ALTERNATIVE MECHANISMS OF STOCHASTICITY IN MAPS WITH DISCONTINUITIES

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The maps with discontinuities dynamic chaos research is made. The borders of stability and bifurcation of cycles cutting are obtained. The structure of the stable cycles tree is determined. A new mechanism of spontaneous transition to chaos caused by non-local bifurcation of stable cycles cutting is found out.

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1. INTRODUCTION

Deterministic chaos [1, 2] is a permanent attribute of many physical systems. Complex irregular behavior, as a rule, arises as a result of system bifurcations at continuous change of its control parameters. The mechanisms and scenarios of this transition make one of the most important problems of modern dynamic systems theory. It's convenient to study them on simple discrete models (maps) [3-5]. Chaotic dynamics of such abstract systems are closely connected with fundamental problems of ergodicity (statistic physics), turbulence (hydrodynamics), non-linear oscillations and waves (radiophysics, biology etc.), quantization, cosmology etc. The reason of this connection is in non-linearity of the differential equations describing every particular case. Non-linearity leads to exponential sensitivity of smooth solutions because of the change of their initial conditions. As a result, "roughened" (at the choice of initial conditions) motion becomes entangled and non-regular. This is the general mechanism of dynamic chaos. But there is another generator of non-regularity – singularities (discontinuities) of different nature. If they are present in dynamic equations, then the flow causes their growth in number. At special conditions they "pollute" the phase space so much that chaos is inevitable. In the given paper on the example of a simple discontinuous piece-wise linear map model a new mechanism of "spontaneous chaotization" is found. The loss of stability here may takes place not continuously, as in all known examples, but as a result of special bifurcation of stable map cycles cutting.

2. MAP WITH DISCONTINUITY

Let us choose a model of one-dimensional piece-wise linear map with a discontinuity and the extrem points. We have two-parameter family (Fig. 1)

$$f(x)=\begin{cases}\mu_1 x, x\in I_1=[0,a];\\ \mu_2 x+(A-2aE)/(1-2a), x\in I_2=[a,1/2-0];\\ \mu_2 x+[1+(A-2(1-a)E)/(1-2a)], x\in I_3=[1/2+0,1-a];\\ \mu_1 x+(1-\mu_1), x\in I_4=[1-a,1];\end{cases} \quad (1)$$

where the multiplicators μ_α that correspond to the natural segment partition $I=\bigcup I_\alpha$, $\alpha=1,2,3,4$; ($\mu_1=\mu_4=A/a$ and $\mu_2=\mu_3=2(A-E)/(1-2a)$); amplitude $A=f(a)$ and top of discontinuity $E=(1+\varepsilon)/2$ with gap ε are introduced. As independent parameters let us choose the pair (A,E). In the quadrant $1/2\le\{A,E\}\le 1$ all the properties of the map (1) will be studied. There is the symmetry $f(x)=1-f(\tilde{x});\tilde{x}=1-x;f(\tilde{x})=\tilde{f}(x);\tilde{f}(\tilde{f}(x))=x$.

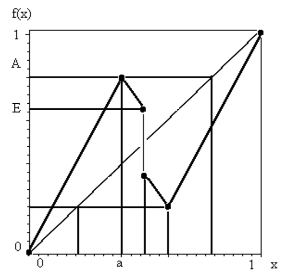

Fig. 1. *Map with a discontinuity*

The map (1) built differs from the "saw" $f(x)=Kx(\bmod 1)$ and "zigzag" $f(x)=Kx+(1-K)\theta(x-1/2)\theta(1-x)$ (θ is a Heavyside function) at $K>1, x\in[0,1]$, and "tent" $f(x)=|A-x|$ at $1/2<A<1$, $x\in[-1/2,x_-]\cup[x_+,1/2]$ with a crack $x_+-x_-=\delta>0$, studied before by the combination of non-monotony and discontinuity factors with the conservation of phase space simple connection (the function itself stands the discontinuity, but I doesn't). Let us note incomplete, $0<\varepsilon<1$, in general case, character of the discontinuity. That's the difference from the maps containing the "addition" of $\bmod 1$.

3. STABILITY

The transition from regular system iteration behavior (1) to the chaotic one should be regarded in full space $Z = I \times \Omega$ – direct product of phase space I to the space of parameters $\Omega = \{A, E\}$. One-dimensionality of the system lets us regard just one of its cuts. Let us map the transition to chaos or stability loss of the system with a diagram of stable cycles (drains) on the plane (A, E). To the structure of cycles plays the main role in the investigation of dynamics.

An arbitrary cycle $C = \{f^i(x^*)\}_{i=0}^{p-1}$ of the map (1) is characterized by a set of defining parameters: cycle period $p \geq 2$ (fixed points are not considered); the kind k and topological type W. The period defines the number of the points of the cycle with the initial point x^*. The kind corresponds to the number of its points $x_i = f^i(x^*)$ ($f^i = f \circ \ldots \circ f$ is the composition of transformations divisible by i) in the regions neighbouring to the discontinuity. Type W corresponds to the path of the cycle passing through the intervals I_α ($\alpha = 1,2,3,4$). As symbolic addresses defining the ways let us take them dual to each other (reversibility) $s = 2,3; u = 4,1$ or $\tilde{s} = 3,2; \tilde{u} = 1,4$. Every way is a sequence of such numeric indexes. The reversibility divides all the cycles to symmetric $C_b = \tilde{C}_b$ and non-symmetric $C_a \neq \tilde{C}_a$ (dual drains). For any cycle $\tilde{\tilde{C}} = C$. The stability of the cycle is defined by its multiplicator (or $m(C) \equiv |\mu(C)|$)

$$\mu(C) = \prod_{i=0}^{p-1} \frac{df(f^i(x^*))}{dx} = \mu_1^{p-k}\mu_2^{k} = \left(\frac{A}{a}\right)^{p-k}\left[\frac{2(A-E)}{1-2a}\right]^k \quad (2)$$

The multiplicators of dual cycles coincide $\mu(C) = \mu(\tilde{C})$, and the ways are connected by a symmetric change $W \to \tilde{W} \Leftrightarrow s,u \to \tilde{s},\tilde{u}$. So, for non-symmetric cycles it's enough to consider any of the types dual to each other.

The limit of stability is defined by the condition $m(C) = 1$. On the parameter plane two lines over and under the diagonal $A = E$ (i.e., for $sign(E - A) = \pm 1$) correspond to it

$$E = E_{\pm}^{p,k}(A) = A\left\{1 \pm a^{\frac{p-k}{k}}\left(\frac{1}{2} - a\right)A^{-\frac{p}{k}}\right\}. \quad (3)$$

As $0 < a < 1/2 \leq A \leq 1$, with the growth of the cycle period its stability zone is reduced. The curves of stability are quickly approaching to one another and the diagonal. On the diagonal $A = E$ all the cycles are stable with null $\mu(A = E) = 0$. Stability zones, limited by the curves (3), do not define the condition of system stability in general. Borders (3) don't indicate if the drains for which they are calculated (co)exist. The existing of the drains is connected with the bifurcation of cutting.

4. CYCLES CUTTING

Cycles cutting is the reconstruction of their structure at special values of control parameters when the cycles of one kind disappear, and the cycles of other kind appear. (In reality these bifurcations are richer, so that only groups of cycles can be deleted. Besides, the cutting of one cycle isn't necessarily accompanied by the appearing of another.) At that, the transition takes place abrupt, so the corresponding multiplicators of the map (in a cycle point) discontinue. Let us note that in all known cycle bifurcations their multiplicator changes continuously. (First of all this concerns the cascade of period doublings [6, 7]. At intermittence the fixed point disappears [8], but the map multiplicator in it is continuous. For a strange attractor [9, 10] this also isn't an exception.) . In general case cutting has non-local character, because it touches the phase space regions far from the bifurcating cycle.

The conditions (parameters) at which the cutting takes place can be defined in two ways. The first way is from the metric considerations. This means that with the change of the parameters the cycle under consideration passes according to its way the intervals of chosen partition I, not leaving them. When this rule is broken for one of the cycle points, $x_\alpha(A^*, E^*) \notin I_\alpha$, the cycle is deleted (at critical parameters A^*, E^*). So the coordinates of the initially existing p-cycle points $x_\alpha(A, E) \in I_\alpha$ smoothly depend on the parameters of the map and are defined from the system of equations

$$f^p(x_i) = x_i;\, x_i \neq f^k(x_i),\, k = 1,\ldots,p-1;\, i = 1,\ldots,p. \quad (4)$$

The defining of cycle coordinates demands "guessing" of its way. The latter defines the explicit form of the equations (4) that contain the composition of the maps. On the other hand, it isn't known a priori if such a cycle exists. That's why the procedure lies in enumeration of possible ways. For piece-wise linear maps its result is unique, because the solution of linear system (4) for a cycle with given period (or way) is unique. But with the growth of cycle period (the number of ways growth exponentially, $M_p \propto 4^p$) such a procedure becomes burdensome. Here geometric methods help.

The cutting is closely connected with Markovian partitions of the map. Among them are distinctive those generated by the properties of the map – critical points and discontinuities. It's easy to verify that the cutting takes place when one of such "special" Markovian partition replaces another. At the same time the way of a "special" partition corresponds to that of the deleted cycle. In dependence of the choice generating such a partition of a special point (and its pre-images or images, which is equivalent for a cycle) we have two types of map cutting conditions (1)

$$A := f^{p-k}(f^k(a))\Big|_{W(f^k(a))} = f^k(a);$$

$$\tilde{A} := f^{p-k}(f^k(1-a))\Big|_{W(f^k(1-a))} = f^k(1-a); \quad (5)$$

$$E; \tilde{E} := f^{p-k}(f^k(1/2 \mp 0))\Big|_{W(f^k(1/2 \mp 0))} = f^k(1/2 \mp 0), \quad (6)$$

where $W(x^*)$ denotes the way, beginning at a special initial point (p and k are the period and the kind of the cycle, the way of a "special" cycle coinciding with the way of the cycle under consideration). According to W the branches of f are placed in the composition. Having solved the equations, we obtain the relations for critical parameters (A^*, E^*). The number of such conditions is $N_p^c = cp$, where c is the full number of special points of the map (1) $c = 4$. Let us note that unlike the cycle with the change of the initial point of cycle its way also changes. This causes change of the sequence order of functions in the compositions (5) and (6) and so of their result. (The composition operation is non-commutative, with the exception of linear functions.)

For some cycles of the map a part of the conditions (5)-(6) becomes equivalent. For drains only two of the independent relations are left

$$A := f^p(a)\big|_{W(a)} = a \Rightarrow A = A_W(a);$$

$$E := f^p(1/2)\big|_{W(1/2)} = 1/2 \Rightarrow E = E_W(A). \qquad (7)$$

Explicit form of such dependencies is not written for concrete examples. Besides, the kind of drains is limited by the values $k = 1, 2$ and there are no more dual drains in one point of the parameter space. (The proof of this theorem is geometrically obvious.)

So, only the drain ways of the following type are possible $W(C_a^s) = suU; W(C_b^s) = suU\tilde{s}\tilde{u}\tilde{U}$ (see naming cycles above), where U is a sequence of added in turns addresses u and $\tilde{u}$ (the former are not less numerous than the latter). Let us choose for more distinctness one of a pair of dual drains, for instance, with the initial address $s = 2$ (and elements $u = 4, \tilde{u} = 1$). Let us build the "stem" of the drain tree – a recurrent sequence of irreducible (non-symmetric)stable cycles

$$W(C_a^s) = [24]; [244]; [2441, 2444]; [24414; 24441, 24444];$$
$$[244141, 244144; 244411, 244414; 244441, 244444]; \qquad (8)$$

and so on and its "branches" of symmetric drains, obtained by the "doubling" of the way,

$$W(C_b^s) = [23]; \{[2431]; \ldots; W(C_b^s) = W(C_a^s)\tilde{W}(C_a^s)\}. \qquad (9)$$

The cascade of drain doublings can be continued (Fig. 2). At that a symmetric drain of the type $suU\tilde{s}\tilde{u}\tilde{U}$ first becomes a non-symmetric one of the type $suUu\tilde{u}\tilde{U}$ ($\tilde{s}$ can be changed to u), and then it is "doubled", and so on. We don't stop on the details of this cascade. Only we shall note that it is generated by the bifurcation of cutting. The doubling takes place inside the stability zone on the plane (A, E), on the lines implicitly specified by the equations (7) of E – type. The module of cycle multiplicator leaps

$$m(C_b^s) = m^2(C_a^s) < m(C_a^s) < 1. \qquad (10)$$

In a continuous system such a cascade is principally impossible. Every irreducible cycle (8) generates its own cascade of doubling for the cycles, whose ways are easy to write. Here dominate (by the area of stability zone occupied) the cascades of the main set of drains of the type 24,244,2444 and so on. For example, 24-cascade (preceded by a 23-drain as a doubling of deleted 1-cycle on the discontinuity $x = 1/2$) is the following (see Fig. 2)

$$24, 2431; 2441, 24413114; 24414114; \ldots. \qquad (11)$$

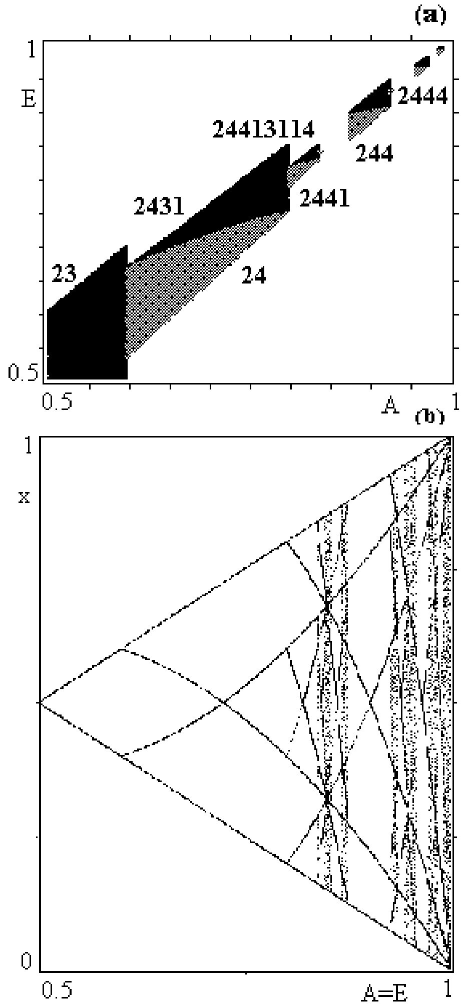

Fig. 2. *(a) Tree of drains with cycles doubling on parameter plane (A,E) and (b) bifurcation diagram on phase plane (x,A=E)*

It isn't difficult to copy out any fragment of the drain tree made and define its borders. If we introduce cycle ordering ($\prec$) along the diagonal (from smaller $D = A \equiv E$ to greater $1/2 \le D \le 1$), than the drains are ordered in ascending period (a symmetric drain precedes a non-symmetric one of the same period). Irreducible (non-symmetric drains), generating those cascades, are ordered according to an easy-checked rule of "monotony"

$$W1 \prec W \prec W4; W11 \prec W1 \prec W \prec W41 \prec W4; \ldots, \qquad (12)$$

where W is the way of an initial arbitrary chosen drain, to which the addresses 4- and 1- are attached according to the "stem" (8) of cycles. The monotony of the full tree ways corresponds to the monotony of their multiplicators

$$W(C_1) \prec W(C_2) \Leftrightarrow 0 \le m(C_1) < m(C_2) < 1, \qquad (13)$$

the equality being untrue on the borders of cutting.

External borders for the given type cycles region are the borders of stability, $E = E_+(A)$ for a symmetric cycle and $E = E_-(A)$ for non-symmetric one, and one of the borders of cutting (7), namely, A -type one. For a symmetric cycle this border will be the right one and for non-symmetric the left one. The inner border corresponds to cutting with period doubling (Fig. 2). Along the diagonal $A = E$ a self-similar structure is formed. For understanding the mechanisms of chaotization its "telescopical" nature (Fig. 2) and the presence of dips (drains cutting) at the transition from a stable region to a chaotic one because of the movement along the diagonal are important. The first dip appears on the border of cycle 23 cutting at $A_{23} = A_{24} = 1 - a$. At the movement inside the 23 drain region, between the stability lines of cycles 24 and 23, so correspondingly $E_\pm^{2,1} = A[1 \pm a(1/2 - a)/A^2]$, $E_\pm^{2,2} = A[1 \pm (1/2 - a)/A]$, the jump stability loss of the system takes place at the cutting of drain 23, $A > 1 - a$. The transition to chaos is spontaneous, because in this region drain 23 can be far from stability loss, $|\mu(23)| < 1$. Analogous thing happens on the rest of the cutting "teeth" of the drains tree. The loss of stability at the movement across the tree, $A = const$, takes place on the stability borders and has ordinary local nature.

5. CHAOS MECHANISMS

The cutting of stable cycles leads to spontaneous chaotizaion of the non-continuous map. Here it is necessary to pass the cutting "tooth" on the drains diagram. Also other changes of control parameters are possible. The deformation in one-parameter map families corresponds (1) to the movement along different lines on the plane (A, E) .

The telescopical structure of the tree leads to the elementary nature of the following transformations: 1) an ordinary stability loss; 2) stability cutting; 3) doubling by cutting. A combination of these acts can give different scenarios of transition to chaos of the following types: drain–chaos, drain–cascade, doublings–chaos, drain–chaos–drain (cascade)–chaos etc. At that the general scheme naturally contains mechanisms of period doubling with drains cutting, spontaneous chaotization and their different combinations, i.e. a special "intermittence" with chaos interruptions by regular gaps.

Among structurally stable (rough) bifurcations, still spontaneous chaotization stands out. Let us note that in some sense it can not only stimulate, but also suppress the general tendency to stability loss (prevent "thermal death"). Really, the border of an ordinary stability zone (in derived smooth systems) is wedged into the chaos zone in the parameter space; so "stable" space is more possible than "unstable". This corresponds to the point of the cutting "tooth". While at the bottom of the "tooth" a chaotic zone is wedged into the stable one. So small fluctuations of external noise are more possible to leave the system stable. Depending on the cutting parameters (the size of the "tooth") this self-regularity mechanism can be important for real physical singular systems.

6. SUMMARY

In conclusion let us name the features of singular dynamic systems discovered in this paper.

1. The bifurcation of cycles cutting. The connection between the cutting conditions and Markovian partitions.
2. Cutting of cycles without a loss of stability, but with period doubling. Drains cascade. Self-similarity of separate cascade elements and whole cascades.
3. Stable cycles cutting. Telescopical structure of drains tree.
4. Spontaneous chaotization mechanism. Characteristics of stochasticity (multiplicators of map, Lyapunov exponent and so on) change in leaps.
5. The variety of scenarios for a transition to stochasticity with a universal role of drains cutting.

Further development of the theory can be connected with the research of essentially non-linear, multidimensional maps and maps with non-trivial phase space topology. Discontinuity factor will be also defining for them.

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