УДК 531.38

©2011. **H.M. Yehia**

# NEW 2D INTEGRABLE FAMILIES WITH A QUARTIC SECOND INVARIANT

The method introduced by the present author in 1986 still proves most effective in constructing integrable 2-D Lagrangian systems, which admit in addition to the energy another integral of motion that is polynomial in velocities. In a previous article (J. Phys. A: Math. Gen., 39, 5807–5824, 2006) we constructed a system, which admits a quartic complementary integral. This system, called by us "master", is the largest known, as it involves 21 parameters, and contains, as special cases of it, almost all previously known systems of the same type that admit a quartic integral. In the present note we generalize the method we used before to construct new several-parameter systems that are not special cases of the master system. A new system involving 16 parameters is introduced and a special case of it admits interpretation in a problem of rigid body dynamics. It gives a unification of certain special versions of known classical integrable cases due to Kovalevskaya, Chaplygin and Goriatchev and other cases recently introduced by the present author.

**Keywords:** *integrable families, invariant, rigid body dinamics.*

**Introduction.** ***Historical*** The famous Kovalevskaya's integrable case of rigid body dynamics was the first example of a mechanical system that admits an integral of motion quartic in velocities [1]. For more than a century this case attracted attention of many specialists, who treated explicit solution in terms of time and gave several modifications and generalizations. Only in the last three decades there appeared a few new integrable systems with a quartic integral, but mainly concerning the motion of a particle in the Euclidean plane under the action of certain potential forces. A short, but nearly complete up to its date, list of those systems can be found in Hietarinta's review paper [2]. A few cases of the same type were obtained later in [3].

In virtue of Maupertuis principle, the motion of a natural mechanical system can be brought into equivalence (in the orbital sense) with the geodesic flow on some Riemannian metric. Metrics on the Riemannian sphere associated with known integrable cases of rigid body dynamics were constructed in [4]. Two families of integrable systems with a quartic integral on $S^2$ were obtained in [5] and [6]. Few more works discussed possible integrable systems with low-degree polynomials on $S^2$ and the hyperbolic plane $H^2$ (see e.g. [7–9]).

The method introduced in our work [10] and developed in several later works, has led to construction of a large number of several-parameter families of integrable systems with a complementary integral ranging from second to fourth degree (see e.g. [13, 14]). This method leads in a natural way to integrable systems on various flat and curved 2-D manifolds. Although our primary interest is in systems on Riemannian manifolds, some of the constructed integrable systems live on pseudo-Euclidean or pseudo-Riemannian manifolds.

The culmination of this method was the construction of the so-called "master" system with a quartic integral [15]. It involves the largest ever number of 21 parameters and covers almost all systems of that type that were known earlier.

In the present article we extend the metod used in [15] to construct certain systems with more general structures. One of the resulting systems that involves 15 parameters and is investigated in some detail. A special case introduces a new integrable problem in rigid body dynamics.

***Construction of integrable systems.*** According to a result of Birkhoff [16], the general natural mechanical system (on an arbitrary 2D Riemannian configuration space) can always be reduced in certain (isometric) coordinates $\xi, \eta$ and time parametrization $\tau$ to the form of a ficticious plane system described by the Lagrangian

$$L = \frac{1}{2}[\xi^{'2} + \eta^{'2}] + U, U = U(\xi, \eta) \tag{1}$$

restricted to its zero-energy level

$$\xi'^2 + \eta'^2 - 2U = 0. \tag{2}$$

The energy constant $h$ for the original system enters linearly as a parameter in the function $U$, which has the structure

$$U = \Lambda(h - V) \tag{3}$$

where $V$ is the potential of the original system and $\Lambda$ is a function that depends on the metric of the configation space.

In [15] (see also [10]), it was proved that if an integral of motion of the mechanical system exists in the form of a polynomial of the fourth degree in velocities, this integral can be reduced while preserving the form of the Lagrangian (1) to the form

$$I = \xi'^4 + P\xi'^2 + Q\xi'\eta' + R = I_0(\text{const}) \tag{4}$$

where $R$ is given by the quadrature

$$R = -\int Q\frac{\partial U}{\partial \xi}d\eta - \int [2P\frac{\partial U}{\partial \xi} + Q\frac{\partial U}{\partial \eta} + 2U\frac{\partial Q}{\partial \eta}]_0 d\xi \tag{5}$$

in which $[\,]_0$ means that the expression in the bracket is computed for $\eta$ taking an arbitrary constant value $\eta_0$ (say), and the other three functions involved are expressed as

$$P = \frac{\partial^2 F}{\partial \xi^2}, Q = -\frac{\partial^2 F}{\partial \xi \partial \eta}, U = -\frac{1}{4}(\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2}) \tag{6}$$

in terms of a single auxiliary function $F$ which satisfies the nonlinear partial differential equation

$$\frac{\partial^2 F}{\partial \xi \partial \eta}(\frac{\partial^4 F}{\partial \xi^4} - \frac{\partial^4 F}{\partial \eta^4}) + 3(\frac{\partial^3 F}{\partial \xi^3}\frac{\partial^3 F}{\partial \xi^2 \partial \eta} - \frac{\partial^3 F}{\partial \eta^3}\frac{\partial^3 F}{\partial \eta^2 \partial \xi}) + 2(\frac{\partial^2 F}{\partial \xi^2}\frac{\partial^4 F}{\partial \xi^3 \partial \eta} - \frac{\partial^2 F}{\partial \eta^2}\frac{\partial^4 F}{\partial \eta^3 \partial \xi}) = 0. \tag{7}$$

The set of solutions of this equation generates all systems of the type (1) having an integral of the form (4) on the zero level of their energy integral. Affecting all possible conformal mappings of the complex $\zeta = \xi + i\eta$ plane followed by a general point transformation to the generalized coordinates $q_1, q_2$ with a suitable change of the time variable we obtain all systems of the general form on 2D Riemannian manifolds, having a quartic integral on the zero level of their energy integral.

To construct a system that would be a priori integrable on all levels of energy, we should receive a function $U$ from (6) that has the structure (3), i.e. involving in a linear way an arbitrary parameter $h$. In that case The auxiliary function should have the form $F = hF_0 + F_1$, where $F_0, F_1$ are functions not depending on $h$. It is shown in [15] how the resulting conditional integrable system can be used to construct another integrable system valid on arbitrary energy level.

**1. The auxiliary equation and certain forms of solutions.** However, it became clear that the original isometric variables $\xi, \eta$ are not practically suitable coordinates for the description of the solution, and that a symmetric separation solution can be more conveniently expressed in terms of the pair of variables $p, q$ related to $\xi, \eta$ by the relations

$$\xi = \int^p \frac{dp}{\sqrt[4]{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}}, \eta = \int^q \frac{dq}{\sqrt[4]{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}. \tag{8}$$

This solution, first announced in [15], can be written as

$$F = F_0 + \nu pq, \tag{9}$$

where $F_0 =$

$$= \int^p \frac{dp}{\sqrt[4]{a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0}} \int^p \frac{(4C_0 + 4C_1p + 4Ap^2 + \frac{1}{4}b_3p^3)dp}{(a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0)^{3/4}} +$$

$$+ \int^q \frac{dq}{\sqrt[4]{a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}} \int^q \frac{(4D_0 + 4D_1p + 4Ap^2 + \frac{1}{4}a_3p^3)dq}{(a_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0)^{3/4}}. \tag{10}$$

It involves 15 free parameters$a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, A, C_0, C_1, D_0, D_1, \nu$. It should be noted that this solution determine the Lagrangian and the quartic integral valid only on the zero level of its energy.

In the present article we try some deformations of the last system to accomodate more terms of the product type, probably at the expense of enforcing certain restrictions on some of the parameters figuring in (10). In fact, we assume F in the form

$$F = F_0 + \sum_{i+j=2}^{4} F_{ij} p^i q^j \tag{11}$$

so that we have replaced the monomial term $pq$ with a fourth-degree polynomial. The limit 4 for the degree was chosen on an experimental basis, as it seemed that fifth-degree polynomials would take incomparable time and computation resources.

The expression (11) is inserted in equation (7). Using the relations (8) and making some manipulations, we obtain a polynomial expression of the sixth degree in the two variables $p, q$ that must vanish, whose coefficients constitute a system of 27 polynomial equations in the 26 parameters involved $\{A, C_0, C_1, D_0, D_1, a_i, i = 0, ..., 4;\ \ b_j,\ \ j = 0, ..., 3,\ \ F_{mn},\ \ 2 \leq m \leq n \leq 4\}$. We have solved this system using the MAPLE computer algebra package. The result was 40 distinct solutions, i.e. 40 working combinations of the parameters that lead to the construction of integrable systems with a quartic integral. One of those solutions reproduces the "master" system. This system is characterized by the preservation of all the 15 parameters $a_i, b_i, A, C_j, D_j$ and $\nu$. The remaining 39 cases are not expected to be all different. In view of the symmetric way in which groups of parameters enter with the two variables $p$ and $q$, some solutions lead to rewriting one and the same system. The final number of different systems turns out to be much less than 40. In a forthcoming article those systems will be classified and put in a form as simple as possible.

In only one more of the fourty cases $a_4 \neq 0$, so that the two polynomials that occur under the fourth degree root signs are of the fourth degree. This case, which differs from the master system, will be considered in detail in the next sections.

**2. The generic restricted case.** We write the Lagrangian and the complementary integral for this case after some transformation to more symmetric form that does not affect the generality of the system

$$L = \frac{1}{2}[\frac{\acute{u}^2}{\sqrt{au^4 + k_1 u^2 + k_0}} + \frac{\acute{v}^2}{\sqrt{av^4 + m_1 v^2 + m_0}}] + U, \tag{12}$$

$$U = -\frac{N[m_1 u^4 + v^2(4au^4 + 3k_1 u^2 + 2k_0)] + \nu uv(2au^2 + k_1) + Ku^2 + D}{\sqrt{au^4 + k_1 u^2 + k_0}} -$$
$$-\frac{N[k_1 v^4 + u^2(4av^4 + 3m_1 v^2 + 2m_0)] + \nu uv(2av^2 + m_1) + Kv^2 + E}{\sqrt{av^4 + m_1 v^2 + m_0}}, \tag{13}$$

$$I = \frac{\acute{u}^4}{au^4 + k_1 u^2 + k_0} +$$
$$+\frac{4\{N[m_1 u^4 + v^2(4au^4 + 3k_1 u^2 + 2k_0)] + \nu uv(2au^2 + k_1) + Ku^2 + D\}}{au^4 + k_1 u^2 + k_0}\acute{u}^2 -$$
$$-8(2Nuv + \nu)\acute{u}\acute{v} - 8(2Nuv + \nu)^2\sqrt{au^4 + k_1 u^2 + k_0}\sqrt{av^4 + m_1 v^2 + m_0} +$$
$$+16N(Dv^2 - Eu^2) - 4\nu(\nu + 4Nuv)(2au^2 v^2 + m_1 u^2 + k_1 v^2) -$$
$$-16N^2[(k_0 + k_1 u^2)v^4 + (m_0 + m_1 v^2)u^4 + 2au^4 v^4] +$$

$$
\begin{aligned}
&+\frac{4}{a^3(au^4+k_1u^2+k_0)}\times\\
&\times\{N^2\{m_1^2k_0(ak_0-k_1^2)+m_1^2[a^3u^8+k_0a^2u^4-k_1^2u^2(k_1+au^2)+ak_0k_1u^2]+\\
&+a^3v^2(2k_0+k_1u^2)(2k_0v^2+k_1u^2v^2-2m_1u^4)\}-\\
&-2N\nu a^3u^3v[m_1(2k_0+k_1u^2)+v^2(4ak_0-k_1^2)]-\\
&-2NDa^2[av^2(2k_0+k_1u^2)+m_1(k_0+k_1u^2)]+\\
&+\nu^2a^3(k_1^2-4ak_0)u^2v^2+Da^3[D+2Ku^2+2\nu uv(2au^2+k_1)]-\\
&-K^2a^2(k_0+k_1u^2)-2K\nu a^3uv(2k_0+k_1u^2)+2KNa[m_1a^2u^6-\\
&-ak_1(-m_1+av^2)u^4-(2k_0a^2v^2-k_1^2m_1)u^2+k_0k_1m_1]\}
\end{aligned}
\tag{14}
$$

in which the prime represents derivative with respect to the independent variable $\tau$ (ficticious time).

The system with the Lagrangian (12) admits the integral (14) only on the zero-energy level of this system

$$
\frac{1}{2}\Big[\frac{\acute{u}^2}{\sqrt{au^4+k_1u^2+k_0}}+\frac{\acute{v}^2}{\sqrt{av^4+m_1v^2+m_0}}\Big]-U=0.
$$

It depends on 10 parameters $a, k_0, k_1, m_0, m_1, D, E, K, \nu, N$, of which the first five enter in both the kinetic energy and potential terms of the Lagrangian and the last five ones enter only in the potential terms and moreover they enter only linearly. The last four parameters constitute a set of energy-like parameters (see e.g. [15]). They are essential in building the unrestricted integrable system. Comparing (12) to its counterpart in the "master" system [15], we find that (12) involves only one new parameter $N$, which is not present in the master system. When $N$ is set equal to zero (12) turns out to be a special case of the master system resulting from one restriction on the coefficients of each of the two fourth-degree polynomials entering under the root sign and two restrictions on the energy -like parameters, namely $C_1 = D_1 = 0$.

**3. Dynamics- The unrestricted generalization.** We now proceed to use those parameters to construct a general integrable system valid on arbitrary energy level out of the restricted one. Introducing new parameters by the relations

$$
\begin{aligned}
&D=h_1-h\alpha_1, \qquad E=h_2-h\alpha_2,\\
&K=h_3-h\alpha_3, \qquad \nu=h_4-h\alpha_4,\\
&N=h_5-h\alpha_5
\end{aligned}
\tag{15}
$$

and performing the change of independent variable to the actual-time parametrization by using the relation

$$
d\tau=\frac{dt}{\Lambda},
\tag{16}
$$

where

$$\Lambda = \frac{\alpha_1+\alpha_3u^2+\alpha_4uv(2au^2+k_1)+\alpha_5[m_1u^4+v^2(4au^4+3k_1u^2+2k_0)]}{\sqrt{au^4+k_1u^2+k_0}}+$$
$$+\frac{\alpha_2+\alpha_3v^2+\alpha_4uv(2av^2+m_1)+\alpha_5[k_1v^4+u^2(4av^4+3m_1v^2+2m_0)]}{\sqrt{au^4+k_1u^2+k_0}} \quad (17)$$

we arrive at the new Lagrangian

$$L=\frac{1}{2}\Lambda[\frac{\dot{u}^2}{\sqrt{au^4+k_1u^2+k_0}}+\frac{\dot{v}^2}{\sqrt{av^4+m_1v^2+m_0}}]-V+h,$$

$$V = \frac{1}{\Lambda}\{\frac{h_1+h_3u^2+h_4uv(2au^2+k_1)+h_5[m_1u^4+v^2(4au^4+3k_1u^2+2k_0)]}{\sqrt{au^4+k_1u^2+k_0}}+$$
$$+\frac{h_2+h_3v^2+h_4uv(2av^2+m_1)+h_5[k_1v^4+u^2(4av^4+3m_1v^2+2m_0)]}{\sqrt{au^4+k_1u^2+k_0}}\} \quad (18)$$

which admits on an arbitrary energy level $h$ the integral resulting from (14) by the substitutions (15) and (16), i.e. $\acute{u}\to\Lambda\dot{u}, \acute{v}\to\Lambda\dot{v}$. The integral will depend on the parameters occuring in the Lagrangian and also on the energy constant $h$. The last constant may be substituted by its expression in terms of the coordinates and velocities to get the final form free of the energy restriction. The resulting system depends on 16 parameters $a, k_0, k_1, m_0, m_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, h_1, h_2, h_3, h_4, h_5$ and $h$, of which the first nine enter in both the kinetic energy and potential terms of the Lagrangian and the last five ones enter only in the potential.

**4. Special cases.** ***Generalization of the cases of Bozis and Wojciechowski.*** Let $a = k_0 = m_0 = 1,\ k_1 = m_1 = -2$. Under the coordinate transformation $p=\sin y,\ q=\sin x$ the Lagrangian (18) takes the form

$$\begin{aligned} L &= \frac{1}{2}[\alpha+\beta\sin x\sin y+\gamma(2\cos^2x\cos^2y-\cos^2x-\cos^2y)+ \\ &+\frac{\delta_1}{\cos^2x}+\frac{\delta_2}{\cos^2y}](\dot{x}^2+\dot{y}^2)- \qquad (19) \\ &-\frac{a+b\sin x\sin y+c(2\cos^2x\cos^2y-\cos^2x-\cos^2y)+\dfrac{d_1}{\cos^2x}+\dfrac{d_2}{\cos^2y}}{\alpha+\beta\sin x\sin y+\gamma(2\cos^2x\cos^2y-\cos^2x-\cos^2y)+\dfrac{\delta_1}{\cos^2x}+\dfrac{\delta_2}{\cos^2y}}. \end{aligned}$$

When $\beta=\gamma=\delta_1=\delta_2=0$ we have, after ignoring an insignificant additive constant

$$L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-[b\sin x\sin y+c(2\cos^2x\cos^2y-\cos^2x-\cos^2y)+\frac{d_1}{\cos^2x}+\frac{d_2}{\cos^2y}]. \quad (20)$$

This system is new. It describes plane motion of a particle in a 4-parameter potential. The complementary integral of this system can be written as

$$\begin{aligned} I &= (\dot{x}^2 + \frac{2d_1}{\cos^2 x})(\dot{y}^2 + \frac{2d_2}{\cos^2 y}) - 2\cos x \cos y(b + 2c \sin x \sin y)\dot{x}\dot{y} + \\ &\quad + \cos^2 x \cos^2 y(b + 2c \sin x \sin y)^2 + 4c(d_1 \cos^2 y + d_2 \cos^2 x). \end{aligned}$$

When $c = 0$ this case reduces to a special version of that found by Bozis [17] and when $c = b = 0$ the system becomes separable and the integral degenerates into the product of two quadratic integrals. A slight variation of the parameters in (20) to be $k_1 = m_1 = 2$ changes trigonometric functions to hyperbolic (or exponential) functions, and thus giving a new system like the type of [18]. The analog of (20) gives a particle in the potential

$$V = b \sinh x \sinh y + c(2\cosh^2 x \cosh^2 y - \cosh^2 x - \cosh^2 y) + \frac{d_1}{\cosh^2 x} + \frac{d_2}{\cosh^2 y}.$$

In a similar way, one can obtain a mix of the two types.by taking $k_1 = -m_1 = 2$.

Some other variations lead to combinations of exponential and trigonometric or hyperbolic functions. For example, let $a = 1,\ k_0 = k_1 = 0,\ m_0 = 1,\ m_1 = 2.$ Under the coordinate transformation $p = e^x,\ \ q = \cosh y$ the Lagrangian takes the form

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V,\ V = ae^{2x} \cosh 2y + be^x \cosh y + ce^{-2x} + \frac{d}{\sinh^2 y}. \tag{21}$$

***Generalization of ystems of the Toda type.*** If in (18) we set $a = 1,\ k_0 = k_1 = m_0 = m_1 = 0$, the Lagrangian takes the form

$$L = \frac{1}{2}\lambda(\dot{x}^2 + \dot{y}^2) - \frac{1}{\lambda}[h_0 + ae^{-2x} + be^{-2y} + ce^{x+y} + de^{2(x+y)}], \tag{22}$$

where

$$\lambda = \alpha_0 + \alpha e^{-2x} + \beta e^{-2y} + \gamma e^{x+y} + \delta e^{2(x+y)} \tag{23}$$

and the integral may be written, after using the energy integral to eliminate $h$, as

$$\begin{aligned} I &= \lambda^4 \dot{x}^2 \dot{y}^2 + 2\lambda^2 [be^{-2y}\dot{x}^2 + ae^{-2x}\dot{y}^2 + (ce^{x+y} + de^{2x+2y})\dot{x}\dot{y}] + \\ &\quad + e^{2x+2y}(c + de^{x+y})^2 + 2d(be^{2x} + ae^{2y}) + 4abe^{-2x-2y}. \end{aligned} \tag{24}$$

***Application to rigid body dynamics.*** We now consider the general problem of motion of a rigid body about a fixed point under the action of a combination of conservative axisymmetric potential forces. The equations of motion for this problem can be written in the Euler–Poisson form:

$$
\begin{aligned}
A\dot{p} + (C-B)qr &= \gamma_2 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_2}, \\
B\dot{q} + (A-C)pr &= \gamma_3 \frac{\partial V}{\partial \gamma_1} - \gamma_1 \frac{\partial V}{\partial \gamma_3}, \\
C\dot{r} + (B-A)pq &= \gamma_1 \frac{\partial V}{\partial \gamma_2} - \gamma_2 \frac{\partial V}{\partial \gamma_1},
\end{aligned}
$$

$$
\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0, \tag{25}
$$

where $A, B, C$ are the principal moments of inertia, $p, q, r$ are the components of the angular velocity of the body and $\gamma_1, \gamma_2, \gamma_3$ are the components of the unit vector $\gamma$ fixed in space in the direction of the axis of symmetry of the force fields applied to the body, all being referred to the principal axes of inertia at the fixed point.

The system (25) admits three integrals:

$$
I_1 = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + V, \tag{26}
$$

$$
I_2 = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3, \tag{27}
$$

$$
I_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{28}
$$

Equations (25) admit an equivalent representation in the Lagrangian form (see e.g. [15]). After ignoring the cyclic angle of precession, around the axis of symmetry of the field on the zero level of the cyclic integral $I_2 = 0$, the Routhian of this mechanical system expressed in the other two Eulerian angles has the form

$$
R = \frac{1}{2} A \Big[ \dot{\theta}^2 + \frac{C \sin^2 \theta \dot{\varphi}^2}{A - (A-C)\cos^2 \theta} \Big] - V, \tag{29}
$$

where $\theta$ is the angle of nutation and $\varphi$ is the angle of proper rotation (about the axis of symmetry of the body). Comparing the structure of this Routhian function to that of the Lagrangian (18) and recalling the procedure followed in a similar situation in [15], we get convinced that they become identical only in the four case $A = B = 2C$.

In fact, setting $a = 1$, $k_1 = 1$, $k_0 = 0$, $m_1 = 2$, $m_0 = 1$, $\alpha_1 = \alpha_4 = = \alpha_5 = 0$, $\alpha_2 = \alpha_3$, and affecting the substitution $u = \dfrac{\cos^2 \theta}{2 \sin \theta}$, $v = \cos \varphi$ and renaming the remaining parameters, we get the Lagrangian

$$
L = \frac{1}{2} [\dot{\theta}^2 + \frac{\sin^2 \theta}{1 + \sin^2 \theta} \dot{\varphi}^2] - V, \tag{30}
$$

$$
V = 2C[a \sin \theta \sin \varphi + b \sin^2 \theta \cos(2\varphi) + \frac{\lambda}{\cos^2 \theta} + \delta \frac{1 + \sin^2 \theta}{\sin^2 \theta \cos^2 \varphi}] \tag{31}
$$

and the integral

$$
\begin{aligned}
I = & \frac{\sin^6\theta\dot{\varphi}^4}{(1+\sin^2\theta)^4} + \frac{\sin^2\theta\dot{\varphi}^2}{(1+\sin^2\theta)^2}[2a\sin\theta\sin\varphi + b(2-3\cos^2\theta) + \\
& + \frac{2\lambda\sin^2\theta}{\cos^2\theta} + \frac{4\delta}{\cos^2\varphi} + \sin^2\theta\dot{\theta}^2] + \\
& + \frac{2\dot{\theta}\dot{\varphi}\sin\theta\cos\theta\cos\varphi}{(1+\sin^2\theta)}(a\sin\theta + 2b\cos^2\theta\sin\varphi) + \dot{\theta}^2[b\cos 2\varphi + \frac{2\delta}{\cos^2\varphi}] + \\
& + \frac{4\delta^2}{\sin^2\theta\cos^4\varphi} + \delta[\frac{4\lambda}{\cos^2\theta\cos^2\varphi} + \frac{4a\sin\varphi}{\sin\theta\cos^2\varphi} + 2b(2\sin^2\theta + \frac{1-3\sin^2\theta}{\sin^2\theta\cos^2\varphi})] + \\
& + (\frac{1}{2}a^2 - 2ab\sin\theta\sin\varphi)(1-2\sin^2\theta\cos^2\varphi) + \\
& + \frac{1}{2}b^2\sin^2\theta[\sin^2\theta(\cos 4\varphi - 1) + 4] + 2\lambda b\tan^2\theta\cos 2\varphi. \qquad (32)
\end{aligned}
$$

To facilitate comparison with other results, we now express the last integrable case of rigid body dynamics in the Euler-Poisson variables, i.e. as a solution of the system (25), in the next

**Theorem 1.** *For $A = B = 2C$ and for the potential*

$$
V = 2C\left[a\gamma_1 + b(\gamma_1^2 - \gamma_2^2) + \frac{\lambda}{\gamma_3^2} + \delta\frac{2-\gamma_3^2}{\gamma_2^2}\right] \qquad (33)
$$

*equations (25) are integrable on the level $I_2 = 0$. The complementary integral has the form*

$$
\begin{aligned}
I \;=\; & (p^2 - q^2 - a\gamma_1 + b\gamma_3^2 - \frac{\lambda(\gamma_1^2 - \gamma_2^2)}{\gamma_3^2})^2 + (2pq - a\gamma_2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2})^2 + \\
& + \frac{\delta}{\gamma_2^2}\left[2(p^2+q^2)\gamma_3^2 - 2a\gamma_1\gamma_3^2 - 2\lambda\gamma_1^2 + 2b + \frac{\delta\gamma_3^4}{\gamma_2^2}\right]. \qquad (34)
\end{aligned}
$$

This case is new. For comparison we provide a table of presently known integrable potentials related to the type (33), which admit a quartic integral under the condition $A = B = 2C$:

| Author- year | Potential |
|---|---|
| Kovalevskaya [1]1889 | $V_1 = a_1\gamma_1 + a_2\gamma_2$ |
| Chaplygin [19] 1903 | $V_2 = b_1(\gamma_1^2 - \gamma_2^2) + b_2\gamma_1\gamma_2$ |
| Goriatchev [20] 1916 | $V_3 = a\gamma_1 + a_2\gamma_2 + b(\gamma_1^2 - \gamma_2^2) + b_1\gamma_1\gamma_2 + \frac{\lambda}{\gamma_3^2}$ |
| Yehia [15] 2006 | $V_4 = b(\gamma_1^2 - \gamma_2^2) + \frac{\lambda}{\gamma_3^2} + \rho(\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6}) + (2-\gamma_3^2)(\frac{\nu}{\gamma_1^2} + \frac{\delta}{\gamma_2^2})$ <br> $V_5 = a\gamma_1 + \frac{\lambda}{\gamma_3^2} + \frac{\varepsilon}{\sqrt{\gamma_1^2+\gamma_2^2}} + \frac{(2-\gamma_3^2)}{\gamma_2^2}(\delta + \mu\frac{\gamma_1}{\sqrt{\gamma_1^2+\gamma_2^2}})$ |

In the potential (33) the parameter $a$ is present in Kovalevskaya's case (1889), $b$ in Chaplygin's case (1903), $\lambda$ in Goriatchev's case (1916) and $\delta$ in both cases announced in our work [15] (2006). The combination (33) is new.


1. *Kowalevski S.V.* Sur le probleme de la rotation d'un corps solide autour d'un point fixe // Acta Math.– 1889. – **12**.– P. 177–232.
2. *Hetarinta J.* Direct Methods for the search of the second Invariant // Phys. Rep., (Review Section of Physics Letters). – 1987.– **147**, No. 2. – P. 87–154.
3. *Karlovini M., Pucacco G., Rosquist K., Samuelsson L.* A unified treatment of quartic invarients at fixed and arbitrary integral // J. Math. Phys. – 2002. – **43**, № 8. – P. 4041–4059.
4. *Bolsinov A.V., Kozlov V.V., Fomenko A.T.* The Maupertuis principle and geodesic fows on a sphere that arise from integrable case of the dynamics of a rigid body // Uspekhi Mat. Nauk. – 1995. – **50**. – P. 3–32. = Russian Math. Surveys. – 1995. – **50**. – P. 473–501.
5. *Selivanova E.N.* New families of conservative systems on $S^2$ possessing an integral of fourth degree in momenta // Ann. Global Anal. Geom. – 1999. – **17**. – P. 201–219.
6. *Hadeler K.P., Selivanova E.N.* On the case of Kovalevskaya and new cases of integrable conservative systems on $S^2$ // Reg. Chaot. Dyn. – 1999. – **4, 3**. – P. 45–52.
7. *Dullin H.R., Matveev V.S.* A new integrable system on the sphere // Math. Res. Lett. – 2004. – **11**. – P. 10001–10008.
8. *Ranada M., Santander M.* Super-integrable systems on the two-dimensional sphere $S^2$ and the hyperbolic plane $H_2$ // J. Math. Phys. – 1999. – **40**. – P. 5026–5057.
9. *Ranada M., Santander M.* On harmonic oscillators on the two-dimensional sphere $S^2$ and the hyperbolic plane $H_2$ // Ibïd. – 2002. – **43**. – P. 431–451.
10. *Yehia H.M.* On the integrability of certain problems in particle and rigid body dynamics // J. Méch. Théor. Appl. – 1986. – **5**, No 1. – P. 55–71.
11. *Yehia H.M.* Two-dimensional conservative mechanical system with quartic second integral // Reg. Chaot. Dyn. – 2006, **11**. – P. 103–122.
12. *Yehia H.M.* Kovalevskaya's integrable case: Generalization and related new results // Ibïd. – 2003. – **8**. – P. 337–348.
13. *Yehia H.M.* Atlas Of Two-Dimensional Irreversible Conservative Lagrangian Mechanical Systems With A Second Quadratic Integral // J. Math. Phys. – 2007. – **48**. – 082902.
14. *Yehia H.M.* On certain two-dimensional conservative mechanical systems with cubic second integral // J. Phys. A: Math. Gen. – 2002. – **35**. – P. 9469–9487.
15. *Yehia H.M.* The Master integrable two-dimensional system with a quartic second integral // Ibïd. – 2006. – **39**. – P. 5807–5824.
16. *Birkhoff G.* Dynamical Systems. – Amer. Math. Soc. Colloq. publ. – 1927. – **9**.
17. *Bozis G.* Compatibility conditions for a non-quadratic integral of motion // Celest. Mech. – 1982. – **28**. – P. 367–80.
18. *Wojciechowski S.* On the integrability of the Calogero-Moser system in an external quartic potential and other many-body systems // Phys. Lett. A. – 1984. – **102**. – P. 85–88.
19. *Chaplygin S.A.* A new particular solution of the problem of motion of a solid in a liquid // Trudy Otdel. Phys. Nauk Obsh. Liub. Estestvozn. – 1903. – **11**, № 2. – P. 7–10.
20. *Goriatchev D.N.* New case of integrability of the Euler dynamical equations // Varshav. Univ. Izvest. – 1916.– No 3. – P. 1–13.


**Х.М. Яхья**

**Новые интегрируемые случаи уравнений динамики с интегралами четвертой степени**

Продолжены исследования, начатые автором в 1986 году, и посвященные изучению условий существования у лагранжевых систем первых интегралов четвертого порядка. Рассматриваемая система характеризуется 16 параметрами. Получена структура лагранжиана, для которой дифференциальные уравнения движения допускают решения, характеризующиеся первым полиномиальным интегралом четвертого порядка. Это позволило обобщить известные случаи интегрируемости Ковалевской, Чаплыгина и Горячева классической задачи о движении твердого тела, имеющего неподвижную точку.

**Ключевые слова:** *лагранжиан, решение, потенциальная функция.*

### Х.М. Яхья

### Нові інтегровні випадки рівнянь динаміки з інтегралами четвертого степеня

Продовжено дослідження, початі автором у 1986 році, і присвячені вивченню умов існування у лагранжевих систем перших інтегралів четвертого порядку. Розглядувана система характеризується 16 параметрами. Одержано структуру лагранжиана, для якої диференціальні рівняння руху припускають розв'язки, що характеризуються першим поліноміальним інтегралом четвертого порядку. Це дозволило узагальнити відомі випадки інтегровності Ковалевської, Чаплигіна і Горячева класичної задачі про рух твердого тіла, яке має нерухому точку.

**Ключові слова:** *лагранжиан, розв'язок, потенціальна функція.*

*Mansoura University, Mansoura, Egypt*
Получено 28.10.11
hyehia@mans.edu.eg