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Morse-Bott functions on manifolds with semi-free circle action

Let W^{2n} be a closed manifold of dimension ≥ 6 with semi-free circle having finitely many fixed points. We study S^1-invariant Morse-Bott functions on W^{2n}. The aim of this paper is to obtain exact values of minimal numbers of singular circles of some indexes of S^1-invariant Morse-Bott functions on W^{2n}.

Keywords: *Semi-free circle action, manifold, Morse-Bott function, Morse number.*

1. S^1-invariant Morse-Bott functions

Let W^{2n} be a closed smooth manifold. Suppose that W^{2n} admits a smooth semi-free circle action with finitely many fixed points. It is known that every isolated fixed point p of a semi-free S^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point p there exist an open invariant neighborhood U of p and a diffeomorphism h from U to an open unit disk D in $\mathbb{C}^n$ centered at origin such that h conjugates the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \dots, 1)$. We will use both complex, $(z_1, \dots, z_n)$, and real coordinates $(x_1, y_1, \dots, x_n, y_n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_i = x_i + \sqrt{-1}y_i$. The pair (U, h) will be called a **standard chart** at the point p. Let $f : W^{2n} \to \mathbb{R}$ be a smooth S^1-invariant function on the manifold W^{2n}. Denote by Σ_f the set of singular points of the function

f. It is clear that the set of isolated singular points $\Sigma_f(p_i) \subset \Sigma_f$ of f coincides with the set of fixed points W^{S^1}.

A **point** $p \in W^{S^1}$ **is nondegenerate** if the Hessian of the function f at p is nondegenerate. For a nondegenerate fixed point p there exist a standard chart (U, h) such that on U the function f is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_\lambda|^2 + |z_{\lambda+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of nondegenerate fixed point p is always even.

Denote by $\Sigma_f(S^1)$ the set singular points of the function f that are disconnected union of circles. These circles will be called **singular**.

A **circle** $s \in \Sigma f_(S^1)$ **is called nondegenerate** if there is an S^1-invariant neighborhood U of s on which S^1 acts freely and such that the point $\pi(s)$ is nondegenerate for the function

$$\pi_*(f) : U/S^1 \to \mathbb{R},$$

induced on U/S^1 by the natural map $\pi : U \to U/S^1$. An invariant version of Morse lemma says that there exist an S^1-invariant neighborhood U of the circle s and coordinates $(x_1, \ldots, x_{2n-1})$ on U/S^1 such that the function $\pi_*(f)$ has the following presentation:

$$\pi_*(f) = \pi_*(f(\pi(s))) - |x_1|^2 - \ldots - |x_\nu|^2 + |x_{\nu+1}|^2 + \ldots + |x_{2n-1}|^2.$$

By definition ν is the **index** of singular circle s.

Definition 1. *A smooth S^1-invariant function $f : W^{2n} \to \mathbb{R}$ on a manifold W^{2n} with a semi-free circle action which has isolated fixed points is called an S^1-invariant Morse-Bott function if each connected component of the singular set Σ_f is either a nondegenerate fixed point or a nondegenerate critical circle,* [3].

Definition 2. *Assume that W^{2n} is the closed manifold with a smooth semi-free circle action which has isolated fixed points $p_1, \ldots, p_k$. For any fixed point p_i there exists a standard chart (U_i, h_i) such that each U_i is diffeomorphic to the unit disk D^{2n} in*

$\mathbb{C}^n$ and that U_i are pairwise disjoint. Take any ***arbitrary integer*** *$\lambda_i = 0, 1, \ldots, n$ and define the following function on $f_i : U_i \to \mathbb{R}$ by*

$$f_i = f_i(p_i) - |z_1|^2 - \ldots - |z_{\lambda_i}|^2 + |z_{\lambda_i+1}|^2 + \ldots + |z_n|^2.$$

Theorem 1. *Every smooth semi-free circle action on a closed manifold with isolated fixed points $p_1, \ldots, p_k$ has an S^1-invariant Morse-Bott function f such that $f = f_i$ on U_i.*

Proof. From results of paper [2] it follows that functions f_i can be extended from U_i to $W^{2n} \setminus \bigcup U_i$. □

Theorem 2. *The number of fixed points of any smooth semi-free circle action on W^{2n} with isolated fixed points is always even and equal to the Euler characteristic, $\chi(W^{2n})$, of the manifold W^{2n}.*

Proof. By Theorem 1 we construct on U_1 the function

$$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2,$$

on U_j, $(j \geq 2)$ the function

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2$$

and extend such functions to S^1-invariant Morse-Bott function f on $W^{2n} \setminus \bigcup U_i$. Since the manifold CP^n is non-cobordant to zero it follows that the number of fixed points of any smooth semi-free circle action on W^{2n} with isolated fixed points is equal to the Euler characteristic $\chi(W^{2n}) = 2k$ of W^{2n}. □

Definition 3. *Let f be an S^1-invariant Morse-Bott function for smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$ on a closed manifold W^{2n}. Suppose that the index of a critical point p_i of f is λ_i.* ***The state of*** *f is the collection of numbers $\lambda_1, \lambda_2, \ldots, \lambda_{2k}$, which we will be denoted by $St_f(\lambda_i)$.*

Remark 1. *From Theorem 1 it follows that for every smooth semi-free circle action on a closed manifold W^{2n} with isolated fixed points $p_1, \ldots, p_{2k}$ and any collection numbers $\lambda_1, \lambda_2, \ldots, \lambda_{2k}$, such that $0 \leq \lambda_i \leq 2n$ there exists an S^1-invariant Morse-Bott functions*

*f on W^{2n} with state $St_f(\lambda_i)$. Such a collection of numbers will be denoted by $St(\lambda_i)$ and called a **state**.*

Definition 4. *Let W^{2n} be a closed smooth manifold with smooth semi-free circle action which has finitely many fixed points. The S^1-**equivariant Morse number** $\mathcal{M}^{\nu}_{S^1}(W^{2n}, St(\lambda_i))$ **of index** ν **of a state** $St(\lambda_i)$ of W^{2n} is the minimum number of singular circles of index ν taken over all S^1-invariant Morse-Bott functions on W^{2n} with state $St(\lambda_i)$.*

*The S^1-**equivariant Morse number** $\mathcal{M}^{\nu}_{S^1}(W^{2n})$ of index ν of W^{2n} is the minimum number of $\mathcal{M}^{\nu}_{S^1}(W^{2n}, St(\lambda_i))$ taken over all states.*

*The S^1-**equivariant Morse number** $\mathcal{M}_{S^1}(W^{2n}, St(\lambda_i))$ **of a state** $St(\lambda_i)$ is the minimum number of singular circles of all indices taken over all S^1-invariant Morse-Bott functions on W^{2n} with state $St(\lambda_i)$.*

*The S^1-**equivariant Morse number** $\mathcal{M}_{S^1}(W^{2n})$ of W^{2n} is the minimum number of $\mathcal{M}_{S^1}(W^{2n}, St(\lambda_i))$ taken over all states.*

There is an unsolved problem: for a manifold W^{2n} with a semi-free circle action which has finitely many fixed points **find exact values of the numbers** $\mathcal{M}^{\nu}_{S^1}(W^{2n}, St(\lambda_i))$**,** $\mathcal{M}^{\nu}_{S^1}(W^{2n})$**,** $\mathcal{M}_{S^1}(W^{2n}, St(\lambda_i))$**, and** $\mathcal{M}_{S^1}(W^{2n})$.

Definition 5. *An S^1-invariant Morse-Bott function f on the manifold W^{2n} with semi-free circle action which has finitely many fixed points is*

***minimal for index** ν **of a state** $St(\lambda_i)$ if the number of singular circles of f of index ν is equal to $\mathcal{M}^{\nu}_{S^1}(W^{2n}, St(\lambda_i))$;*

***minimal for index** ν if the number of singular circles of f of index ν is equal to $\mathcal{M}^{\nu}_{S^1}(W^{2n})$;*

***minimal for state** $St(\lambda_i)$ if the number of all singular circles of f is equal to $\mathcal{M}_{S^1}(W^{2n}, St(\lambda_i))$;*

***minimal** if the number of all singular circles of f is equal to $\mathcal{M}_{S^1}(W^{2n})$.*

Theorem 3. *Let W^{2n} $(2n > 5)$ be a closed smooth simply-connected manifold admits a smooth semi-free circle action with isolated fixed points $p_1, \dots, p_{2k}$. Then on the manifold W^{2n} for the state $St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})$ there exists a minimal (minimal for index ν) S^1-invariant Morse-Bott function g for the state*

$$St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})$$

and

$$\begin{aligned}
&\mathcal{M}_{S^1}(W^{2n}, St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})) = \\
&\quad = \sum_{i=1}^{n-1} \mu(H_i((W^{2n}/S^1) \setminus (p_{l+1} \cup \dots \cup p_{2k}), p_1, \dots, p_l, \mathbb{Z}) + \\
&+ \sum_{i=2}^{n-2} \mu(Tors(H_i((W^{2n}/S^1) \setminus (p_{l+1} \cup \dots \cup p_{2k}), p_1, \dots, p_l, \mathbb{Z}),
\end{aligned}$$

$$\begin{aligned}
&\Big(\mathcal{M}^{\nu}_{S^1}(W^{2n}, St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})) = \\
&\quad = \mu\big(H_\nu((W^{2n}/S^1) \setminus (p_{l+1} \cup \dots \cup p_{2k}), p_1, \dots, p_l, \mathbb{Z})\big) + \\
&+ \mu\big(Tors(H_{\nu-1}((W^{2n}/S^1) \setminus (p_{l+1} \cup \dots \cup p_{2k}), p_1, \dots, p_l, \mathbb{Z}))\big)\Big),
\end{aligned}$$

where $0 \le l \le 2k$ ($\mu(H)$ – minimal number of generators of group H).

Proof. Choose an invariant neighborhood U_i of the point p_i diffeomorphic to the unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Consider the manifold $V^{2n} = (W^{2n} \setminus U)/S^1$. It is clear that its boundary is a disconnected union of complex projective spaces

$$\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \dots \cup \mathbb{C}P_k^{2n-2}.$$

The set W^{2n}/S^1 is simply-connected. It is easy to see using van Kampen theorem that $(W^{2n} \setminus U)/S^1$ is simply-connected as well. From S. Smale's theorem [4] is follows that on $(W^{2n} \setminus U)/S^1$ there exists a minimal Morse function which we used to construct an S^1-invariant Morse-Bott function g for state

$$St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})$$

on the manifold W^{2n}. The values of

$$\mathcal{M}_{S^1}\big(W^{2n}, St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})\big)$$

and

$$\mathcal{M}^{\nu}_{S^1}\big(W^{2n}, St(\underbrace{0, \dots, 0}_{l}, \underbrace{2n, \dots, 2n}_{2k-l})\big)$$

follow from S. Smale's theorem and simple homology calculation. □

Remark 2. *Using diagrams technique,* [1]*, one can give estimates for equivariant Morse number for other states. This will be made in forthcoming paper.*

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