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CANONICAL VARIABLES IN MHD FLOWS WITH SHOCKS AND OTHER BREAKS

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The rigorous method of introduction of hamiltonian variables in MHD flows with discontinuities is presented. Appropriate boundary conditions are obtained and proved to give conventional continuity conditions at the boundary.

Introduction

The most general form of physical laws is those expressed by variational principle. A variational principle for fluid motion description in terms of Lagrangian variables has been developed, see, for instance, [1], [2] and literature cited therein. Formulation of variational principle in Eulerian description requires introduction of subsidiary fields, known as Lagrange markers. Apparently the most widely used procedure is the use of additional fields introduced by means of constraints, cf. [3] that leads to the fluid velocity excluding from the set of independent variables. The velocity representation arising is analogous to the Weber's one [2]. In spite of subsidiary variables appearing, the use of canonical variables leads to essential simplifications allowing to exploit highly-developed standard methods, cf. [4] – [8]. Hamiltonian variables for fluid with free boundary were introduced in [9]. Lately this results were generalized for any type of discontinuities, including shocks [10], [11]. Here we develop the variational principle and present corresponding canonical variables for MHD flows. The general variational principle presented describes any possible type of MHD discontinuity surfaces.

Variational principle

Let us assume that discontinuity surface separating two regions of continuous motion is defined by equation

$$R(\mathbf{r},t)=0 . \qquad (1)$$

Suppose the surface has no edges, thus locally Eq. (1) may be written in explicit form:

$$R(\mathbf{r},t)=z-\zeta(\mathbf{r}_\perp,t), \quad \mathbf{r}_\perp=(x,y). \qquad (2)$$

We start with the Hamiltonian

$$H=H_V+H_\Sigma , \qquad (3)$$

$$H_V=\int dt\int d\mathbf{r}\, H_V ,\ H_V=ñ\left(\frac{v^2}{2}+\varepsilon\right)+\frac{H^2}{8\pi}-\Phi \mathrm{div}\mathbf{H}, \quad (4)$$

$$H_\Sigma=\int dt\int d\mathbf{r}_\perp H_\Sigma ,\ H_\Sigma=\Psi\mathbf{u}\nabla R-\{\mathbf{G}\}\nabla R-\mathbf{k}\{\mathrm{rot}\,\mathbf{S}\}, \quad (5)$$

$$\mathbf{G}\equiv ñ\,\mathbf{v}'\Gamma+f\mathbf{H}+m\mathbf{S}, \quad \Gamma=\gamma+g\varphi+\boldsymbol{\eta}\boldsymbol{\mu}. \qquad (6)$$

Here ñ is the fluid density and ε is internal energy; braces indicate the jump of the corresponding quantity at the boundary $R=0$: $\{X\}\equiv X_1-X_2$; $\mathbf{v}'=\mathbf{v}-\mathbf{u}$ denotes the fluid velocity relative to the boundary one, $\mathbf{u}\equiv\mathbf{u}(\mathbf{r}_\perp,t)$. The latter may be obtained from Eq. (1) by differentiating it with respect to time: $dR/dt=\partial R/\partial t+\mathbf{u}\nabla R=0$, $\mathbf{u}\equiv d\mathbf{r}/dt|_{R=0}$. H_V represents the volume Hamiltonian and H_Σ the surface one. Lagrange multiplier $\boldsymbol{\Phi}$ guarantees nondivergent character of the magnetic field $\mathbf{H}$. The fluid velocity $\mathbf{v}$ is expressed via other fields by the representation

$$ñ\,\mathbf{v}+ñ\,\nabla\varphi+\mathbf{ë}\,\nabla\mathbf{ì}+\sigma\nabla s+[\mathbf{H}\,\mathrm{rot}\,\mathbf{S}]=0, \qquad (7)$$

where φ is scalar potential, $\boldsymbol{\mu}$ is the vector field of Lagrange markers and s is entropy density.[1]

The set of volume canonical coordinates Q and conjugated momenta P is as follows:

$$Q=(Q,\mathbf{S}),\ \ Q=(\varphi,\ \mathbf{ì},s);\ \ P=(P,\mathbf{H}),\ \ P=(ñ,\ \mathbf{ë},\sigma). \ (8)$$

The surface Hamiltonian H_Σ does not affect equations of motion but is essential for the boundary conditions. The surface canonical coordinate is ζ with conjugated momentum Ψ, other surface variables are Lagrange multipliers for constraints that guarantee correct boundary conditions. They also may be treated as canonical variables but with identically zero momenta. Terms in Γ, Eq. (6), provide correspondingly the mass, velocity potential and Lagrange markers fluxes continuity.

Thus the total action containing both volume and surface parts is as follows

$$A=A_V+A_\Sigma=\int dt\int d\mathbf{r}\left[\Sigma P\dot{Q}-H_V\right]+\int dt\int d\mathbf{r}_\perp\left[\Psi\dot{\zeta}-H_\Sigma\right] (9)$$

Variation of the action with respect to the volume values of canonical variables Q, P, and Φ leads to the set of Hamiltonian equations together with $\mathbf{H}$ nondivergency condition:

$$\frac{\partial\varphi}{\partial t}=\frac{\delta H}{\delta ñ}=w-\mathbf{v}\nabla\varphi-\frac{v^2}{2},\ \ \frac{\partial ñ}{\partial t}=-\frac{\delta H}{\delta\varphi}=-\mathrm{div}(ñ\,\mathbf{v}); \qquad (10)$$

$$\frac{\partial\mathbf{ì}}{\partial t}=\frac{\delta H}{\delta\lambda}=-(\mathbf{v}\nabla)\mathbf{ì},\ \ \frac{\partial\lambda_m}{\partial t}=-\frac{\delta H}{\delta\mu_m}=-\mathrm{div}(\lambda_m\mathbf{v}); \qquad (11)$$

$$\frac{\partial s}{\partial t}=\frac{\delta H}{\delta\sigma}=-\mathbf{v}\nabla s,\ \ \frac{\partial\sigma}{\partial t}=-\frac{\delta H}{\delta s}=-ñ\,T-\mathrm{div}(\mathbf{v}\sigma); \qquad (12)$$

$$\frac{\partial\mathbf{S}}{\partial t}=\frac{\delta H}{\delta\mathbf{H}}=\frac{\mathbf{H}}{4\pi}+\nabla\Phi-[\mathrm{rot}\,\mathbf{S},\mathbf{v}],\ \frac{\partial\mathbf{H}}{\partial t}=-\frac{\delta H}{\delta\mathbf{S}}=\mathrm{rot}\,[\mathbf{vH}], \quad (13)$$

$$\mathrm{div}\mathbf{H}=-\frac{\delta H}{\delta\Phi}=0 , \qquad (14)$$

where T is temperature. From Eqs. (10) – (14) follow the conventional MHD equations.

It should be noted that there exist some restrictions on the boundary values of generalized potentials

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[1] The term with entropy is necessary for consideration of shocks, cf. [10, 11], while the introduction of the term $\boldsymbol{\lambda}\nabla\boldsymbol{\mu}$ along with $[\mathbf{H}\,\mathrm{rot}\,\mathbf{S}]$ (in contrast with reduced representation, cf., for instance, [4, 5]) guarantees correct passage to the limit of conventional hydrodynamics.

$\varphi, \sigma, \boldsymbol{\mu}, \lambda$ and $\mathbf{S}$ that should be taken into account if the boundary exists. These restrictions follow from the jumps of the corresponding volume equations and define time evolution of the jumps in natural assumption that corresponding surface sources are absent:

$$\partial\{\varphi\}/\partial t + \{v'_n \partial_n \varphi\} = \{w - v^2/2\} \quad (15)$$

$$\partial\{\text{ì}\}/\partial t + \{v'_n \partial_n \text{ì}\} = 0, \quad (16)$$

$$\partial\{\text{ë}/\text{ñ}\}/\partial t + \{v'_n \partial_n (\text{ë}/\text{ñ})\} = 0, \quad (17)$$

$$\partial\{\sigma/\text{ñ}\}/\partial t + \{v'_n \partial_n (\sigma/\text{ñ})\} = -\{T\}, \quad (18)$$

$$\partial\{\mathbf{S}\}/\partial t = \{\mathbf{H}/4\pi + \nabla\Phi + [\mathbf{v}, \operatorname{rot}\mathbf{S}] + (\mathbf{u}\nabla)\mathbf{S}\}. \quad (19)$$

If some of these variables do not undergo jump (as, for instance, it is for φ, ì, $\text{ë}/\text{ñ}$ in the shock case) then the corresponding equation presents necessary time independent condition on the boundary values of the volume variables and their spatial derivatives.

Fulfilling volume equations we obtain the following residual (surface) variation of the action

$$\delta A = \delta A\,|_{surf} \equiv (\delta A)_{bound} + \delta A_{\Sigma},$$

$$(\delta A)_{bound} = \int dt \int d\mathbf{r}_\perp \{P\dot{Q} - H\}\delta\zeta + \quad (20)$$

$$+ \int dt \int d\mathbf{r}_\perp \{(\mathbf{v}'\nabla R) P\delta Q + \Phi(\delta\mathbf{H}, \nabla R) - (\mathbf{H}\delta\mathbf{S})(\mathbf{u}\nabla R)\}.$$

Here the second integral arises from integrations by parts, it contains boundary values of the volume momenta P and coordinates Q. The first integral corresponds to the variation of the discontinuity surface.

The variational principle formulation should be supplemented by the set of quantities that are varied independently at the surface. The suitable choice is not unique, we accept it[2] consisting of i) all surface variables; ii) both sides boundary values of variations of all volume momenta, entropy, and of spatial derivatives of all volume coordinates; iii) boundary values of variations of φ, ì, $\mathbf{S}$ from one side of the surface. Thus we suppose absence of jumps when crossing the surface for the latter quantities, i.e.,

$$\{\delta\varphi\} = \{\delta\mu\} = \{\delta\mathbf{S}\} = 0.$$

In this assumptions the functional derivatives of the action with respect to the boundary values of all hydrodynamic momenta P, spatial derivatives of hydrodynamic coordinates ∇Q and $\operatorname{rot}\mathbf{S}$ gives us equations:

$$\delta\nabla Q \Rightarrow P\Gamma\nabla R = 0, \quad (21)$$

$$\delta P \Rightarrow \Gamma\nabla Q\nabla R = 0,$$

$$\text{ä}\operatorname{rot}\mathbf{S} \Rightarrow \Gamma[\mathbf{H}, \nabla R] = 0$$

that are fulfilled simultaneously from each side of the surface if $\Gamma = 0$. The case $\Gamma \neq 0$ is trivial one, because all hydrodynamic momenta P vanish on the boundary as it follows from (21), consequently $\text{ñ} = 0$ and the fluid is absent.

[2] This choice allows to consider all possible types of discontinuities in the framework of one variational principle. If one restricts to the special types of discontinuities, then less subsidiary constraints required and more simple principles can be formulated

Variation with respect to the rest variables leads to the set of boundary conditions:[3]

$$\delta\gamma \Rightarrow \{j\} = 0, \quad j \equiv \text{ñ}(\mathbf{v}'\nabla R), \quad (22)$$

$$\text{ä}\,\text{ì} \Rightarrow \{(\text{ë} + \text{ñ}\,\text{ç})(\mathbf{v}'\nabla R)\} = 0, \quad (23)$$

$$\delta\eta \Rightarrow \{\text{ñ}\,\text{ì}(\mathbf{v}'\nabla R)\} = 0, \quad (24)$$

$$\delta g \Rightarrow \{\text{ñ}\,\varphi(\mathbf{v}'\nabla R)\} = 0, \quad (25)$$

$$\delta s \Rightarrow \sigma(\mathbf{v}'\nabla R) = 0, \quad (26)$$

$$\delta f \Rightarrow \{H_n\} = 0, \quad (27)$$

$$\delta m \Rightarrow \{S_n\} = 0, \quad (28)$$

$$\delta\mathbf{k} \Rightarrow \{\operatorname{rot}\mathbf{S}\} = 0, \quad (29)$$

$$\delta\mathbf{H} \Rightarrow f + \Phi = 0, \quad (30)$$

$$\delta\mathbf{S} \Rightarrow \{\mathbf{H}v'_n - \mathbf{v}H_n\} + m\nabla R = 0, \quad (31)$$

$$\delta\mathbf{u} \Rightarrow -\Psi\nabla R - \{\rho\Gamma\}\nabla R = 0, \quad (32)$$

$$\delta\zeta \Rightarrow \{P\dot{Q} - H + \operatorname{div}\mathbf{G}\} - \dot{\Psi} - \nabla_\perp(\mathbf{u}_\perp\Psi) = 0. \quad (33)$$

Two last equations for ζ and Ψ are of the Hamiltonian form:

$$\dot{\zeta} = \delta H/\delta\Psi = \mathbf{u}\nabla R,$$

$$\dot{\Psi} = -\delta H/\delta\zeta = \{P\dot{Q} - H + \operatorname{div}\mathbf{G}\} - \nabla_\perp(\mathbf{u}_\perp\Psi).$$

Note here that Eq. (32) with (21) taken into account leads to vanishing of the conjugated to the boundary displacement ζ momentum: $\Psi = 0$, for all types of discontinuities and thus the last equation takes a form

$$\{P\dot{Q} - H + \operatorname{div}\mathbf{G}\} = 0.$$

From the presented boundary conditions follow conventional MHD conditions, namely fluid momentum flux continuity:

$$\{p + \text{ñ}\,v_n'^2 + \mathbf{H}_\tau^2/8\pi - H_n^2/8\pi\} = 0, \quad (34)$$

$$\{\text{ñ}\,v'_n\mathbf{v}'_\tau + \mathbf{H}_\tau H_n/4\pi\} = 0, \quad (35)$$

and the energy flux continuity:

$$\{\text{ñ}\,v'_n(v'^2/2 + w) + [v'_n H^2 - H_n(\mathbf{v}'\mathbf{H})]/4\pi\} = 0. \quad (36)$$

Here p denotes fluid pressure. Although Eqs. (22) – (33) involve large number of subsidiary variables, nevertheless it may be checked they do not contain extraneous to the conventional MHD cases (for short we will not discuss this point in details), but describe all possible types of discontinuities intrinsic to MHD: slide, contact, shock, and rotational discontinuity. Leaving out the procedure of splitting Eqs. (21), (22) – (33) let us discuss below the special cases.

Shocks

Let us start with the shock discontinuity, where $j \neq 0$ and $\{\text{ñ}\} \neq 0$. First of all, if the mass flux is nonzero, it should be continuous due to (22). Along with equations (24), (25), (27) – (29) that are introduced by surface constraints

[3] Note that it is convenient to vary the surface term $\{\mathbf{G}\nabla R\}$ included in H_Σ in the form of volume integral, namely $\int d\mathbf{r}_\perp\{\mathbf{G}\}\nabla R = \int d\mathbf{r}\operatorname{div}\mathbf{G}$, assuming the surface functions $\gamma, g, \text{ç}, f, m$ be prolonged to the volume along normal $\mathbf{n} \equiv \nabla R/|\nabla R|$, i.e.,

$$\partial_n\gamma = \partial_n g = \partial_n\text{ç} = \partial_n f = \partial_n m = 0.$$

$$\{\mathtt{ì}\}=\{\varphi\}=\{H_n\}=\{S_n\}=\{\mathrm{rot}\,\mathbf{S}\}=0\,, \qquad (37)$$

we get the continuity of the tangential component of the electric field, Eq. (31),

$$j\{\mathbf{H}_\tau/ñ\}=H_n\{\mathbf{v}'_\tau\}. \qquad (38)$$

Eq. (23) defines the variable **ç** value and, in turn, leads to $\{\text{ë}/ñ\}=0.$ For $j\neq 0$ Eq. (26) leads to the boundary value of entropy momentum vanishing, $\sigma|_\zeta=0.$ The normal component of Eq. (31) yields $m=0.$ Eq. (33) with identity

$$\{P\dot{Q}-H\}=\{p+H_\tau^2/8\pi+\mathrm{div}(\Phi\mathbf{H})\},$$

which follows from the volume equations, together with Eqs. (22) – (33) and (19) leads to the continuity of the normal component of the fluid momenta's flux

$$\{p+ñ\,v_n'^2+\mathbf{H}_\tau^2/8\pi\}=0\,.$$

Let us prove that the tangential component of the momenta's flux (35) is also continuous. The first of the volume equations (13) and Eq. (38) can be presented in the following form:

$$(j+H_n a)\{\mathbf{H}_\tau/ñ\}=H_n^2\{1/ñ\}\mathrm{rot}_\tau\mathbf{S}\,,$$

$$\{\mathbf{H}_\tau/4\pi\}-a^2\{\mathbf{H}_\tau/ñ\}=(j-H_n a)\{1/ñ\}\mathrm{rot}_\tau\mathbf{S}$$

with $a=\mathrm{rot}_n\mathbf{S}$. Excluding $1/ñ$ jump from these equations we obtain that jumps of vectors $\mathbf{H}_\tau$ and $\mathbf{H}_\tau/ñ$ are parallel:

$$H_n^2\{\mathbf{H}_\tau/4\pi\}=j^2\{\mathbf{H}_\tau/ñ\}. \qquad (39)$$

It may be checked that substituting of tangential velocity jump $\{\mathbf{v}'_\tau\}=\{\mathbf{v}_\tau\}=-\{[\mathbf{H},\mathrm{rot}\,\mathbf{S}]_\tau/ñ\}$ to Eq. (35) and subsequent excluding of $\{1/ñ\}$ leads to the same equation (39). Thus the tangential component of the momenta's flux is also continuous.

Analogously, it can be checked that the energy flux is also continuous (we suppress proof for short).

Rotational discontinuities

Let us consider another type of discontinuities, $j\neq 0,$ $H_n\neq 0,$ but without density jump, $\{ñ\}=0$, consequently, $v'_n=0$, $\{v_n\}=0$

Note that Eq. (38) yields equation

$$\{\mathbf{H}_\tau\}=(H_n/v'_n)\{\mathbf{v}'_\tau\} \qquad (40)$$

that means that velocity and magnetic filed tangentional components jumps are codirectional. Using (40) and (35) we get $j=H_n/\sqrt{4\pi ñ}$. Boundary conditions for the variables reiterate the previous case with $\{1/ñ\}=0$ taken into account, so, evidently, the proof of the hydrodynamic fluxes continuity is analogous.

Contact discontinuities

If the mass flux vanishes from one side of the discontinuity surface $j=0$, than, as it is clear from it's continuity, it equals zero from the another one. This case corresponds to contact discontinuity if $H_n\neq 0$. Boundary conditions become much more simpler in these cases, as Eqs. (22) – (26) are satisfied identically. From Eqs. (31), (35), and (33) follow the continuity of the tangential components of velocity, magnetic field, and pressure: $\{\mathbf{v}'_\tau\}=0$, $\{\mathbf{H}_\tau\}=0$, $\{p\}=0.$ These equations are known to describe contact discontinuity. Note that surface tension effects are easily described by including the corresponding term to the surface Hamiltonian density:

$$H_\Sigma\to H_\Sigma+H_\alpha,\quad H_\alpha=\int d\mathbf{r}_\perp\alpha\left(\sqrt{1+(\nabla_\perp\zeta)^2}-1\right),$$

where α denotes surface tension coefficient.

Slide discontinuities

This case corresponds to $j=0$, $H_n=0.$ Then from Eq. (33) we obtain the continuity of normal component of the fluid momenta's flux

$$\{p+\mathbf{H}_\tau^2/8\pi\}=0.$$

It is rather simple to check up and momenta's flux tangential component continuity (35). Thus, we may assert that the set of boundary conditions describes slides.

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