter concentrated in that part of fractal realization X which is cut out by B. The measure μ is defined on the base of space refinement by the formula

$$\mu(\mathbf{X}\cap B)=\inf\{s(\frac{L}{N^m})|\{\mathbf{x}: A_x^{(m)}\cap B\cap\mathbf{X}\neq\varnothing\}|\} \qquad (4)$$

where the function $s(\lambda)$ is positive at $\lambda>0$ and such that $\lim_{\lambda\to 0} s(\lambda) = 0$. That function is determined precisely to asymptotic equivalence. We name the measure μ defined by the way as *measure of s -type*. Obviously, μ is a stochastic measure since its values depend on the realization X. The value μ(Ḅ X) may be considered as a random field on Λ when collection of domains B is restricted by the ensemble $\{B:\ A_0^{(n)} + \mathbf{z}\ ;\ \mathbf{z}\in\Lambda\}$.

Essentially, the s -type measure may be nontrivial for each point **z** only at special choice of the function s to within asymptotic equivalence. The non-triviality is implied in the sense that the measure is finite and nonzero. Just the measure type is the metrical characterization of fractal. However, it may be changed for each fixed fractal realization X when $\mathbf{z}\in\Lambda$ varying. Besides, it may properly depend on the realization X. We shall imply the metrical uniformity of fractal as such a property when the measure type does not depend both on X and on **z**. In this case, we shall speak that measure is uniform and has a nonrandom type.

For topologically uniform fractal with dimension D, the function $s(\cdot)$ may be chosen in the form $s(\lambda)= \lambda^D g(\lambda)$, where $g(\lambda)>0$ and it is a function slowly varying at zero,

$$\lim_{\lambda\to 0}\frac{g(\lambda + C)}{g(\lambda)} = 1\,. \qquad (5)$$

Presence of uniform measure having nonrandom type on given fractal permits to introduce by identical manner a stochastic integral on each realization X,

$$\int_X v(\mathbf{x})\mu(d\mathbf{x}) = \inf_n \sum_{\mathbf{x}\in\square_n} v(\mathbf{x})\mu(A^n_{\mathbf{x}}\cap X)\,.$$

In turn, it permits to introduce some fractal bundles, i.e. to introduce fields of physical values on X. At this, each physical value is connected not with points $\mathbf{z}\in X$ but with fractal part contained in each domain $B\subset\Lambda$, i.e. the field is implied as a linear functional V[v] of any function $v(\mathbf{z})$ on Λ,

$$V[v] = \int_{X\cap B} v(\mathbf{x})\mu(d\mathbf{x})\,. \qquad (6)$$

Let us remark that one may to require some special properties when fractal medium modelling. Namely, fractal may have or may have no such global topological property that is named the percolation. Also, it may have the isotropy property in average and/or, analogously, the self-similarity one. They are named, correspondingly, the *stochastic isotropy* and the *stochastic self-similarity*. Mathematically, they are expressed in the following form

$$\mathbf{P}\{R\,\Gamma\} = \mathbf{P}\{\Gamma\},\qquad \mathbf{P}\{\lambda\,\Gamma\} = \mathbf{P}\{\Gamma\},$$

where R is an arbitrary space rotation matrix $R\Gamma = \{RX\}$ and $0<\lambda<1$, $\lambda\,\Gamma=\{\lambda X\}$. However, presence of stochastic self-similarity is possible only for special choice of similarity factors λ.

4. SIMPLEST STOCHASTIC MODELS

We shall consider random point fields in the cube $\Lambda= [0, L]^d \subset R^d$, d=3. Let $H\subset\square_m$. Then for each $l < m$, this set is represented in the form of a disjoint union of some subsets $S_l(\mathbf{x},H)$, $\mathbf{x}\in K_l(H)$ where $S_l(\mathbf{x},H) =\{\mathbf{y}\in H: K_l(\mathbf{y}) = \mathbf{x}\}$. They are originals of points **x** at the roughening operation. It is obvious that

$$P_{m+1}(H) = Q_m(H|\,K_m(H))P_m(K_m(H)), \qquad (7)$$

$$Q_m(H|K_m(H))=\mathbf{P}\{X:K_{m+1}(X)=H|K_m(X)=G\}.$$

For each m = 1, 2,... and for each $\mathbf{x}\in\square_m$ at fixed number m, it is defined by uniform way the "similarity" map T : $S_m(\mathbf{x},\square_{m+1}) \to \square_1$ which consists of the coincidence of the cell $A^{(m)}_{\mathbf{x}}$ with the cell $A^{(m)}_{\mathbf{0}}$ and of the consequent extension it in N^m times up to the coincidence with Λ.

Let us consider now the so-called random point fields with markovian refinements. The distinctive their property consists in the explicit construction conditional probability in the following form

$$Q_m(H,\cdot G) = \prod_{\mathbf{x}\in G} q(TS_m(\mathbf{x},H))\,, \qquad (8)$$

where the function $q(\cdot) > 0$ is defined on such a collection of all subsets of the set $\Re_1$ which does not contain the empty set $\varnothing$ This function is the probability distribution on the pointed out space of elementary random events, i.e.

$$\sum_{\varnothing\neq\boldsymbol{\sigma}\subset\Re_1} q(\boldsymbol{\sigma}) = 1\,.$$

The simplest class of random point fields models is realized by formulas (7), (8) when $q(\boldsymbol{\sigma}) = p(|\boldsymbol{\sigma}|)$ where $|\boldsymbol{\sigma}|$ is the number of elements in $\boldsymbol{\sigma}$ and

$$p(l) = C(\frac{p}{1-p})^l \cdot l^\alpha,\quad l = 1, 2, ..., N^d, \qquad (9)$$

where $\alpha\geq 0$, $1 > p > 0$ and the constant C is determined on the base of the normality condition

$$\sum_{l=1}^{N^d} p(l)\binom{N^d}{l} = 1 .$$

For random point fields defined by formulas (7)–(9), it has proved [13] that their fractal dimension D is nonrandom and it is the same for all regions of the fractal under consideration. The dimension D may be calculated in such case on the base of the formula

$$D = \frac{\ln \sum_{\varnothing \neq \boldsymbol{\sigma} \subset \Re_1} |\boldsymbol{\sigma}| \, q(\boldsymbol{\sigma})}{\ln N} .$$

In particular, the above mentioned example of stochastic fractal that is determined by the function (9) at $\alpha=1$, we found

$$D = \frac{\ln(1 + p(N^d - 1))}{\ln N} .$$

5. OBSERVABLES

At last, we shall indicate how physical properties of fractal structures appear. One of important observable geometric value is the structure factor $I(\mathbf{k})$ being a function of wave vector $\mathbf{k}$. It is measured experimentally by dispersion of electromagnetic radiation and is defined analogously to [4],

$$I(\mathbf{k})=\langle \mu^{(-1)}(X) \int_X \exp\{-i\mathbf{k}(\mathbf{x}-\mathbf{y})\}\mu(d\mathbf{x})\mu(d\mathbf{y}) \rangle .$$

In general case, statistical moments of random function

$$\int_X \exp\{-i\mathbf{k}\mathbf{x}\}\mu(d\mathbf{x})$$

depending on $\mathbf{k}$ are such observables. Thus, probability distribution of this function is observable. From this point of view, main theoretical problem is evaluation of all possible statistical characterizations of that distribution. They are defined on the base of the characteristic functional

$$\square[u]=\square\exp\{i\int u(\mathbf{k})d\mathbf{k}\int_X \exp\{-i\mathbf{k}\mathbf{x}\}\mu(d\mathbf{x})\}\rangle , \quad (10)$$

where $u(\mathbf{k})$ is an arbitrary continuous, rapidly decreasing function on $\square^3$. Denoting $\bar{u}(\mathbf{x})$ its Fourier image, we see that moments $<\mu(d\mathbf{x}_1)...\mu(d\mathbf{x}_n)>$ of the stochastic measure $\mu(\cdot)$ are subjected to determination. Their calculation is fulfilled by averaging on the base of probability distributions

$$Q_n[\mathbf{x}_1,...,\mathbf{x}_n]=\sum_{H\in \square_{\square}: A^n_{\mathbf{x}_i} \subset H, i=1,...,n} P_m(H), \quad m\le n \qquad (11)$$

which conserve their sense when passing to limit both on subdivision order m and on L (the thermodynamic limit).

6. PROGRAM

In connection with the approach to theoretical study of fractal media projected in this communication, following principal questions may be set up.

1. How to synthesize random point fields with guarantee of their topological and metrical uniformities in frameworks of introduced mathematical concepts?
2. How to construct models having those or others remarkable geometric properties, i.e. stochastic isotropy and/or local stochastic self-similarity, presence or absence of percolation? In last case, study of percolation phase diagram in the model parameter space is necessary.
3. How to calculate the structure factor by effective way?
4. How to construct phenomenological thermodynamics for media that is described by mathematical models of introduced type? Apparently, surface tension must play peculiar role in this case. (We are grateful to S.V. Peletminskii who pointed out this fact to us.)
5. How to build statistical mechanics of physical values fluctuations (including nonequilibruim ones) on stochastic fractals?
6. How to construct macroscopic electrodynamics and elasticity theory for media being stochastic fractals?

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