Journal of Mathematical Physics, Analysis, Geometry
2007, vol. 3, No. 3, pp. 342–364

Homogenization of a Linear Nonstationary Navier–Stokes Equations System with a Time-Variant Domain with a Fine-Grained Boundary

N.K. Radyakin

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine

E-mail:radyakin@ilt.kharkov.ua

Received May 5, 2006

The problem of distortion of viscous incompressible fluid with a great number of solid particles with given velocities is considered. The diameters of particles and the distance between them tend to zero, and the number of particles tends to infinity. The asymptotic behavior of the solutions of the linear system of Navier–Stokes equations is considered. In a homogenized model there appears an additional term containing the strength tensor of a single particle.

Key words: Navier–Stokes equations, solid body dynamics, homogenization, suspension.
Mathematics Subject Classification 2000: 35B27, 35Q30, 74Q10, 76M30.

1. Introduction

This problem appeared in relation to the construction of a homogenized model of suspensions of small solid particles in a viscous incompressible fluid. Originally it is a rather complex one; it is described by the Navier–Stokes equations and the equations of solid body dynamics for particles. It is necessary to study the asymptotic behavior of the solutions of this system when the radii of particles and the distances between them tend to zero. A direct analysis of the problem faces some difficulties, as the domain occupied by the fluid is not known beforehand. It is natural to divide the problem into two parts, the first of which is to study the asymptotic behavior of the carrier fluid, disturbed by small particles, the trajectories of which are known. This problem for the linear NS system is solved in this paper. Our asymptotic analysis allowed to determine a homogenized system of equations for the carrier fluid. In fact, this system happens to be incomplete and for its completion it is necessary to study the influence of the carrier fluid on the particles (to find the Stocks-forces). A similar result is announced in [4].

2. Problem Statement and Formulation of the Main Result

Consider a large number of small solid particles Q_ε^i that move in a fluid filling the volume $\Omega \subset \mathbb{R}^3$. The mass centers of the particles move according to given trajectories $\vec{x}_\varepsilon^i(t)$ while the particles themselves rotate around $\vec{x}_\varepsilon^i(t)$ with given angular velocities $\vec{\theta}_\varepsilon^i(t)$. Assume that the functions $\vec{x}_\varepsilon^i(t)$, $\vec{\theta}_\varepsilon^i(t)$, as well as the surfaces ∂Q_ε^i of the particles and the border $\partial\Omega$ belong to the class C^2. Assume also that the particles while moving do not collide with each other and with the boundary $\partial\Omega$ (they remain at positive distances from each other).

Let us introduce the notation: $Q_\varepsilon^i(t)$ is a location of the particle with number i at the moment t (the domain occupied by the particle), Q_ε^{iT} is a trace of the particle i in $\mathbb{R}^4 = \mathbb{R}^3 \times [0,\infty)$ after moving for time T; $\partial Q_\varepsilon^{iT}$ is a side surface of the trace; $\Omega_T = \Omega \times (0,T]$; $\Omega_\varepsilon^T = \Omega_T \setminus \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^{iT}$; $\Omega_\varepsilon(t) = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^i(t)$; $Q_\varepsilon(t) = \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^i(t)$; $\Omega_\varepsilon(0) \equiv \Omega_\varepsilon$.

Consider the following initial boundary valued problem in domain Ω_ε^T:

$$\vec{u}_{\varepsilon t} - \nu\Delta\vec{u}_\varepsilon = -\nabla p_\varepsilon + \vec{g}(x,t), \ \operatorname{div}\vec{u}_\varepsilon = 0, \ (x,t) \in \Omega_\varepsilon^T; \tag{2.1}$$

$$\vec{u}_\varepsilon(x,t) = \vec{v}_\varepsilon^i(t) + \vec{\theta}_\varepsilon^i(t) \times (\vec{x} - \vec{x}_\varepsilon^i), \ (x,t) \in \partial Q_\varepsilon^{iT}; \tag{2.2}$$

$$\vec{u}_\varepsilon(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T]; \tag{2.3}$$

$$\vec{u}_\varepsilon(x,0) = \vec{U}_\varepsilon(x), \ x \in \Omega_\varepsilon, \tag{2.4}$$

where $\nu > 0$ is the viscosity of fluid, $\vec{u}_\varepsilon(x,t)$ and $p_\varepsilon(x,t)$ are the fields of velocities and pressures in the fluid, $\vec{x}_\varepsilon^i(t)$ is a center of mass of the particle i; $\vec{v}_\varepsilon^i(t) = \dot{\vec{x}}^i{}_\varepsilon(t)$ is a velocity of the center of mass, $\vec{\theta}_\varepsilon^i(t)$ is an instantaneous angular velocity of particle i, $\vec{U}_\varepsilon(x) \in W_2^{3/2}(\Omega_\varepsilon)$ is a divergent free vector function (the initial velocity field of the fluid), which satisfies the following conditions: $\vec{U}_\varepsilon(x) = \vec{v}_\varepsilon^i(0) + \vec{\theta}_\varepsilon^i(0) \times (\vec{x} - \vec{x}_\varepsilon^i(0))$ for $x \in \partial Q_\varepsilon^i(0)$; $\vec{U}_\varepsilon(x) = 0$ for $x \in \partial\Omega$, $\vec{g}(x,t) \in L_2(\Omega_T)$ is a volume force, which act on the fluid.

This problem describes a linear approximation of evolution in the time of the velocity field $\vec{u}_\varepsilon$ and pressures p_ε of the viscous incompressible fluid, disturbed by the solid particles moving in it by given trajectories. The boundary conditions (2.2) and (2.3) correspond to the condition of adhesion on the moving solid particles $Q_\varepsilon^i(t)$ and nonmoving boundary $\partial\Omega$.

As it is known from [1, 2], there exists a unique solution $\{\vec{u}_\varepsilon(x,t),\ p_\varepsilon(x,t)\}$ of the problem (2.1)–(2.4) such that $\vec{u}_\varepsilon \in W_2^{2,1}(\Omega_\varepsilon^T)$, $p_\varepsilon \in L_2(\Omega_\varepsilon^T)$, and the time interval T does not depend on the number of particles N_ε and on their sizes. Let us extend the vector of the velocity of the fluid $\vec{u}_\varepsilon$ onto the sets Q_ε^{iT} in accordance with the equalities (2.2) and consider the asymptotic behavior of the extended $\vec{u}_\varepsilon$ as $\varepsilon \to 0$, i.e., when the radii of particles tend to zero as $O(\varepsilon^3)$ and the number of particles N_ε increases as $O(\varepsilon^{-3})$.

We assume that the instantaneous angular velocities of the particles $\vec{\theta}_\varepsilon^i(t)$ and their derivatives are uniformly bounded with respect to ε. Assume also that the centers of masses of particles move according to the trajectories $\vec{x}_\varepsilon^i(t) = \vec{\Phi}(\vec{\xi}_\varepsilon^i, t)$, $i = 1, 2, \ldots, N_\varepsilon$, where $\vec{\Phi}(x,t)$ is a twice differentiable vector function on $\mathbb{R}^3 \times [0,T]$. This function is a one-one map of Ω into Ω for any $t \in [0,T]$, in the following way $\vec{\Phi}(x,0) = \vec{x}$. Let $\vec{\xi}_\varepsilon^i$ be the location of the center of mass of particle i in the moment $t = 0$. Suppose that the points $\vec{\xi}_\varepsilon^i$ are initially located in the domain $\Omega' \subset \Omega$, such that for each $t \in [0,T]$, δ-neighborhood $\Phi_\delta(\Omega', t)$ ($\delta = \max\limits_{1 \le i \le N_\varepsilon} (d_\varepsilon^i)^{2/3}$) of the domain $\Phi(\Omega', t)$ belongs to Ω. Here and further by d_ε^i we denote the external diameter of the set Q_ε^i.

In order to describe the asymptotic behavior of the solution $\vec{u}_\varepsilon(x)$ as $\varepsilon \to 0$, let us define the stress tensor of the sets Q_ε^i, which characterizes their mass and orientation in space.

Let Q be a bounded closed set in $\mathbb{R}^3$ with a smooth boundary ∂Q. Now let us consider the following boundary problem in $\mathbb{R}^3 \setminus Q$:

$$\Delta \vec{v}(x) = \nabla p(x), \ \operatorname{div} \vec{v} = 0, \ x \in \mathbf{R}^3 \setminus Q,$$

$$\vec{v}(x) = \vec{e}^k, \ x \in \partial Q, \ \vec{v}(x) = O(1/|x|), |x| \to \infty. \tag{2.5}$$

From the results described in [1] it follows that there is a unique solution $\vec{v}^k$ of this problem with the finite energy $\|\nabla \vec{v}^k\|_{L_2(\mathbf{R}^3 \setminus Q)} < \infty$. Suppose

$$C_{kl}(Q) = \int\limits_{\mathbf{R}^3 \setminus Q} (\nabla \vec{v}^k, \nabla \vec{v}^l) dx = \int\limits_{\mathbf{R}^3 \setminus Q} \sum_{i,j=1}^{3} \frac{\partial v_i^k}{\partial x_j} \frac{\partial v_i^l}{\partial x_j} dx, \quad k, l = 1, 2, 3. \tag{2.6}$$

It is obvious that the matrix $\{C_{kl}(Q)\}_{k,l=1}^3$ does not depend on the shifts of Q, and because of the linearity of the problem (2.5), when rotated, Q transforms as a second rank tensor:

$$C_{kl}(\Pi Q) = \sum_{i,j=1}^{3} C_{ij} \Pi_{ik} \Pi_{jl}, \tag{2.7}$$

where $\{\Pi_{ik}\}_{i,k=1}^3$ is the rotation matrix. It is easy to see that under homothetic contraction of Q the components of this tensor decrease proportionately to the diameter of Q.

We define $\{C_{kl}(Q)\}_{k,l=1}^3$ as a stress tensor of the set Q as it is similarly to newtonian capacity (see [3]) characterizes both the massiveness of the set Q and its orientation in space.

We denote by $C(Q_\varepsilon^i) = \{C_{kl}(Q_\varepsilon^i)\}_{k,l=1}^3$ a stress tensor of the set Q_ε^i; $\sum\limits_{(G)} C(Q_\varepsilon^i)$ is the sum over those values of the index i, for which Q_ε^i is strictly inside the domain $G \subset \Omega$; $T(Q_\varepsilon^i)$ is the minimal ball containing Q_ε^i; r_ε^i is the distance from $T(Q_\varepsilon^i)$ to $\bigcup\limits_{j\neq i}^{N_\varepsilon} T(Q_\varepsilon^j) \bigcup \partial\Omega$; $d_\varepsilon^i = \sup\limits_{x',x''\in Q_\varepsilon^i} |x'-x''|$ is the diameter of the set Q_ε^i.

The main result of the paper is the following theorem:

Theorem 1. *Let the following conditions hold as $\varepsilon \to 0$:*

1) $\lim\limits_{\varepsilon\to 0} \max\limits_{1\le i\le N_\varepsilon} d_\varepsilon^i = 0$, $d_\varepsilon < C_1\varepsilon^3$ $(C_1 > 0)$.

2) For each arbitrary $G \subset \Omega$ and each $t \in [0,T]$

$$\lim_{\varepsilon\to 0} \sum_{(G)} C_{kl}(Q_\varepsilon^i(t)) = \int_G C_{kl}(x,t)dx,$$

where $C(x,t) = \{C_{kl}(x,t)\}_{k,l=1}^3$ is a continuous tensor in Ω_T (the sum $\sum\limits_{(G)}$ is over those values of index i, for which $Q_\varepsilon^i(t) \in G$).

3) For each t: $t \in [0,T]$ $d_\varepsilon^i < C(r_\varepsilon^i(t))^3$, where C does not depend on i, ε and t.

4) The sequence of the initial vector functions $\{\vec{U}_\varepsilon(x), \varepsilon \to 0\}$, extended to the set $\bigcup\limits_{i=1}^{N_\varepsilon} Q_\varepsilon^i(0)$ by using the equation (2.2), weakly converges in $W_2^1(\Omega)$ to a vector function $\vec{U}(x) \in W_2^1(\Omega)$.

Then the sequence $\{\vec{u}_\varepsilon(x,t), \varepsilon \to 0\}$ of solutions of the problem (2.1)–(2.4) converges in $L_2(\Omega_T)$ to the solution $\vec{u}(x,t)$ of the following problem:

$$\vec{u}_t - \nu\Delta\vec{u} + \nu C(x,t)[\vec{u} - \vec{W}(x,t)] = -\nabla p + \vec{g}(x,t), \tag{2.8}$$

$$\mathrm{div}\vec{u}(x,t) = 0, \ (x,t) \in \Omega_T;$$

$$\vec{u}(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T]; \tag{2.9}$$

$$\vec{u}(x,0) = \vec{U}(x), \ x \in \Omega, \tag{2.10}$$

where $\vec{W}(x,t) = \vec{\Phi}_t(\xi,t)|_{\xi=\Phi^{-1}(x,t)}$ and Φ^{-1} is an inverse mapping onto $\vec{\Phi}$.

R e m a r k 1. Let the angular velocities $\vec{\theta}_\varepsilon^i(t)$ change smoothly with transition from one particle to another, namely the following equality is true: $\vec{\theta}_\varepsilon^i(t) = \vec{\theta}(\vec{x}_\varepsilon^i, t)$, where $\vec{\theta}(x,t)$ is a vector function continuous on x and differentiable with respect to t, and $\vec{x}^i_\varepsilon$ is the position of the center of mass of particle i. Then tensor $C(x,t)$ from condition 2 of Th. 1 can be computed according to the following formula:

$$C_{kl}(x,t) = \sum_{i,j=1}^{3} C_{ij}(\Phi^{-1}(x,t),0)\Pi_{ik}(\Phi^{-1}(x,t),t) \times \Pi_{jl}(\Phi^{-1}(x,t),t)$$

$$\times \left| \det \frac{\partial \vec{\Phi}(x,t)}{\partial x} \right|^{-1} \chi_\Omega(\Phi^{-1}(x,t)),$$

where $\chi_\Omega(x)$ is a characteristic function of the domain Ω, and $\Pi(x,t)$ is a matrix, the columns $\vec{\Pi}_k(x,t)$ of which are the solutions of the Cauchy $\dot{\vec{\Pi}}_k(x,t) = \vec{\theta}(x,t) \times \vec{\Pi}_k(x,t)$, $\Pi_{jk}(x,0) = \delta_{jk}$. One can see it easily by using (2.7) and taking into account that the rotation operator of a solid particle (matrix $\Pi^i(t)$) satisfies the following equation $\dot{\Pi}^i(t) = \vec{\theta}^i(t) \times \Pi^i(t)$, where $\vec{\theta}^i(t)$ is the instantaneous angular velocity of particle i, and the vector product is applied to column vectors of matrix $\Pi^i(t)$. Thus it is sufficient to compute the limit in condition 2 only for $t = 0$.

3. Additional Statements

In this section we establish some additional statements (Lems. 1, 2, and 3) and derive apriori estimates for $\nabla \vec{u}_\varepsilon(x,t)$ and $\vec{u}_{\varepsilon t}(x,t)$ in $L_2(\Omega^T_\varepsilon)$, which we will use later in the proof of Th. 1. Before formulating Lem. 1, we introduce the following notation. Let $\{\vec{v}^{k,i}(x,t), p^{k,i}(x,t)\}$ be the solution of the problem (2.5), when $Q = Q^i_\varepsilon(t)$, and the vector of velocity $\vec{v}^{k,i}(x,t)$ is extended to $Q^i_\varepsilon(t)$ by the equality $\vec{v}^{k,i}(x,t) = \vec{e}^k$ ($\vec{e}^k$ is an ort of the axis x_k). Let us introduce the vector functions $\vec{\tilde{v}}^{k,i}(x,t)$ to satisfy the equalities:

$$\text{rot}\vec{\tilde{v}}^{k,i}(x,t) = \vec{v}^{k,i}(x,t), \tag{3.1}$$

as $|x - x^i(t)| \le r^i_\varepsilon(t) + d^i_\varepsilon$, and define the vector functions

$$\vec{w}_\varepsilon(x,t) = \sum_{i=1}^{N_\varepsilon} \left\{ \text{rot} \sum_{k=1}^{3} \vec{\tilde{v}}^{k,i}(x,t) v^i_k(t) \varphi^i_\varepsilon(x - x^i(t),t) \right.$$

$$\left. + \frac{1}{2}\text{rot} \sum_{k=1}^{3} \theta^i_k(t) \vec{e}^k \sum_{j \ne k} \frac{(x_j - x^i_j(t))^2}{2} \hat{\varphi}^i_\varepsilon(x - x^i(t),t) \right\}, \tag{3.2}$$

$$\vec{W}_\varepsilon(x,t) = \Delta \vec{w}_\varepsilon(x,t)$$

$$-\nabla \sum_{i=1}^{N_\varepsilon} \sum_{k=1}^{3} p^{k,i}(x,t) v^i_k(t) \varphi^i_\varepsilon(x - x^i(t),t), \ (x,t) \in \Omega^T_\varepsilon, \tag{3.3}$$

$$\vec{W}_\varepsilon(x,t) = 0,\ (x,t) \in \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^{iT},$$

where $v_k^i(t)$ and $\theta_k^i(t)$ are the components of the vectors of velocities of centers of the mass $\vec{v}_\varepsilon^i(t)$ and the angular velocity $\vec{\theta}_\varepsilon^i(t)$ of particle i; $x_j^i(t)$ is one of the components of the vector $\vec{x}^i(t) = \vec{x}_\varepsilon^i(t)$ of the center of mass $Q_\varepsilon^i(t)$ of particle i, $\varphi_i(x,t)$ and $\hat{\varphi}_\varepsilon^i(t)$ are the patch functions:

$$\varphi_\varepsilon^i(x,t) = \varphi\left(\frac{|x| - d_\varepsilon^i}{r_\varepsilon^i(t)}\right);\ \hat{\varphi}_\varepsilon^i(x) = \varphi\left(\frac{|x| - d_\varepsilon^i}{d_\varepsilon^i}\right); \tag{3.4}$$

and $\varphi(t)$ is a twice continuously differentiable function, that equals 1 for $t \le 0$ and 0 for $t > 1/2$.

It follows from conditions 1 and 3 of Th. 1 that for sufficiently small ε $r_\varepsilon^i(t) \gg d_\varepsilon^i$, $t \in [0,T]$, so, taking into account (3.2) and (3.4), we establish that the vector function $\vec{w}_\varepsilon(x,t)$ satisfies the boundary conditions (2.2), (2.3) and is a divergent free function.

Lemma 1. *Let conditions 1 and 3 of Th. 1 hold. Then $\vec{w}_\varepsilon(x,t)$ converges to zero in $L_2(\Omega_T)$ and for any fixed $t \in [0,T]$ in $L_2(\Omega)$; the derivatives $\vec{w}_{\varepsilon t}(x,t)$ and $\nabla\vec{w}_\varepsilon(x,t)$ converge weakly to zero in $L_2(\Omega_T)$ and are bounded in $L_2(\Omega)$ for any fixed t. If, further, condition 2 of Th. 1 holds, then $\vec{W}_\varepsilon(x,t)$ converges weakly in $L_2(\Omega_T)$ to a vector function $C(x,t)\vec{W}(x,t)$, where $\vec{W}(x,t) = \vec{\Phi}_t(\xi,t)|_{\xi=\Phi^{-1}(x,t)}$, and matrix $C(x,t)$ is defined in condition 2.*

P r o o f. Because of the properties of functions $\varphi_\varepsilon^i(x,t)$, $\hat{\varphi}_\varepsilon^i(x)$, and equality (3.1) and representation (3.2), it follows that

$$\|\vec{w}_\varepsilon\|^2_{L_2(\Omega_T)} \le C\int_0^T \sum_{i=1}^{N_\varepsilon}\left[(d_\varepsilon^i)^2 r_\varepsilon^i(t) + \frac{(d_\varepsilon^i)^4}{r_\varepsilon^i(t)} + (d_\varepsilon^i)^5\right] dt,$$

where the constant C depends only on $\max\limits_{t,i}|\vec{\theta}^i(t)|$, $\max\limits_{x,t}|\vec{\Phi}(x,t)|$.

Taking into account that $d_\varepsilon^i = o(r_\varepsilon^i)$, $r_\varepsilon^i < C$ and using the Cauchy–Schwartz inequality, we obtain

$$\|\vec{w}_\varepsilon\|^2_{L_2(\Omega_r)} \le CT\max_i d_\varepsilon^i \max_t\left\{\sum_{i=1}^{N_\varepsilon}\frac{(d_\varepsilon^i)^2}{(r_\varepsilon^i(t))^3}\right\}^{1/2} \max_t\left\{\sum_{i=1}^{N_\varepsilon}\frac{(r_\varepsilon^i(t))^3}{2}\right\}^{1/2}$$

$$\le CTd_\varepsilon AB, \tag{3.5}$$

where we denote

$$A = \sup_{\varepsilon} \max_{t} \left\{ \sum_{i=1}^{N_\varepsilon} \frac{(d_\varepsilon^i)^2}{(r_\varepsilon^i(t))^3} \right\}^{1/2}, \quad B = \sup_{\varepsilon} \max_{t} \left\{ \sum_{i=1}^{N_\varepsilon} \frac{(r_\varepsilon^i(t))^3}{2} \right\}^{1/2}, \tag{3.6}$$

$$d_\varepsilon = \max_i d_\varepsilon^i, \quad \varepsilon < 1.$$

According to the definition of $r_\varepsilon^i(t)$ and condition 3 of Th. 1, A and B do not depend on ε and t. As the balls with radii $r_\varepsilon^i(t)/2$ and centers in points $x^i(t)$ do not intersect, it follows from (3.5) according to conditions 1 and 3 of Th. 1 that

$$\lim_{\varepsilon\to 0} \|\vec{w}_\varepsilon\|_{L_2(\Omega_T)} = 0. \tag{3.7}$$

Similarly, we obtain the following estimate for $\nabla \vec{w}_\varepsilon(x,t)$

$$\max_t \|\nabla \vec{w}_\varepsilon\|^2_{L_2(\Omega)} \le C \sum_{i=1}^{N_\varepsilon} \left[d_\varepsilon^i + \frac{(d_\varepsilon^i)^2}{r_\varepsilon^i(t)} \right] \le CAB < \infty. \tag{3.8}$$

It follows from condition 3 of Th. 1 and (3.8) that $\nabla \vec{w}_\varepsilon(x,t)$ is uniformly bounded on ε in $L_2(\Omega_T)$ and for each fixed t in $L_2(\Omega)$.

The definition of the operator of rotation of the i-th particle $\Pi^i(t)$ and the linearity of problem (2.5) imply the following equalities:

$$\vec{v}^{k,i}(x,t) = \sum_{j=1}^{3} \Pi^i(t) \vec{v}^{j,i}([\Pi^i(t)]^{-1}(x - x^i(t), 0)(\vec{e}^k, \Pi^i(t)\vec{e}^j),$$

$$\vec{\tilde{v}}^{k,i}(x,t) = \sum_{j=1}^{3} \Pi^i(t) \vec{\tilde{v}}^{j,i}([\Pi^i(t)]^{-1}(x - x^i(t), 0)(\vec{e}^k, \Pi^i(t)\vec{e}^j). \tag{3.9}$$

Now, using equalities (3.1), (3.2) and (3.9), the properties of the functions $\varphi_\varepsilon^i(x,t)$, $\hat{\varphi}_\varepsilon^i(x)$ and taking into account that vector functions $\vec{\theta}^i(t)$, $\vec{v}^i(t) = \dot{\vec{x}}^i$ are bounded in $C^1([0,T])$ uniformly by epsilon, we obtain the following inequality:

$$\max_t \|\vec{w}_{\varepsilon t}(x,t)\|^2_{L_2(\Omega)} \le C \max_t \sum_{i=1}^{N_\varepsilon} \left[d_\varepsilon^i + \frac{(d_\varepsilon^i)^2}{r_\varepsilon^i(t)} (\dot{r}_\varepsilon^i(t))^2 \right].$$

It is obvious that $|\dot{r}_\varepsilon^i(t)| \le \max_j |\vec{v}^j(t)| + \max_j \|\Pi^j(t)\| d_\varepsilon^j$, so the right-hand side of this inequality can be estimated similarly to (3.8). As a result, the derivatives $\vec{w}_\varepsilon$ are bounded in $L_2(\Omega)$ with respect to t and x and in $L_2(\Omega_T)$ uniformly by ε.

It follows from the uniform boundness of $\vec{w}_{\varepsilon t}$ and $\nabla \vec{w}_\varepsilon$ in $L_2(\Omega_T)$ and equalities (3.6) that $\vec{w}_\varepsilon$ weakly converges to zero in $W_2^1(\Omega_T)$ and in accordance with the imbedding theorem strongly converges to zero in $L_2(\Omega)$ for each $t \in [0,T]$. Thus, the first part of Lem. 1 is proved.

Now consider a vector function $\vec{W}_\varepsilon(x,t)$, defined by the equalities (3.2) and (3.3). Using the equality (3.1) and taking into account that $\vec{v}^{k,i}(x,t)$ is the solution of the problem (2.5) with $Q = Q^i_\varepsilon(t)$, and due to the divergent free of $\vec{v}^{k,i}(x,t)$: $\Delta\vec{v}^{k,i}(x,t) = -\text{rotrot}\vec{v}^{k,i}(x,t) = -\text{rot}\vec{v}^{k,i}(x,t)$, it is not difficult to obtain the following inequality:

$$|\vec{W}_\varepsilon(x,t)| \le C\sum_{i=1}^{N_\varepsilon}\left\{\sum_{k=1}^{3}|p^{k,i}(x,t)||\nabla\varphi^i_\varepsilon(x-x^i(t),t)| + \sum_{k=1}^{3}\sum_{l=0}^{1}|D^l\vec{v}^{k,i}(x,t)||D^{3-l}\varphi^i_\varepsilon(x-x^i(t),t)| + 2\sum_{k=1}^{3}\sum_{l=0}^{1}|D^l\vec{v}^{k,i}(x,t)||D^{2-l}\varphi^i_\varepsilon(x-x^i(t),t| + 6\sum_{l=0}^{2}|x-x^i(t)|^l|D^{l+1}\hat{\varphi}^i_\varepsilon(x-x^i(t))|\right\}, \tag{3.10}$$

where $C = \max\limits_i \max\limits_t \left\{|\vec{v}^i(t)|; \vec{\theta}^i(t)|\right\}$, p $|D^l\vec{u}| = \sum\limits_{|\alpha|=l}\left(\sum\limits_{i=1}^{3}|D^\alpha u_i|^2\right)^{1/2}$.

Using this inequality, as well as the estimates (3.5), (3.7), (3.8), Lem. 1 and taking into account that $d^i_\varepsilon = o(r^i_\varepsilon(t))$, $r^i_\varepsilon(t) < C$, we obtain

$$\|\vec{W}_\varepsilon(x,t)\|_{L_2(\Omega_T)} \le C\left\{\int_0^T\sum_{i=1}^{N_\varepsilon}\frac{(d^i_\varepsilon)^2}{(r^i_\varepsilon(t))^3}dt + T\sum_{i=1}^{N_\varepsilon}d^i_\varepsilon\right\} \le CT[A^2 + AB],$$

where C does not depend on ε, and A and B were defined in (2.6). This inequality and the conditions 3 of Th. 1 imply that $\vec{W}_\varepsilon(x,t)$ is bounded in $L_2(\Omega_T)$ uniformly with respect to ε.

Let $\vec{\Psi}(x,t)$ be the arbitrary vector function from $C^2(\Omega_T)$ and

$$J_\varepsilon = \int_0^T\int_\Omega(\vec{W}_\varepsilon(x,t), \vec{\Psi}(x,t))dxdt. \tag{3.11}$$

Considering (3.2)–(3.3) we represent J_ε in the form $J_\varepsilon = J^1_\varepsilon + J^2_\varepsilon$, where

$$J^1_\varepsilon = -\int_0^T\int_{\Omega_\varepsilon(t)}\sum_{i=1}^{N_\varepsilon}\sum_{k=1}^{3}(\Delta[\vec{v}^{k,i}(x,t)\varphi^i_\varepsilon(x-x^i(t),t)]$$

$$-\nabla[p^{k,i}(x,t)\varphi^i_\varepsilon(x-x^i(t),t),\vec{\Psi}(x^i(t),t))v^i_k(t)dxdt,$$

and the following inequality is obtained for the integral J^2_ε

$$|J^2_\varepsilon| \le C\int\limits_0^T\sum_{i=1}^{N_\varepsilon}\left\{(d^i_\varepsilon)^4 + d^i_\varepsilon r^i_\varepsilon(t) + d^i_\varepsilon[r^i_\varepsilon]^2\right\}dt$$

$$\le CT\left[d^{1/3}_\varepsilon A^{2/3}B^{4/3} + d^{2/3}_\varepsilon A^{2/6}B^{10/6} + d^3_\varepsilon AB\right],$$

this, together with conditions 1 and 3 of Th. 1, implies that

$$\lim_{\varepsilon\to 0} J^2_\varepsilon = 0. \tag{3.12}$$

Furthermore, taking into account that $\{\vec{v}^{k,i}(x,t),p^{k,i}(x,t)\}$ is the solution of the problem (2.5) with $Q=Q^i_\varepsilon(t)$, the integral J^1_ε can be represented as follows

$$J^1_\varepsilon = -\int\limits_0^T\sum_{i=1}^{N_\varepsilon}\left\{\sum_{k,l=1}^{3}\int\limits_{\mathbf{R}^3}(\nabla\vec{v}^{k,i}(x,t),\nabla\vec{v}^{l,i}(x,t))dx\cdot v^i_k(t)\vec{\Psi}(x^i(t),t)\right\}dt$$

$$= -\int\limits_0^T\sum_{i=1}^{N_\varepsilon}\sum_{k,l=1}^{3}C_{kl}(Q^i_\varepsilon(t))v^i_k(t)\vec{\Psi}(x^i(t),t)dt.$$

Hence by virtue of condition 2 of Th. 1, the equalities $\vec{v^i}(t) = \dot{\vec{x^i}}(t) = \vec{v}(x,t)|_{x=x^i(t)}$, $\vec{v}(x,t) = \vec{\Phi}_t(\xi,t)|_{\xi=\vec{\Phi}^{-1}(x,t)}$ and the smoothness of functions $\vec{\Psi}$ and $\vec{v}$, it follows that

$$\lim_{\varepsilon\to 0} J^1_\varepsilon = -\iint\limits_{\Omega_T}\Big(C(x,t)\vec{v}(x,t),\vec{\Psi}(x,t)\Big)dxdt.$$

Because the vector functions $\vec{W}_\varepsilon(x,t)$ are bounded in $L_2(\Omega_T)$ uniformly with respect to ε, it follows from the equalities (3.11) and (3.12) that $\vec{W}_\varepsilon(x,t)$ converge weakly in $L_2(\Omega_T)$ to $-C(x,t)\vec{v}(x,t)$ as $\varepsilon\to 0$. Lemma 1 is proved.

To prove Th. 1 we have to use the estimates in the space $L_2(\Omega)$ of the partial derivatives with respect to x and t of the vector function $\vec{v}_\varepsilon(x,t) = \vec{u}_\varepsilon(x,t) - \vec{w}_\varepsilon(x,t)$, where $\vec{u}_\varepsilon$ is the solution of problem (2.1)–(2.4) and $\vec{w}_\varepsilon$ is defined in (3.2). To obtain these estimates consider the boundary volume problem for the function $\vec{v}_\varepsilon(x,t)$

$$\vec{v}_{\varepsilon t} - \nu\Delta\vec{v}_\varepsilon = -\nabla q_\varepsilon + \vec{g}_\varepsilon(x,t),\ \ \mathrm{div}\vec{v}_\varepsilon = 0,\ \ (x,t)\in\Omega^T_\varepsilon; \tag{3.13}$$

$$\vec{v}_\varepsilon(x,t) = 0, \ \ (x,t) \in \left(\bigcup_{i=1}^{N_\varepsilon} \partial Q_\varepsilon^i \bigcup \partial\Omega \right) \times [0,T]; \tag{3.14}$$

$$\vec{v}_\varepsilon(x,0) = \vec{V}_\varepsilon(x), \ \ x \in \Omega_\varepsilon, \tag{3.15}$$

where

$$\vec{g}_\varepsilon = \vec{g}(x,t) - \vec{w}_{\varepsilon t} + \nu(\Delta\vec{w}_\varepsilon - \nabla \sum_{i=1}^{N_\varepsilon} \sum_{k=1}^{3} p^{k,i}(x,t) v_k^i(t) \varphi_\varepsilon^i(x - x^i(t),t);$$

$$q_\varepsilon = p_\varepsilon(x,t) + \nu \sum_{i=1}^{N_\varepsilon} \sum_{k=1}^{3} p^{k,i}(x,t) v_k^i(t) \varphi_\varepsilon^i(x - x^i(t),t);$$

$$\vec{V}_\varepsilon = \vec{U}_\varepsilon(x) - \vec{w}_\varepsilon(x,0).$$

Multiplying (3.13) by $\vec{v}_\varepsilon(x,t)$ and integrating over Ω_ε^T, we get

$$\int_0^T \int_{\Omega_\varepsilon(t)} [(\vec{v}_{\varepsilon t}, \vec{v}_\varepsilon) - \nu(\Delta\vec{v}_\varepsilon, \vec{v}_\varepsilon)] \, dxdt$$

$$= \int_0^T \int_{\Omega_\varepsilon(t)} (\nabla q_\varepsilon, \vec{v}_\varepsilon) dxdt + \int_0^T \int_{\Omega_\varepsilon(t)} (\vec{g}_\varepsilon, \vec{v}_\varepsilon) dxdt. \tag{3.16}$$

Since $\operatorname{div}\vec{v}_\varepsilon = 0$, then from (2.16) using boundary condition (3.14) and the initial condition (3.15), we have

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\nabla q_\varepsilon, \vec{v}_\varepsilon) dxdt = 0;$$

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\Delta\vec{v}_\varepsilon, \vec{v}_\varepsilon) dxdt = -\int_0^T \int_{\Omega_\varepsilon(t)} |\nabla\vec{v}_\varepsilon|^2 dxdt;$$

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\vec{v}_{\varepsilon t}, \vec{v}_\varepsilon) dxdt = \frac{1}{2} \int_0^T \frac{d}{dt} \int_{\Omega_\varepsilon(t)} |\vec{v}_\varepsilon|^2 dxdt$$

$$= \frac{1}{2} \int_{\Omega_\varepsilon(t)} |\vec{v}_\varepsilon(x,T)|^2 dx - \frac{1}{2} \int_{\Omega_\varepsilon(0)} |\vec{V}_\varepsilon(x)|^2 dx. \tag{3.17}$$

Let us extend $\vec{v}_\varepsilon$ by zero on the set $\bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^{iT}$. Then, for any $t \in [0,T]$: $\vec{v}_\varepsilon(x,t) \in \overset{\circ}{W}{}_2^1(\Omega)$. Now Friedreich's inequality implies $\|\vec{v}_\varepsilon\|^2_{L_2(\Omega_T)} \le C\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}$. Therefore,

$$\int_0^T \int_{\Omega_\varepsilon(t)} (\vec{g}_\varepsilon, \vec{v}_\varepsilon)dxdt \le \delta\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} + \frac{C^2}{4\delta}\|\vec{g}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}, \qquad (3.18)$$

where C is a constant, which does not depend on ε, and $\delta > 0$ is an arbitrary positive number.

Now from (3.16)–(3.18), we have

$$\frac{1}{2}\int_{\Omega_\varepsilon(t)} |\vec{v}_\varepsilon(x,T)|^2dx + (1-\delta)\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} \le \frac{C^2}{4\delta}\|\vec{g}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} + \frac{1}{2}\|\vec{V}_\varepsilon\|_{L_2(\Omega_\varepsilon(0))}.$$

We recall that $\vec{g}_\varepsilon = \vec{g} - \vec{w}_{\varepsilon t} + \nu\vec{W}_\varepsilon$, $\vec{V}_\varepsilon = \vec{U}_\varepsilon - \vec{w}_\varepsilon$. Now the last inequality along with Lem. 1 and condition 4 of Th. 1 imply that the partial derivatives of the vector function $\vec{v}_\varepsilon$ with respect to x are uniformly bounded in the space $L_2(\Omega_\varepsilon^T)$ (with respect to ε), i.e.,

$$\|\nabla\vec{v}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)} \le C. \qquad (3.19)$$

To obtain a similar estimate for $\vec{v}_{\varepsilon t}$, we make use of the following lemmas.

Lemma 2. *Let $f_\varepsilon(x) = f_\varepsilon(x,t)$ for any $t \in [0,T]$ be a function from $\overset{\circ}{W}{}_2^1(\Omega)$ and $f_\varepsilon(x,t) = f_{i\varepsilon} = f_{i\varepsilon}(t)$ in $Q_\varepsilon^i = Q_\varepsilon^i(t)$ (here t is considered as a parameter). Let the conditions 1 and 3 of Th. 1 be satisfied.*

Then, the following inequalities hold $\sum_{i=1}^{N_\varepsilon} |f_{i\varepsilon}|^2 d_\varepsilon^i < C\int_\Omega |\nabla f_\varepsilon|^2dx$, where C is a positive constant that does not depend on ε and t.

P r o o f. Denote by $v_\varepsilon^i(x) = v_\varepsilon^i(x,t)$ the solution of the Robin problem in the sets $Q_\varepsilon^i = Q_\varepsilon^i(t)$:

$$\Delta v_\varepsilon^i(x) = 0, \ \ x \in \mathbf{R}^3 \setminus Q_\varepsilon^i;$$

$$v_\varepsilon^i(x) = 1, \ \ x \in Q_\varepsilon^i; \ \ v_\varepsilon^i(x) \to 0, \ |x| \to \infty. \qquad (3.20)$$

The Dirichlet integral of the solution of this problem is called the newtonian capacity of the set Q_ε^i and is denoted by C_ε^i. Moreover,

$$2\pi d^i 1_\varepsilon \le C_\varepsilon^i \le 2\pi d_\varepsilon^i, \qquad (3.21)$$

where d'^i_ε and d^i_ε are the interior and exterior diameters of Q^i_ε, respectively. In what follows we make use of the inequalities (see [3]):

$$|D^\alpha v^i_\varepsilon| \le A \frac{d^i_\varepsilon}{\rho^{1+|\alpha|}}, |\alpha| = 0, 1, 2; \tag{3.22}$$

$$\int\limits_{T_b} |D^\alpha v^i_\varepsilon|^2 dx \le A \left\{ (d^i_\varepsilon)^2 b^{1-2|\alpha|} + (d^i_\varepsilon)^{3-2|\alpha|} \right\} \quad (|\alpha| = 0, 1), \tag{3.23}$$

where $\rho = \rho(x)$ is the distance from the point x to the minimal ball $T(Q^i_\varepsilon)$ containing Q^i_ε, T_b is the ball of radius b, which is concentric to $T(Q^i_\varepsilon)$ and $b > d^i_\varepsilon$.

For any $t \in [0, T]$, consider a function $u_\varepsilon(x) = u_\varepsilon(x, t)$ (here t is a parameter), which is the solution of the following Dirichlet problem:

$$\Delta u_\varepsilon(x) = 0, \;\; x \in \Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} Q^i_\varepsilon;$$

$$u_\varepsilon(x) = f_\varepsilon(x), \;\; x \in \partial Q^i_\varepsilon, \;\; i = 1, \ldots, N_\varepsilon; \;\; u_\varepsilon(x) = 0, \;\; x \in \partial\Omega. \tag{3.24}$$

We extend $u_\varepsilon(x)$ in Q^i_ε by setting $u_\varepsilon(x) = f^i_\varepsilon$ as $x \in Q^i_\varepsilon$, $i = 1, \ldots, N_\varepsilon$. It is well known that u_ε minimizes the Dirichlet integral and, therefore,

$$\|\nabla u_\varepsilon\|_{L_2(\Omega)} \le \|\nabla f_\varepsilon\|_{L_2(\Omega)}. \tag{3.25}$$

Since $u_\varepsilon \in \overset{\circ}{W}{}^1_2(\Omega)$, then it satisfies Friedreich's inequality

$$\|u_\varepsilon\|_{L_2(\Omega)} \le C \|\nabla u_\varepsilon\|_{L_2(\Omega)} \tag{3.26}$$

and the multiplicative inequality

$$\|u_\varepsilon\|_{L_r(\Omega)} \le (48)^{\alpha/6} \|\nabla u_\varepsilon\|^\alpha_{L_2(\Omega)} \|u_\varepsilon\|^{1-\alpha}_{L_2(\Omega)}, \tag{3.27}$$

where $r \in [2, 6]$, $\alpha = 3/2 - 3/r$, and C is a constant depending on the domain Ω only.

It follows from (3.25)–(3.27) that

$$\|u_\varepsilon\|_{L_4(\Omega)} \le C \|\nabla f_\varepsilon\|_{L_2(\Omega)}, \tag{3.28}$$

where C is a constant that does not depend on ε and t.

We set $u_\varepsilon(x) = \hat{u}_\varepsilon(x) + w_\varepsilon(x)$, where $\hat{u}_\varepsilon = \sum_{i=1}^{N_\varepsilon} f^i_\varepsilon v^i_\varepsilon(x) \varphi^i_\varepsilon(x)$, $v^i_\varepsilon(x)$ is the solution of the problem (3.20), $\varphi^i_\varepsilon(x)$ is a patch function defined earlier in (2.4).

It is clear that $w_\varepsilon = 0$ for $x \in \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^i \bigcup \partial\Omega$ and, therefore due to (3.24) w_ε is orthogonal to u_ε with respect to the Dirichlet scalar product. Then, it follows from the development of u_ε, $\|\nabla w_\varepsilon\|^2_{L_2(\Omega)} = -(\nabla \hat{u}_\varepsilon, \nabla w_\varepsilon)_{L_2(\Omega)}$. In the right-hand side of this equality we apply the integration by parts and Hölder's inequality. We have

$$\|\nabla w_\varepsilon\|^2_{L_2(\Omega)} = \int_\Omega \sum_{i=1}^{N_\varepsilon} f_\varepsilon^i \Delta(v_\varepsilon^i \cdot \varphi_\varepsilon^i) \cdot w_\varepsilon dx$$

$$\leq \left\{ \int_\Omega \left| \sum_{i=1}^{N_\varepsilon} f_\varepsilon^i \Delta(v_\varepsilon^i \cdot \varphi_\varepsilon^i) \right|^{4/3} dx \right\}^{3/4} \left\{ \int_\Omega |w_\varepsilon|^4 dx \right\}^{1/4}.$$

We estimate the first term in the right-hand side of this inequality using (3.22). Taking into account the properties of φ_ε^i, (3.20) and Hölder's inequality, we get

$$\|\nabla w_\varepsilon\|^2_{L_2(\Omega)} \leq C \left\{ \sum_{i=1}^{N_\varepsilon} \frac{(d_\varepsilon^i)^2}{(r_\varepsilon^i)^3} \right\}^{1/2} \left\{ \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i \right\}^{1/2} \|w_\varepsilon\|_{L_4(\Omega)}. \tag{3.29}$$

The representation of u_ε and (3.28) imply that $\|w_\varepsilon\|_{L_4(\Omega)} \leq \|\nabla f_\varepsilon\|_{L_2(\Omega)} + \|\hat{u}_\varepsilon\|_{L_4(\Omega)}$. On the other hand, from the representation of $\hat{u}_\varepsilon$ we deduce that $\hat{u}_\varepsilon(x) \in W_2^{0,1}(\Omega)$.

To estimate the second term in the right-hand side we apply (3.27). Then we estimate each term using (3.22), (3.23), and the inequality $d_\varepsilon^i < r_\varepsilon^i < C$. We obtain

$$\|w_\varepsilon\|_{L_4(\Omega)} \leq C\|\nabla f_\varepsilon\|_{L_2(\Omega)} + C \max_i d_\varepsilon^i \left\{ \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i \right\}^{1/2}.$$

Plugging this inequality in (3.29) and taking into account condition 3 of Th. 1, we finally get

$$\|\nabla w_\varepsilon\|^2_{L_2(\Omega)} \leq (\delta + C \max_i d_\varepsilon^i) \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i + C/\delta \|\nabla f_\varepsilon\|^2_{L_2(\Omega)}, \tag{3.30}$$

where C is a constant which does not depend on ε, t, and δ is an arbitrary positive number.

Let us estimate the Dirichlet norm of the function $\hat{u}_\varepsilon$ from below. We have

$$\|\nabla \hat{u}_\varepsilon\|^2_{L_2(\Omega)} = \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 \|\nabla v_{i\varepsilon}\|^2_{L_2(\Omega)} + \Delta_\varepsilon, \tag{3.31}$$

where

$$\Delta_\varepsilon = \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 \int_\Omega |\nabla v_\varepsilon^i|^2 \left[(\varphi_\varepsilon^i)^2 - 1\right] dx$$

$$+ \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 \|v_\varepsilon^i \nabla \varphi_\varepsilon^i\|^2_{L_2(\Omega)} + 2 \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 \int_\Omega v_\varepsilon^i \varphi_\varepsilon^i (\nabla v_\varepsilon^i, \nabla \varphi_\varepsilon^i) dx.$$

The definition of the newtonian capacity C_ε^i, inequality (3.21) and condition 1 of Th. 1 imply

$$\sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 \|\nabla v_\varepsilon^i\|^2_{L_2(\Omega)} = \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 C_\varepsilon^i \geq A \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i,$$

where A is a positive constant that does not depend on ε and t.

It follows from condition 3 of Th. 1 that $r_\varepsilon^i \geq B(d_\varepsilon^i)^{2/3}$ $(B > 0)$. Then, using (3.22), we have

$$|\Delta_\varepsilon| \leq C_1 \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i| \frac{(d_\varepsilon^i)^2}{r_\varepsilon^i} \leq C \max_i (d_\varepsilon^i)^{1/3} \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i,$$

where $C = C_1 B^{-1}$ does not depend on ε and t. Thus, according to (3.31)

$$\|\nabla \hat{u}_\varepsilon\|^2_{L_2(\Omega)} \geq \left[A - C \max_i (d_\varepsilon^i)^{1/3}\right] \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i. \tag{3.32}$$

Now (3.25), (3.30) and (3.32) imply the inequality

$$\left[A - C \max_i (d_\varepsilon^i)^{1/3} - C \max_i d_\varepsilon^i - \delta\right] \sum_{i=1}^{N_\varepsilon} |f_\varepsilon^i|^2 d_\varepsilon^i \leq C(1 + 1/\delta) \|\nabla f_\varepsilon\|^2_{L_2(\Omega)},$$

where $\varepsilon > 0$ is an arbitrary positive number and A and C are positive constants that do not depend on ε, t and δ.

Now from this inequality along with condition 1 of Th. 1 we immediately obtain the statement of Lem. 2.

Lemma 3. *Let conditions 1, 3 of Th. 1 be fulfilled for the sets* $Q_\varepsilon^i = Q_\varepsilon^i(t)$, $i = 1, \ldots, N_\varepsilon$.

Then for any $\Psi_\varepsilon(x) = \Psi_\varepsilon(x,t) \in L_2(\Omega)$, *such that:*

1) $\int_\Omega \Psi_\varepsilon(x) dx = 0$;

2) $\Psi_\varepsilon(x) = 0$ *for* $x \in \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^i$*;*

there exists a vector function $\vec{z}_\varepsilon(x) = \vec{z}_\varepsilon(x,t) \in W_2^{0,1}(\Omega)$ *such that for any* $t \in [0,T]$

$$\mathrm{div}\vec{z}_\varepsilon(x) = \Psi_\varepsilon(x), \;\; x \in \Omega; \tag{3.33}$$

$$\vec{z}_\varepsilon(x) = 0, \;\; x \in \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^i(t) \bigcup \partial\Omega; \tag{3.34}$$

$$\|\vec{z}_\varepsilon(x)\|_{W_2^1(\Omega)} \leq C\|\Psi_\varepsilon\|_{L_2(\Omega)}, \tag{3.35}$$

where C *is a constant that does not depend on* ε *and* t*.*

P r o o f. For any $t \in [0,T]$, we construct a vector function $\vec{f}_\varepsilon(x) = \vec{f}_\varepsilon(x,t) \in \overset{\circ}{W}{}_2^1(\Omega)$, such that $\mathrm{div}\vec{f}_\varepsilon(x) = \Psi_\varepsilon(x)$, $x \in \Omega$; $\vec{f}_\varepsilon(x) = \vec{f}_\varepsilon^i$, $x \in Q_\varepsilon^i$ $(i = 1,\dots,N_\varepsilon)$; $\|\vec{f}_\varepsilon\|_{W_2^1(\Omega)} \leq C\|\Psi_\varepsilon\|_{L_2(\Omega)}$.

We set

$$\vec{z}_\varepsilon(x,t) = \vec{f}_\varepsilon(x,t) - \sum_{i=1}^{N_\varepsilon} \mathrm{rot}\left[\sum_{k=1}^{3} \vec{v}^{ki}(x,t) f_{k\varepsilon}^i(t)\varphi_\varepsilon^i(x - x^i(t),t)\right], \tag{3.36}$$

where $f_{k\varepsilon}^i$ are the components of the constant vectors $\vec{f}_\varepsilon^i$, the vector function $\vec{v}^{ki}$ and the patch function $\varphi_\varepsilon^i(x,t)$ are the same as in (3.2). Taking into account the properties of $\vec{f}_\varepsilon$, $\vec{v}^{ki}$ and φ_ε^i, it is easy to see, that $\vec{z}_\varepsilon$ satisfies (3.33), (3.34). It remains to show that the estimate (3.35) holds true. From (3.36) we have

$$\|\vec{z}_\varepsilon\|_{W_2^1(\Omega)}^2 \leq 2\left\{\|\vec{f}_\varepsilon\|_{W_2^1(\Omega)} + C\sum_{i=1}^{N_\varepsilon} |\vec{f}_\varepsilon^i|^2 \left[d_\varepsilon^i + \frac{(d_\varepsilon^i)^2}{r_\varepsilon^i}\right]\right\},$$

where C is a constant that does not depend on ε and t.

It is clear that $\|\vec{f}_\varepsilon\|_{W_2^1(\Omega)} \leq C\|\Psi_\varepsilon\|_{L_2(\Omega)}$. The last two inequalities along with conditions 2, 3, of Th. 1 and Lem. 2 imply (2.35). Lemma 3 is proved.

Let us now estimate the partial derivative $\vec{v}_{\varepsilon t}$ of the solution $\vec{v}_\varepsilon$ of the problem (3.13)–(3.15).

Let $\vec{l}_\varepsilon(x,t) = \{l_\varepsilon^k(x,t), k = 0,1,2,3\}$ be a vector field in Ω_T, which is tangent to the lateral surface of Ω_ε^T and such that

$$l_\varepsilon^0(x,t) \equiv 1, \;\; |D_x^k \vec{l}_\varepsilon(x,t)| < C, \;\; (k = 0,1), \tag{3.37}$$

where C is a constant not depending on ε.

It is easy to see that there exists a vector field satisfying these properties. In fact, consider a vector function $\vec{\Phi}_\varepsilon(x,t)$, such that for any $t \in [0,T]$

$$\vec{\Phi}_\varepsilon(\vec{\xi},t) = \vec{\Phi}(\vec{\xi},t) + \sum_{i=1}^{N_\varepsilon} \left[\vec{\Phi}(\vec{\xi}^i_\varepsilon,t) - \vec{\Phi}(\vec{\xi},t) + \Pi^i_\varepsilon(\vec{\xi} - \vec{\xi}^i_\varepsilon)\right] \varphi\left(\frac{|\vec{\xi} - \vec{\xi}^i_\varepsilon|}{d^i_\varepsilon}\right), \quad (3.38)$$

where $\vec{\xi}^i_\varepsilon$ denotes the center of mass of the i-th particle at $t = 0$, $\vec{\Phi}(x,t)$ is a vector function giving the motion to the centers of masses, $\Pi^i_\varepsilon(t)$ are the relation operators generated by the rotation of the particles around their centers of mass $\vec{\Phi}(\vec{\xi}^i_\varepsilon,t)$, $\varphi(y)$ is a twice differentiable function, such that $\varphi(y) = 1$ for $y \le 1$, $\varphi(y) = 0$ for $y > 3/2$.

According to the properties of the function $\vec{\Phi}(x,t)$ and conditions 1, 3 of Th. 1, for any $t \in [0,T]$ and ε sufficiently small, $\vec{x} = \vec{\Phi}_\varepsilon(\vec{\xi},t)$ is a one-to-one map into Ω. Moreover, there exists a continuously differentiable inverse map $\vec{\xi} = [\Phi_\varepsilon]^{-1}(x,t)$. We set

$$l^k_\varepsilon(x,t) = \frac{\partial \Phi^k_\varepsilon(\vec{\xi},t)}{\partial t}|_{\xi=[\Phi_\varepsilon]^{-1}(x,t)}, \;\; l^0_\varepsilon(x,t) \equiv 1, \quad k = 0,1,2,3. \quad (3.39)$$

The map $\vec{x} = \vec{\Phi}(\vec{\xi}^i_\varepsilon,t) + \Pi^i_\varepsilon(t)(\vec{\xi} - \vec{\xi}^i_\varepsilon)$ maps $Q^i_\varepsilon(0)$ in $Q^i_\varepsilon(t)$, and the functions $\varphi^i_\varepsilon(x,t) = \varphi\left(\frac{|[\Phi_\varepsilon]^{-1}(x,t) - \vec{\xi}^i_\varepsilon|}{d^i_\varepsilon}\right)$ equal 1 in Q^{iT}_ε and 0 in Q^{jT}_ε $(j \ne i)$. Then the vector field $\vec{l}_\varepsilon(x,t)$ is tangent to $\bigcup_{i=1}^{N_\varepsilon} \partial Q^{iT}_\varepsilon$. Since $\vec{\Phi}(\vec{\xi},t)$ for any $t \in [0,T]$ maps $\partial\Omega$ on $\partial\Omega$ and $\varphi^i_\varepsilon(x,t) = 0$ on $\partial\Omega \times [0,T]$, then $\vec{l}_\varepsilon(x,T)$ is tangent to $\partial\Omega \times [0,T]$. The rotation operator $\Pi^i_\varepsilon(t)$ satisfies the equation $\dot{\Pi}^i_\varepsilon(t) = \vec{\theta}^i_\varepsilon(t) \times \Pi^i_\varepsilon(t)$ and $\vec{\Phi}(x,t)$, $\vec{\theta}^i(t)$, $i = 1,\ldots,N_\varepsilon$, and $\varphi(y)$ are sufficiently smooth. Then we apply (3.38), (3.39) and finally obtain (3.37).

Let us denote by $\dfrac{d}{dl_\varepsilon}$ the derivative with respect to the vector field $\vec{l}_\varepsilon,\ldots,$

$$\frac{d}{dl_\varepsilon} = \frac{\partial}{\partial t} + l^1_\varepsilon(x,t)\frac{\partial}{\partial x_1} + l^2_\varepsilon(x,t)\frac{\partial}{\partial x_2} + l^3_\varepsilon(x,t)\frac{\partial}{\partial x_3}$$

and set $\Psi_\varepsilon(x,t) = \operatorname{div}\dfrac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon}$, where $\vec{v}_\varepsilon(x,t)$ is the solution of (3.13)–(3.15), extended by zero in $\bigcup_{i=1}^{N_\varepsilon} Q^{iT}_\varepsilon \bigcup \partial\Omega$.

Since the field $\vec{l}_\varepsilon(x,t)$ is tangent to $\bigcup_{i=1}^{N_\varepsilon} \partial Q^{iT}_\varepsilon \bigcup \partial\Omega$, thus it follows from the properties of $\vec{v}_\varepsilon$ that $\dfrac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} = 0$ in $\bigcup\limits_{i=1}^{N_\varepsilon} \partial Q_\varepsilon^{iT} \bigcup \partial\Omega$, p $\int\limits_{\Omega(t)} \Psi_\varepsilon(x,t)dx = 0 \ \forall t \in [0,T]$,

$\Psi_\varepsilon(x,t) = 0$ for $(x,t) \in \bigcup\limits_{i=1}^{N_\varepsilon} Q_\varepsilon^{iT}$ and

$$\Psi_\varepsilon(x,t) = \sum_{i,k=1}^{3} \frac{\partial l_\varepsilon^i}{\partial x_k}\frac{\partial v_{k\varepsilon}}{\partial x_i}. \tag{3.40}$$

Thus Ψ_ε satisfies the conditions of Lem. 2 and, therefore, there is a vector function $\vec{z}(x,t) \in \overset{\circ}{W}{}_2^1(\Omega)$ such that $\forall t \in [0,T]$: $\mathrm{div}\vec{z}_\varepsilon = \Psi_\varepsilon(x,t)$, $x \in \Omega$; $\vec{z}_\varepsilon(x,t) = 0$; $x \in \bigcup\limits_{i=1}^{N_\varepsilon} Q_\varepsilon^i(t) \bigcup \partial\Omega$

$$\|\vec{z}_\varepsilon\|_{W_2^1(\Omega)} \le C\|\Psi_\varepsilon\|_{L_2(\Omega)}.$$

This inequality, (3.37) and (3.40) imply

$$\int_0^T \|\vec{z}_\varepsilon(x,t)\|^2_{W_2^1(\Omega)}dt < C\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_T)}, \tag{3.41}$$

where C is a constant that does not depend on ε.

We multiply the equation (3.14) by $\dfrac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon$ and integrate over Ω_ε^T. We have

$$\iint\limits_{\Omega_\varepsilon^T} \left(\vec{v}_{\varepsilon t}, \frac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) dxdt = \nu \int_0^T \int\limits_{\Omega_\varepsilon(t)} \left(\Delta\vec{v}_\varepsilon, \frac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) dxdt$$

$$-\int_0^T \int\limits_{\Omega_\varepsilon(t)} \left(\nabla q_\varepsilon, \frac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) dxdt + \iint\limits_{\Omega_\varepsilon^T} \left(\vec{g}_\varepsilon, \frac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) dxdt. \tag{3.42}$$

Using (3.37) and (3.41) we estimate the left-hand side of (3.42) as follows

$$\iint\limits_{\Omega_\varepsilon^T} \left(\vec{v}_{\varepsilon t}, \frac{\partial \vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) dxdt = \iint\limits_{\Omega_\varepsilon^T} |\vec{v}_{\varepsilon t}|^2 dxdt$$

$$+\sum_{k=1}^{3} \iint\limits_{\Omega_\varepsilon^T} \left(\vec{v}_{\varepsilon t}, l_\varepsilon^k \frac{\partial \vec{v}_\varepsilon}{\partial x_k}\right) dxdt - \iint\limits_{\Omega_\varepsilon^T} (\vec{v}_{\varepsilon t}, \vec{z}_\varepsilon) dxdt$$

$$\geq \|\vec{v}_{\varepsilon t}\|^2_{L_2(\Omega_\varepsilon^T)} - C_1\|\vec{v}_{\varepsilon t}\|_{L_2(\Omega_\varepsilon^T)}\|\nabla\vec{v}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)}$$

$$-\|\vec{v}_{\varepsilon t}\|_{L_2(\Omega_\varepsilon^T)}\|\vec{z}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)} \geq (1-\delta)\|\vec{v}_{\varepsilon t}\|^2_{L_2(\Omega_\varepsilon^T)} - C/\delta\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}, \tag{3.43}$$

where δ is an arbitrary positive number and C is a constant that is independent of ε.

In a similar way for the last term in the right-hand side in (3.42), we have

$$\left|\iint\limits_{\Omega_\varepsilon^T}\left(\vec{g}_\varepsilon, \frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right)dxdt\right| \leq \delta\|\vec{v}_{\varepsilon t}\|^2_{L_2(\Omega_\varepsilon^T)}$$

$$+\frac{1}{4\delta}\|\vec{g}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} + C\|\vec{g}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)}\|\nabla\vec{v}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)}. \tag{3.44}$$

Since $\operatorname{div}\left(\frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right) = 0$ and $\frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon = 0$ on $\partial\Omega_\varepsilon(t)$, then integrating by parts, we get

$$\int_0^T\int\limits_{\Omega_\varepsilon(t)}\left(\nabla\vec{q}_\varepsilon, \frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right)dxdt = 0 \tag{3.45}$$

and

$$\int_0^T\int\limits_{\Omega_\varepsilon(t)}\left(\Delta\vec{v}_\varepsilon, \frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon} - \vec{z}_\varepsilon\right)dxdt$$

$$= -\int_0^T\int\limits_{\Omega_\varepsilon(t)}\left(\nabla\vec{v}_\varepsilon, \nabla\frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon}\right)dxdt + \int_0^T\int\limits_{\Omega_\varepsilon(t)}(\nabla\vec{v}_\varepsilon, \nabla\vec{z}_\varepsilon)dxdt. \tag{3.46}$$

We estimate the second term in the right-hand side with the help of (3.41). Thus

$$\int_0^T\int\limits_{\Omega_\varepsilon(t)}(\nabla\vec{v}_\varepsilon, \nabla\vec{z}_\varepsilon)dxdt \leq C\|\nabla\vec{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}, \tag{3.47}$$

and the first term can be represented in the form

$$\int_0^T\int\limits_{\Omega_\varepsilon(t)}\left(\nabla\vec{v}_\varepsilon, \nabla\frac{\partial\vec{v}_\varepsilon}{\partial l_\varepsilon}\right)dxdt = \frac{1}{2}\int\limits_{\Omega_\varepsilon^T}\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(l_\varepsilon^k|\nabla\vec{v}_\varepsilon|^2)dxdt$$

$$-\frac{1}{2}\int\limits_{\Omega_\varepsilon^T}|\nabla\vec{v}_\varepsilon|^2\sum_{k=1}^{3}\frac{\partial l_\varepsilon^k}{\partial x_k}dxdt + \int\limits_{\Omega_\varepsilon^T}\sum_{i,k=1}^{3}\frac{\partial l_\varepsilon^i}{\partial x_k}\left(\frac{\partial\vec{v}_\varepsilon}{\partial x_i}, \frac{\partial\vec{v}_\varepsilon}{\partial x_k}\right)dxdt = J_1+J_2+J_3. \tag{3.48}$$

Due to (3.37), we have

$$|J_2| + |J_3| \leq C\|\nabla \vec{v}_\varepsilon\|^2_{L_2(\Omega^T_\varepsilon)}.) \tag{3.49}$$

The vector field $\vec{l}_\varepsilon(x,t)|\nabla \vec{v}_\varepsilon|^2$ is tangent to the lateral surface of Ω^T_ε and $l^0_\varepsilon \equiv 1$. Then applying to J_1 the theorem of Gauss–Ostrogradski, we get

$$J_1 \equiv \frac{1}{2}\int\limits_{\Omega^T_\varepsilon} \sum_{k=0}^{3} \frac{\partial}{\partial x_k}(l^k_\varepsilon |\nabla \vec{v}_\varepsilon|^2)dxdt$$

$$= \frac{1}{2}\int\limits_{\Omega_\varepsilon(T)} |\vec{v}_\varepsilon(x,T)|^2 dx - \frac{1}{2}\int\limits_{\Omega_\varepsilon(0)} |\nabla \vec{v}_\varepsilon(x,0)|^2 dx. \tag{3.50}$$

Now from (3.42), (3.45), (3.46), (3.48), (3.50) and the estimates (3.43), (3.44), (3.47), (3.49) we obtain

$$(1-2\delta)\|\vec{v}_{\varepsilon t}\|^2_{L_2(\Omega^T_\varepsilon)} + \frac{\nu}{2}\int\limits_{\Omega_\varepsilon(T)} |\nabla \vec{v}_\varepsilon(x,T)|^2 dx$$

$$\leq \left(\frac{1}{4\delta} + C\right)\|\vec{g}_\varepsilon\|^2_{L_2(\Omega^T_\varepsilon)} + C\left(\frac{1}{\delta}+1\right)\|\nabla \vec{v}_\varepsilon\|^2_{L_2(\Omega^T_\varepsilon)} + \frac{\nu}{2}\int\limits_{\Omega_\varepsilon(0)} |\nabla \vec{V}_\varepsilon|^2 dx,$$

where $\delta > 0$ is an arbitrary positive number and C does not depend on ε and δ.

We set $\delta = 1/4$. Recall that $\vec{g}_\varepsilon = \vec{g} - \vec{w}_\varepsilon + \nu \vec{W}_\varepsilon$, $\vec{V}_\varepsilon = \vec{U}_\varepsilon - \vec{w}_\varepsilon$, where $\vec{w}_\varepsilon$ and $\vec{W}_\varepsilon$ are defined in (3.2), (3.3). Now using Lem. 1, condition 4 of Th. 1, and (3.19) we conclude that the derivatives $\vec{v}_{\varepsilon t}$ are bounded in $L_2(\Omega_T)$ uniformly in ε, i.e.,

$$\|\vec{v}_{\varepsilon t}\|_{L_2(\Omega^T_\varepsilon)} < C. \tag{3.51}$$

4. Proof of Theorem 1

For any $t \in [0,T]$ $\vec{v}_\varepsilon(x,t) \in \overset{\circ}{W}{}^1_2(\Omega_\varepsilon)$. We extend $\vec{v}_\varepsilon$ by zero in $\bigcup_{i=1}^{N_\varepsilon} Q^{iT}_\varepsilon$. Due to (3.19), (3.51) we see that the sequence $\{\vec{v}_\varepsilon, \varepsilon \to 0\}$ is bounded in $\overset{\circ}{W}{}^1_2(\Omega_T)$ and, therefore, it is weakly compact in this space. Then there is a subsequence $\{\vec{v}_{\varepsilon_k}, \varepsilon_k \to 0\}$ which converges weakly in $\overset{\circ}{W}{}^1_2(\Omega_T)$ to a vector function $\vec{v}(x,t) \in \overset{\circ}{W}{}^1_2(\Omega_T)$. Due to the imbedding theorem this subsequence converges strongly in $L_2(\Omega_T)$ to the vector function $\vec{v}(x,t)$. Moreover, it converges strongly in $L_2(\Omega_T)$

(uniformly with respect to $t \in [0,T]$). Let us show that $\vec{v}$ is a solution of the following initial boundary value problem:

$$\vec{v}_t - \nu\Delta\vec{v} + \nu C(x,t)\vec{v} = -\nabla p + \vec{F}(x,t), \ \text{div}\vec{v} = 0, \ (x,t) \in \Omega_T; \tag{4.1}$$

$$\vec{v}(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \ \vec{v}(x,0) = \vec{U}(x), \ x \in \Omega; \tag{4.2}$$

where $\vec{F} = g(x,t) + \nu C(x,t)\vec{W}(x,t)$ ($\vec{W}(x,t) = \vec{\Phi}_t(\xi,t)|_{\xi=\Phi^{-1}(x,t)}$), the matrix $C(x,t)$ and the vector function $\vec{U}$ are defined in conditions 2 and 4 of Th. 1, respectively.

It is clear that problem (4.1)–(4.2) has a unique solution. Therefore, the whole sequence of extended functions $\{v_\varepsilon(x,t), \varepsilon \to 0\}$ converges weakly in $\overset{\circ}{W}{}_2^1(\Omega_T)$ and strongly in $L_2(\Omega_T)$ и $L_2(\Omega)$ ($\forall t \in [0,T]$) to $v(x,t)$.

Let us introduce the linear resolving operator $\tilde{\Delta}_\varepsilon$, defined by $\tilde{\Delta}_\varepsilon\vec{u}_\varepsilon = \vec{\Psi}_\varepsilon(x)$, where $\vec{u}_\varepsilon(x)$ is a solution of the following problem:

$$\Delta\vec{u}_\varepsilon(x) - \nabla p_\varepsilon(x) = \vec{\Psi}_\varepsilon(x), \quad \text{div}\vec{u}_\varepsilon(x) = 0, \quad x \in \Omega_\varepsilon(t); \tag{4.3}$$

$$\vec{u}_\varepsilon(x) = 0, \quad x \in \partial\Omega_\varepsilon(t). \tag{4.4}$$

The energy space is denoted by $\overset{\circ}{J}(\Omega)$. It is defined as a closure in $L_2(\Omega_\varepsilon)$ of the divergent free vector functions with a compact support in Ω_ε.

We recall that $L_2(\Omega) = \overset{\circ}{J}(\Omega_\varepsilon) \oplus G(\Omega_\varepsilon)$, where the subspace $G(\Omega_\varepsilon)$ consists of the gradients of the single valued functions from $W_2^1(\Omega_\varepsilon)$. The domain $D(\tilde{\Delta}_\varepsilon)$ of $\tilde{\Delta}_\varepsilon$ is the set of all the solutions of (4.3) corresponding to various $\vec{\Psi}_\varepsilon \in \overset{\circ}{J}(\Omega_\varepsilon)$. It is shown by O.A. Ladyzhenskaya that the operator Δ_ε determines the one-to-one correspondence between $D(\tilde{\Delta}_\varepsilon)$ and $\overset{\circ}{J}(\Omega_\varepsilon)$. It is adjoint and negatively definite on $D(\tilde{\Delta}_\varepsilon)$ (see [1]). The inverse operator $(\tilde{\Delta}_\varepsilon)^{-1} \equiv \tilde{R}_\varepsilon$ is a compact self adjoint operator. We extend this operator by the linearity to the whole $L_2(\Omega_\varepsilon)$ by setting $\tilde{R}_\varepsilon\vec{g} = 0$ for $g \in G(\Omega_\varepsilon)$ and in the space $L_2(\Omega)$ define the operator $\tilde{R} = I_\varepsilon R_\varepsilon P_\varepsilon$, where P_ε is the restriction operator from $L_2(\Omega)$ to $L_2(\Omega_\varepsilon)$, i.e., $\forall\vec{f} \in L_2(\Omega)$, $P_\varepsilon\vec{f}[x] = \vec{f}(x)$ for $x \in \Omega_\varepsilon$; I_ε is the imbedding operator from $L_2(\Omega_\varepsilon)$ to $L_2(\Omega)$, i.e.,

$$\forall\vec{f_\varepsilon} \in L_2(\Omega_\varepsilon), \ I_\varepsilon\vec{f_\varepsilon}[x] = \begin{cases} \vec{f_\varepsilon}(x), & \text{for} \quad x \in \Omega_\varepsilon, \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega_\varepsilon. \end{cases}$$

It is easy to see that R_ε is a compact selfadjoint operator in $L_2(\Omega)$. In a similar way we introduce the operator $\tilde{\Delta}^C$ that determines the one-to-one correspondence between $\vec{u}(x)$ of the solution of the problem

$$\Delta\vec{u} - C(x)\vec{u} - \nabla p = \vec{\Psi}(x), \quad \text{div}\vec{u} = 0, \quad x \in \Omega, \tag{4.5}$$

$$\vec{u}(x) = 0, \quad x \in \partial\Omega, \tag{4.6}$$

and the right-hand sides of $\vec{\Psi} \in \overset{\circ}{J}(\Omega)$. As in [1], one can show that the operator $\tilde{\Delta}^C$ determines the one-to-one correspondence between its domain $D(\tilde{\Delta}^C) = \{\vec{u} \in \overset{\circ}{J}(\Omega) : \tilde{\Delta}^C u \in \overset{\circ}{J}(\Omega)\}$ and $\overset{\circ}{J}(\Omega)$. It is selfadjoint and negatively definite. Its inverse operator $R = (\tilde{\Delta}^C)^{-1}$ is compact and selfadjoint in $\overset{\circ}{J}(\Omega)$. We extend R by the linearity on the whole space $L_2(\Omega) = \overset{\circ}{J}(\Omega) \oplus G(\Omega)$ by setting $R\vec{g} = 0$ for $\vec{g} \in G(\Omega)$. The following theorem holds:

Theorem 2. *Let the conditions of Th. 1 be fulfilled. Then, for any* $\vec{f} \in L_2(\Omega)$ *the sequence* $\{R_\varepsilon \vec{f}, \varepsilon \to 0\}$ *converges in* $L_2(\Omega)$ *to* $R\vec{f}$, *i.e.,* $\|R_\varepsilon \vec{f} - R\vec{f}\|_{L_2(\Omega)} \to 0$ *as* $\varepsilon \to 0$.

The proof of Th. 2 follows from [4].

Denote by $\mathbf{R}_\varepsilon^t$ and $\mathbf{R}_t$ the resolving operators of problem (4.3), (4.4) (for $\Omega_\varepsilon = \Omega_\varepsilon(t)$) and (4.5), (4.6) (for $C(x) = C(x,t)$), respectively. These operators are compact and selfadjoint in $L_2(\Omega)$. Let $\vec{v}_\varepsilon(x,t)$ be a solution of (3.13)–(3.15), extended by zero in the set $Q_\varepsilon^T = \bigcup_{i=1}^{N_\varepsilon} Q_\varepsilon^{iT}$). For any $t \in [0,T]$ $\vec{v}_\varepsilon(x,t)$ can be represented as $\vec{v}_\varepsilon(x,t) = \mathbf{R}_\varepsilon^t \vec{f}_{\varepsilon t}[x]$, where $\vec{f}_{\varepsilon t} \equiv \vec{f}_\varepsilon(x,t) = \vec{v}_{\varepsilon t}(x,t) - \vec{g}_\varepsilon(x,t) = \vec{v}_{\varepsilon t} - \vec{g} + \vec{w}_{\varepsilon t} - \nu \vec{W}_\varepsilon$. Let $\vec{\varphi}(x,t)$ be an arbitrary vector function in Ω_T. Denote by $(\cdot,\cdot)_\Omega$ the scalar product in $L_2(\Omega)$. Taking into account that for any t the operator $\mathbf{R}_\varepsilon^t$ is selfadjoint in $L_2(\Omega)$, and $\vec{\varphi}(x,t) = \vec{\varphi}_t(x) \in L_2(\Omega)$, we have

$$\int_0^T \int_\Omega (\vec{v}_\varepsilon(x,t), \vec{\varphi}(x,t))dxdt = \int_0^T (\mathbf{R}_\varepsilon^t \vec{f}_{\varepsilon t}, \vec{\varphi}_t)_\Omega dt$$

$$= \int_0^T (\vec{f}_{\varepsilon t}, \mathbf{R}_\varepsilon^t \vec{\varphi}_t)_\Omega dt = \int_0^T (\vec{f}_{\varepsilon t}, \mathbf{R}_t \vec{\varphi}_t)_\Omega dt + \int_0^T (\vec{f}_{\varepsilon t}, \mathbf{R}_\varepsilon^t \vec{\varphi}_t - \mathbf{R}_t \vec{\varphi}_t)_\Omega dt. \tag{4.7}$$

According to Lem. 1, $\vec{g}_\varepsilon(x,t)$ converges weakly in $L_2(\Omega_T)$ to the vector function $\vec{g}(x,t) + \nu C(x,t)\vec{W}(x,t)$ and the subsequence $\{\vec{v}_{\varepsilon_k}(x,t), \varepsilon \to 0\}$ converges weakly in $W_2^1(\Omega_T)$ to $\vec{v}(x,t)$ as $\varepsilon \to 0$. The function $\vec{f}_\varepsilon(x,t) = \vec{v}_\varepsilon(x,t) - \vec{g}_\varepsilon(x,t)$ converges weakly in $L_2(\Omega_T)$ to $\vec{v}_t(x,t) - \vec{g}(x,t) - \nu C(x,t)\vec{W}(x,t) = \vec{f}(x,t) = \vec{f}_t$ as $\varepsilon = \varepsilon_k \to 0$. Therefore,

$$\lim_{\varepsilon=\varepsilon_k \to 0} \int_0^T (\vec{f}_{\varepsilon t}, \mathbf{R}_t \vec{\varphi}_t)_\Omega dt = \lim_{\varepsilon=\varepsilon_k \to 0} \int_0^T \int_\Omega (\vec{f}_\varepsilon(x,t) \mathbf{R}_t \vec{\varphi}(x,t))dxdt$$

$$= \int_0^T \int_\Omega (\vec{f}(x,t) \mathbf{R}_t \vec{\varphi}(x,t))dxdt = \int_0^T (\vec{f}_t, \mathbf{R}_t \vec{\varphi}(x,t))_\Omega dt = \int_0^T (\mathbf{R}_t \vec{f}_t, \vec{\varphi}_t)_\Omega dt$$

$$= \int_0^T \int_\Omega (\mathbf{R}_t \vec{f}_t[x], \vec{\varphi}(x,t))dxdt. \tag{4.8}$$

Here we make use of the fact that $\mathbf{R}_t$ is selfadjoint in $L_2(\Omega)$. Then using for any $t \in [0,T]$ Th. 2 and taking into account the uniform boundness of $\|\mathbf{R}_\varepsilon^t \varphi_t\|_{L_2(\Omega)}$ on ε, t we get

$$\left| \int_0^T (f_{\varepsilon t}, \mathbf{R}_\varepsilon^t \varphi_t - \mathbf{R}_t \varphi_t)_\Omega dt \right|$$

$$\leq \|f_\varepsilon\|_{L_2(\Omega_T)} \left\{ \int_0^T \|\mathbf{R}_\varepsilon^t \varphi_t - \mathbf{R}_t \varphi_t\|^2_{L_2(\Omega)} dt \right\}^{1/2} \to 0, \tag{4.9}$$

as $\varepsilon \to 0$.

Thus due to (4.7)–(4.9)

$$\lim_{\varepsilon=\varepsilon_k \to 0} \int_0^T \int_\Omega (\vec{v}_\varepsilon(x,t), \vec{\varphi}(x,t))dxdt = \int_0^T \int_\Omega (\mathbf{R}_t \vec{f}_t[x], \vec{\varphi}(x,t))dxdt.$$

Since $\vec{v}_\varepsilon$ converges in $L_2(\Omega_T)$ to $\vec{v}$, p $\vec{\varphi}$ as $\varepsilon = \varepsilon_k \to 0$, and $\vec{\varphi}$ is an arbitrary continuous vector function, then $\vec{v}(x,t) = \mathbf{R}_t \vec{f}_t[x] = \mathbf{R}_t(\vec{v}_t(x,t) - \vec{g}(x,t) - \nu C(x,t)\vec{W}(x,t)$. By the definition of $\mathbf{R}_t$ this means that $\vec{v}$ satisfies (4.1) and boundary condition (4.2). The vector function $\vec{v}_\varepsilon$ converges weakly in $\overset{\circ}{W}{}_2^1(\Omega_T)$ to $\vec{v}$ as $\varepsilon = \varepsilon_k \to 0$ and, therefore, in $L_2(\Omega)$ uniformly with respect to t. Then it follows from condition 4 of Th. 1 and Lem. 1 that $v(x,0) = U(x)$. Therefore, $\vec{v}$ is the solution of problem (4.1)–(4.2).

Consider now $\vec{u}_\varepsilon$ of the solution of (2.1)–(2.4). Since $\vec{u}_\varepsilon(x,t) = \vec{v}_\varepsilon(x,t) + \vec{w}_\varepsilon(x,t)$, then taking into account $\vec{v}_\varepsilon(x,t) \underset{W_2^1(\Omega_T)}{\rightharpoonup} \vec{v}(x,t)$ and Lem. 1, we conclude that $\vec{u}_\varepsilon$ converges in $L_2(\Omega_T)$ to the vector function $\vec{u}(x,t) = \vec{v}(x,t)$ as $\varepsilon \to 0$. According to (4.1)–(4.2) this vector function is the solution of (2.8)–(2.10). Theorem 1 is proved.

References

[1] *O.A. Ladyzhenskaya,* The Mathematical Theory of Viscous Incompressible Flow. New York, London, Paris, Gordon and Breach Sci. Publ. **XVIII**, (1969).

[2] *O.A. Ladyzhenskaya,* Initial-Boundary Problem for Navier–Stokes Equations in Domains with Time-Varying Boundaries. — Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. Steklov, Leningrad **11** (1968), 97–128. (Russian)

[3] *V.A. Marchenko and E.Ya. Khruslov*, Boundary Value Problems in Domains with a Fine-Grained Boundary. Naukova Dumka, Kiev, 1974. (Russian)

[4] *V.A. Lvov*, The Convergence of the Solutions of a System of Initial Boundary Value Problems in Domains with Moving Fine-Grained Boundary. — *Dokl. AN USSR* (1987), No. 7. 21–24. (Russian)

[5] *Eh.B. Bykhovskij and N.V. Smirnov*, On Orthogonal Decomposition of the Space of Vector Functions Quadratically Summable in the Given Domain and the Operators of Vector Analysis. — *Inst. Math. AN USSR* **59** (1960), 5–36. (Russian)

[6] *E.Ya. Khruslov and Z.F. Nazyrov*, Perturbation of the Thermal Field of Moving Small Particles. Studies in the Theory of Operators and their Applications. Naukova Dumka, Kiev, 1979. (Russian)