# THE SECOND BORN APPROXIMATION IN THEORY OF BREMSSTRAHLUNG OF RELATIVISTIC ELECTRONS AND POSITRONS IN CRYSTAL

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The formulae for the cross section of bremsstrahlung by relativistic electrons and positrons taking into account the contribution of the second Born approximation are obtained. The dependence of the radiation cross section in the field of atomic plane on the sign of charge of the particle is considered.

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In this paper we consider the second Born correction to the process of bremsstrahlung of high energy electrons and positrons in an external field. The account of the second Born approximation leads to dependence of the radiation cross section on the charge sign of radiating particle. It is demonstrated that contribution of the second Born approximation can be substantial for the case of coherent interaction of radiating particle with atoms of a crystal.

## 1. DIFFERENTIAL CROSS SECTION OF THE RADIATION PROCESS

The cross section of the bremsstrahlung of electrons and positrons in an external field is determined by the relation [1]

$$d\sigma = \frac{e^2}{4(2\pi)^4 \omega \varepsilon \varepsilon'} \delta(\varepsilon - \varepsilon' - \omega) \overline{|M|^2} d^3 p' d^3 k , \qquad (1)$$

where $(\varepsilon, p)$ and $(\varepsilon', p')$ are the energy and the momentum of the initial and final particles, $\omega$ and $k$ are the frequency and the wave vector of the radiated wave, $\delta(\varepsilon - \varepsilon' - \omega)$ is the delta-function that determines the energy conservation under radiation. According to the rules of diagram technique [1] the squared matrix element in (1) can be written with the account of the contribution of the second Born approximation in the form

$$|M|^2 = |M_1|^2 U_g^{\,2} - 2U_g \,\mathrm{Re} \int M_1 M_2^* U_q U_{g-q} \frac{d^3 q}{(2\pi)^3} , \qquad (2)$$

where $U_g$ is the Fourier component of the potential energy of the electron (positron) in an external field, $g_\mu = (0, g)$ is the 4-momentum transferred to the external field (it is assumed that the external field is stationary), $g_\mu = p_\mu - p'_\mu - k_\mu$, $M_1$ and $M_2$ are the matrix elements which determine contributions of the first and the second Born approximations (see Fig. 1):

$$M_1 = \overline{u'} \left[ b\hat{e} - \frac{\hat{e}\hat{g}\gamma_0}{2\varepsilon\sigma_g} - \frac{\gamma_0 \hat{g}\hat{e}}{2\varepsilon' \tau_g} \right] u , \qquad (3)$$

$$M_2 = \overline{u'} \left\{ \hat{e} \frac{\hat{p} + m - \hat{g}}{2p\sigma_g} \gamma_0 \frac{1 - \frac{\hat{q}\gamma_0}{2\varepsilon}}{v\sigma_q} - \frac{1 + \frac{\gamma_0 \hat{q}'}{2\varepsilon'}}{v'\tau_{q'}} \hat{e} \frac{1 - \frac{\hat{q}\gamma_0}{2\varepsilon}}{v\sigma_q} + \right.$$

$$\left. + \frac{1 + \frac{\gamma_0 \hat{q}'}{2\varepsilon'}}{v'\tau_{q'}} \gamma_0 \frac{\hat{p}' + m + \hat{g}}{2p'\tau_g} \hat{e} \right\} u , \qquad (4)$$

where $e_\mu$ is the photon polarization vector, $v$ and $v'$ are the initial and final velocities of the electron, $q'_\mu = g_\mu - q_\mu$. The values $b$, $\sigma_g$ and $\tau_g$ in $M_1$ and $M_2$ are determined by the relations

$$b = \frac{1}{\sigma_g} - \frac{1}{\tau_g} , \qquad \sigma_g = g_{||} - \frac{g^2}{2p} ,$$

$$\tau_g = g_{||} + \vec{n}_\perp \vec{g} + \frac{g^2}{2p'} , \qquad (5)$$

where $n = p'/p'$ is the unit vector along the momentum $p'$ direction, and $n_\perp$ are the components of this vector orthogonal to the $p$.

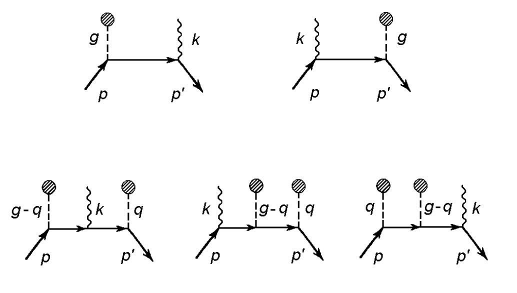


***Fig. 1.*** *Feynman diagrams corresponding to the first and the second Born approximations in the process of bremsstrahlung in an external field*

The matrix element of the radiation process depends on the momentum transferred to the external field $g$ in an explicit form. The cross section itself can be also expressed directly through the transferred momentum (and also through the angle $\vartheta$ between the vectors $k$ and $p$). Such presentation is especially convenient in the range of small values of the transferred momentum $g_\perp << m$, because it is possible to make an expansion in the matrix element by the powers of $g_\perp$ in this case. Transformation to the new variables is described in [2,3]. The differential cross section in this case takes the form:

$$d\sigma = \frac{e^4}{(2\pi)^4}\frac{\varepsilon'}{\varepsilon}\overline{|M|^2}\frac{\delta}{m^2}\frac{d\omega}{\omega}\frac{dy}{\sqrt{1-y^2}}d^3g\,, \qquad (6)$$

where $\delta = \omega m^2/2\varepsilon\varepsilon'$. The variable $y$ is connected to $\vartheta$ by the relation

$$(\varepsilon\vartheta/m)^2 = f + y\sqrt{a}\,, \qquad -1 \le y \le 1\,, \qquad (7)$$

where

$$a = \frac{4g_\perp^2}{m^2\delta}\left(g_\parallel - \delta - \frac{g_\perp^2}{2\varepsilon}\right),$$

$$f = \frac{1}{\delta}\left(g_{||} - \delta - \frac{g_\perp^2}{2\varepsilon} + \frac{g_\perp^2\delta}{m^2}\right).$$

From the fact that the value $a$ in the radical in (7) must be positive one can conclude that

$$g_\parallel \ge \delta + g_\perp^2/2\varepsilon\,. \qquad (8)$$

Note that Eq. (7) determines the possible values of the radiation angle $\vartheta$ under given values of $g_\parallel$ and $g_\perp$.

Eq. (4) can be simplified that makes the procedure of summing over polarization of interacting particles more easy than for original Eq. (4). Neglecting the terms of the order of $m^2/\varepsilon^2$, we obtain after some calculations the following expression for $M_2$:

$$M_2 = \bar{u}'\left\{\left[Q_1 - \frac{\omega}{\varepsilon\varepsilon'\tau_g}\left(\hat{e} + \frac{\gamma_0\hat{g}\hat{e}}{2\varepsilon'}\right)\right]\frac{\vec{q}_\perp\vec{q}'_\perp}{2\varepsilon\sigma_q\sigma_{q'}} + Q_2\right\}u\,, (9)$$

where $Q_1$ is the spinor structure of $M_1$,

$$Q_1 = \hat{e}b - \frac{\hat{e}\hat{g}\gamma_0}{2\varepsilon\sigma_g} - \frac{\gamma_0\hat{g}\hat{e}}{2\varepsilon'\tau_g}\,,$$

and

$$Q_2 = -\frac{\hat{e}\hat{g}\hat{q}_\perp}{4\varepsilon^2\sigma_g\sigma_q} - \frac{\hat{q}'_\perp\hat{g}\hat{e}}{4\varepsilon'^2\tau_g\tau_{q'}} + \frac{\gamma_0\hat{q}'_\perp\hat{e}\hat{q}_\perp\gamma_0}{4\varepsilon\varepsilon'\tau_{q'}\sigma_q}\,.$$

After summing over polarizations of final particles and averaging over polarization of initial particle we obtain with accuracy to terms of order of $m^2/\varepsilon^2$ the following equations for the values $\overline{|M_1|^2}$ and $\overline{M_1M_2^*}$ in (1):

$$\overline{|M_1|^2} = \frac{2}{g_\parallel^2}\left[\left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}\right)g_\perp^2 - 2m^2b^2\right], \qquad (10)$$

$$\overline{M_1M_2^*} = \frac{\vec{q}_\perp\vec{q}'_\perp}{2\varepsilon\sigma_q\sigma_{q'}}\left\{\overline{|M_1|^2} - \frac{2\omega}{\varepsilon'\tau_g}\left[2(p'\cdot p - 2m^2)b - \right.\right.$$
$$\left.\left. - \frac{\varepsilon}{\varepsilon'}\frac{\vec{g}^2}{\tau_g} + \vec{p}'_\perp\cdot\vec{g}\left(\frac{\varepsilon+\varepsilon'}{\varepsilon'\sigma_g} - \frac{2\varepsilon}{\varepsilon'\tau_g}\right)\right] - 8\frac{\varepsilon}{\varepsilon'}b\right\}, \qquad (11)$$

Substituting these equations into (1), we obtain after the integration over $y$ and expansion on $g_\perp/m$ the following expression for the cross section of the radiation with account of the second Born approximation:

$$d\sigma = \frac{e^2}{2\pi^2}\frac{\varepsilon'}{\varepsilon}\frac{\delta}{m^2}\frac{d\omega}{\omega}\frac{g_\perp^2}{g_{||}^2}dg_\parallel g_\perp dg_\perp\left\{F|U_g|^2 + \right.$$
$$+\frac{1}{(2\pi)^3\varepsilon}\left[F + \frac{\omega}{\varepsilon'}\left(1 - 4\frac{\delta}{g_\parallel}\left(1-\frac{\delta}{g_\parallel}\right) + \frac{\omega^2}{2\varepsilon\varepsilon'}\left(1-\frac{\delta}{g_\parallel}\right)\right)\right]\times$$
$$\left.\times U_g\,\mathrm{Re}\int d^3q\frac{(\vec{g}_\perp - \vec{q}_\perp)\vec{q}_\perp}{(g_{||}-q_{||}+i0)(q_{||}+i0)}U_qU_{g-q}\right\}, \qquad (12)$$

where

$$F = 1 + \frac{\omega^2}{2\varepsilon\varepsilon'} - 2\frac{\delta}{g_\parallel}\left(1 - \frac{\delta}{g_\parallel}\right).$$

Let us consider some particular cases of (12). If the condition $\delta << q_{\parallel eff}$ is satisfied, where $q_{\parallel eff}$ are the characteristic values of the longitudinal component of the momentum $q$ in (12), we can neglect the dependence of $U_g$ and $U_{g-q}$ on $g_\parallel$ in (12). After integration over $g_\parallel$ we obtain that

$$d\sigma(g_\perp) = dw(g_\perp)\left(1 + \frac{3}{4}\frac{\omega^2}{\varepsilon\varepsilon'}\right)\cdot\frac{d^2g_\perp}{(2\pi)^2}\left\{|U_g|^2 - \right.$$

$$\times\left(1 + O\left(\frac{g_{||eff}}{q_{||eff}}\right)\right), \qquad (13)$$

where

$$dw(g_\perp) = \frac{2e^2}{3\pi}\frac{\varepsilon'}{\varepsilon}\frac{g_\perp^2}{m^2}\frac{d\omega}{\omega}\,. \qquad (14)$$

For $\omega << \varepsilon$ Eq. (13) corresponds to the product of the radiation probability $dw/d\omega$ and the cross section of elastic scattering of the particle in the external field $d\sigma_{el}$ with account of contribution of the second Born approximation,

$$d\sigma_{el}(g_\perp) = \frac{d^2g_\perp}{4\pi^2}\times$$
$$\times\left\{|U_g|^2 - \frac{1}{\varepsilon}U_g\,\mathrm{Re}\int\frac{d^3q}{(2\pi)^3}\frac{\vec{q}_\perp\vec{q}'_\perp}{(q_\parallel - i0)^2}U_qU_{q'}\right\}.$$

For the Coulomb field of the nucleus with charge $Z|e|$ the last equation transforms to the form

$$d\sigma_{el}(g_\perp) = \frac{\pi Z^2 e^4 d\Omega}{\varepsilon^2 \vartheta^4}\left\{1 - \frac{e}{|e|}\frac{\pi Z e^2}{2}\vartheta\right\},$$

where the scattering angle $\vartheta \approx g_\perp / p$. The last result coincides with the corresponding result of the paper [4] obtained by different method. For arbitrary external field the formula for $d\sigma_{el}$ was obtained in [5,6].

Note that radiation of electrons in Coulomb field exceeds slightly the radiation of positrons. That is due to the fact that electron attracted by the nucleus moves in the region with larger gradient of the potential than the positron.

So in the range of frequencies $\omega \sim \varepsilon$ the theorem about factorization of the radiation cross section, according to which

$$d\sigma \approx dw(g_\perp) d\sigma_{el}(g_\perp), \tag{15}$$

is justified with an accuracy to the correction which determines the contribution of the second Born approximation.

## 2. THE CROSS SECTION FOR RADIATION OF RELATIVISTIC ELECTRONS AND POSITRONS IN THE FIELD OF ATOMIC PLANE IN A CRYSTAL

We can see that dependence of the radiation cross section on the particle charge sign in the case of radiation of high energy electrons and positrons in the field of single atom is rather small. Different situation arises for coherent interaction of relativistic particles with atoms of crystal lattice. In this case, due to the coherent effect the dependence of the radiation cross section on the particle charge sign can be substantially amplified in comparison with analogous dependence of the radiation cross section in an amorphous medium. The attention to this fact was paid in [7] during consideration of contribution of the second Born approximation into coherent radiation cross section of relativistic electrons in the field of atomic plane of the crystal. It was demonstrated that in considered case the relative contribution of the second Born approximation into coherent radiation cross section is determined by the parameter

$$\alpha_p = \frac{Ze^2 R}{\varepsilon a^2 \theta^2} \sim \frac{\theta_c^2}{\theta^2} \tag{16}$$

which represents by the order of value the ratio of the squared critical angle of plane channeling [8] to the squared angle of incidence $\theta$ of the beam to the atomic plane (here $Z|e|$ is the charge of the nucleus of crystal lattice atom, $R$ is the screening radius of the atomic potential, $a$ is the average distance between atoms in the crystal plane). In this case the Born expansion of the radiation cross section is valid if $\alpha_p << 1$. The parameter $\alpha_p$ rapidly increases with $\theta$ decrease. Under $\alpha_p \sim 1$ the account of effects of channeling and above-barrier motion of particles in respect to the crystal atomic plane is necessary [3,8,9].

So consider the coherent radiation of electrons and positrons in the field of continuous potential of one of the atomic planes in a crystal under incidence of the beam under small angle $\theta$ to this plane. The potential energy of the particle in continuous potential of the plane is determined by Eq. (8,9)

$$U(x) = \frac{1}{L_y L_z}\int dydz \sum_{n=1}^{N} u(\vec{r} - \vec{r}_n), \tag{17}$$

where $u(r - r_n)$ is the particle potential energy in field of the single atom of crystal plane located in the point $r_n$, $L_y$ and $L_z$ are the linear dimensions of the plane and $x$ is the coordinate, orthogonal to the atomic plane of the crystal (summation in (17) is made over all atoms of the crystal plane). Taking the atomic potential in the form of the screened Coulomb potential, and

$$u(r) = \frac{Ze|e|}{r} e^{-r/R},$$

we find the expression for the Fourier component of (17):

$$U_g = (2\pi)^2 \delta(g_z)\delta(g_y)\frac{1}{a_y a_z} u_g, \tag{18}$$

where $a_y$ and $a_z$ are the distances between atoms in the plane along the axes $y$ and $z$, and

$$u_g = \frac{4\pi Ze|e|}{\vec{g}^2 + R^{-2}}.$$

Substituting the Fourier component (18) into (12), we obtain the following expression for the radiation cross section

$$\begin{aligned} d\sigma = {} & Z^2 \alpha^3 16\pi \frac{N}{a_y a_z}\frac{\varepsilon'}{\varepsilon}\frac{\delta}{m^2}\frac{d\omega}{\omega}\frac{dg_x}{\theta^2} \times \\ & \times \left\{\left[1 + \frac{\omega^2}{2\varepsilon\varepsilon'} - 2\frac{\delta}{g_x\theta}\left(1 - \frac{\delta}{g_x\theta}\right)\right]\frac{1}{(g_x^2 + R^{-2})^2} + \right. \\ & + \frac{e}{|e|}\frac{2Z\alpha}{\varepsilon\, a_y a_z}\frac{1}{g_x^2 + R^{-2}} \times \\ & \times \left[1 + \frac{\omega^2}{2\varepsilon\varepsilon'} - 2\frac{\delta}{g_x\theta}\left(1 - \frac{\delta}{g_x\theta}\right) + \right. \\ & \left. + \frac{\omega}{\varepsilon'}\left(1 - 4\frac{\delta}{g_x\theta}\left(1 - \frac{\delta}{g_x\theta}\right) + \frac{\omega^2}{2\varepsilon\varepsilon'}\left(1 - \frac{\delta}{g_x\theta}\right)\right)\right] \times \\ & \left. \times \frac{1}{\theta^2}\frac{2\pi R}{g_x^2 + 4R^{-2}}\right\}. \end{aligned} \tag{19}$$

Here we have used the fact that in the case under consideration $g_{||} \approx \theta g_x$. The value $g_x$ here covers the range $g_x \geq \delta/\theta$. Under $\omega << \varepsilon$ Eq. (19) transforms to the corresponding result of the paper [7]. Note that in the case of interaction of the particle with continuous potential of the plane the radiation cross section cannot be presented in the form (15) for any photon frequencies. This is due to the fact that elastic scattering on the continuous plane can take place only to some fixed angles to the plane [5,6] because of energy and momentum conservation laws in the process of elastic scattering.

Eq. (19) demonstrates that for all frequencies the cross section of radiation by positrons turns out larger than the cross section of radiation by electrons, in difference to the case of radiation in Coulomb field. This result can be explained by the following way. The sign of the effect is determined by competition of two factors: (i) the electron is attracted to the plane and moves in the region with larger gradient of the potential than the positron, that leads to increase of radiation; (ii) in distinct to the positron, it spends less time in the region with large gradient of the potential, that leads to decrease of radiaton. In Coulomb field the first factor plays the determinative role, in the field of atomic plane - the second one.

Eq. (19) demonstrates also that radiation spectrum $\omega\, d\sigma / d\omega$ posesses the maximum in the range of frequencies satisfying the condition

$$\frac{2\varepsilon(\varepsilon-\omega)}{m^2\omega} \sim \frac{2R}{\theta}\,. \tag{20}$$

With the particle energy growth the position of this maximum moves to the region of high frequencies. For $\varepsilon \sim m^2 R/\theta$ the maximum is located in the region of frequencies for which the effect of recoil under radiation is substantial. The parameter (16) that determines dependence of the cross section on the particle charge sign for $\varepsilon \sim m^2 R/\theta$ takes the form

$$\alpha_p \sim \frac{Ze^2}{m^2 a^2 \theta}\,.$$

So in the range of energies under consideration with decrease of $\theta$ the dependence of coherent radiation cross section on the charge sign of the particle becomes substantial in the whole range of frequencies of radiated photons.

## 3. COHERENT RADIATION ON A SET OF ATOMIC PLANES IN THE SECOND BORN APPROXIMATION

The cross section of bremsstrahlung on the crystal is determined by relation [3]:

$$d\sigma = N(d\sigma_{coh} + d\sigma_{incoh})\,,$$

where $N$ is the whole number of atoms in crystal, $d\sigma_{coh}$ is the coherent part of radiation cross section caused by interference of radiation produced on different atoms regularly arranged in the crystal, $d\sigma_{incoh}$ is the incoherent part caused by thermal spread of atom positions in the crystal. For the case of interaction of the particle with the set of parallel atomic planes in the crystal, we can obtain the equation for $d\sigma_{coh}$ from (19) by change of integration over $dg_x$ to the summation over $(g_x)_n = \frac{2\pi}{a_x} n$:

$$\int_{\delta/\theta}^{\infty} dg_x \ldots \to \frac{2\pi}{a_x} \sum_{g_x \ge \delta/\theta} \ldots\,,$$

where $a_x$ is the distance between atomic planes. The cross section of coherent radiation of 1 GeV positrons and electrons incident under the angle $\theta = 4 \cdot 10^{-4}$ radians to the <011> plane of the Si crystal is shown on the Fig. 2. We can see that the difference between radiation cross sections for positrons and electrons in the case illustrated is of order of 10%.

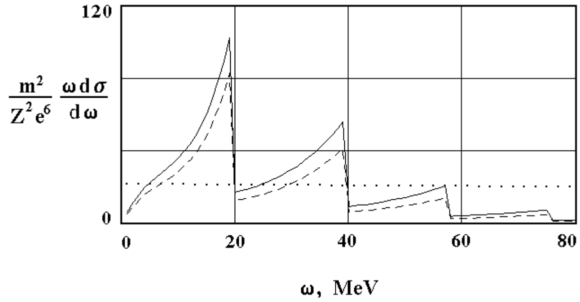


*Fig. 2. The cross section of coherent radiation of 1 GeV positrons (solid line) and electrons (dashed line) incident under the angle $\theta = 4 \cdot 10^{-4}$ radians to the <011> plane of the Si crystal. The dotted line shows the Bethe-Heitler cross section*

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