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Curvature forms and Curvature functions for 2-manifolds with boundary

We obrain that any 2-form and any smooth function on a given 2-manifold with boundary can be realized as the curvature form and the gaussian curvature function of some Riemmanian metric, respectively.

Keywords: *Curvature form,Curvature function, Gauss Bonnet formula, Manifold with boundary.*

1. Introduction

For 2-manifolds, possibly, with boundary the classical Gauss Bonnet formula asserts a relationship between the Euler characteristic of a manifold, its gaussian curvature, and the geodesic curvature of the boundary. This is the only known obstruction on a given 2-form on a manifold to be the curvature form of some Riemmanian metric. Nevertheless, it imposes a constraint on the sign of a function for being the curvature function of a metric. The problem of prescribing curvature forms on closed 2-manifolds was solved by Wallach and Warner [4]. They showed that the Gauss Bonnet formula is a necessary and sufficient condition on a 2-form to be a curvature form. Later, the problem of prescribing curvature functions has been studied by some authors and completely solved for closed manifold by Kazdan and Warner [2]. They proved that any smooth function which satisfies Gauss Bonnet sign condition, is the gaussian curvature of some Riemmannian metric.

In this paper we deal with 2-manifolds with boundary and the problems of prescribing curvature forms and curvature functions which were put to the author by Volodymyr Sharko. In contrast with the case when manifolds have nonempty boundary no obstruction on 2-forms and functions arises. It turns out that any 2-form and smooth function can be realized as the curvature form and curvature function of a metric respectively, this is a surprising phenomena.

2. Preliminaries and the main results

If we want to study manifolds with boundary we are often faced with the extension problems, we handle these problems by gluing manifolds together, providing desired extensions using the elementary techniques of differential topology. At first, we shall consider forms then, the same method will be used for functions.

Let M be a connected, compact and oriented 2-manifold with smooth boundary. Now, glue a 2-disk D^2 to M to get a 2-manifold without boundary $\widetilde{M}$, suitably oriented, joined together along boundaries. Now we shall have occasion to extend forms from M to the whole manifold, the existence of extension is an obvious corollary of the theorem 1.4 [3], that is, if ω_1 and ω_2 are given 2-forms on M and D^2 respectively, (here we just consider 2-forms but in general it is true for arbitrary forms) which are locally represented as $\omega_1 = f_{12}\, dx^1\Lambda\ dx^2$ and $\omega_2 = g_{12}\, dy^1\Lambda dy^2$ in collar neighborhoods of their boundaries then we can piece together functions f_{12} and g_{12} in bicollar neighborhood as the same as of the theorem 1.4 [3] and get a smooth function on $\widetilde{M}$ and hence a smooth 2-form $\widetilde{\omega}$ on $\widetilde{M}$ whose restrictions to M and D^2 are ω_1 and ω_2 respectively.

Lemma 1. *Let ω be a given 2-form then for any arbitrary nonzero real number a there exists an extension $\bar{\omega}$ of ω to D^2 such that $\int_{D^2}\bar{\omega} = a$.*

Proof. Let $\widetilde{\omega}$ be an arbitrary extension such that $\int_{D^2}\widetilde{\omega} \neq 0$. We construct a 2-form $\bar{\omega}$ using bump function such that in an open neighborhood of the boundary coincides with $\widetilde{\omega}$ and $\int_{D^2}\bar{\omega} = a$. Let U be an open neighborhood of the boundary and V be an open neighborhood of the boundary of the disk $\widetilde{D^2}$ with the smaller radius contained in D^2. Let $f dx^1 \Lambda dx^2$ be a local representation of $\widetilde{\omega}$ and g be a bump function which equals to the identity in U and vanishes in V now, put $\widetilde{\widetilde{\omega}} = g\ \widetilde{\omega}$, $\int_{D^2}\widetilde{\widetilde{\omega}} = k \neq 0$ and $\int_U \widetilde{\widetilde{\omega}} = k_1, \int_\Omega \widetilde{\widetilde{\omega}} = k_2$ and $\int_{\widetilde{D^2}}\widetilde{\widetilde{\omega}} = k_3$ where Ω is a space between U and V. Now define a new function h which is equal to the identity in U and $\frac{a-k_1}{k_2+k_3}g$ elsewhere. Set $\bar{\omega} = h\widetilde{\widetilde{\omega}}$. (Notice that we always can choose neighborhoods and function g in order to $k_2 + k_3 \neq 0$). □

As an evident consequence of this lemma we have the following corollary.

Corollary 1. *For any* 2*-form* ω *on* M *there exists an extension* $\widetilde{\omega}$ *such that*

$$\int_{\widetilde{M}} \widetilde{\omega} = 2\pi\chi(\widetilde{M})$$

Theorem 1. *Let* M *be a connected, compact and oriented* 2*-manifold with smooth boundary then any* 2*-form* ω *on* M *is the curvature form of some Riemannian metric* g *on* M*.*

Proof. There exits an extension $\widetilde{\omega}$ of ω such that

$$\int_{\widetilde{M}} \widetilde{\omega} = 2\pi\chi(\widetilde{M})$$

by corollary 1, then employing the theorem of Wallach and Warner [4] for $\widetilde{\omega}$, we get a Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ which its restriction to M is an expected metric. □

Remark 1. *Note that in what, discussed and follows we just consider manifolds having only one boundary component, but, in general, when boundary consists of more than one component the* *theorems remain valid, we just need to glue D^2 to each component to get a closed manifold.*

Since, we integrate a function, not a 2-form, this fact leads us to proceed with the same approach, and expect the similar result for functions, however, we can ask a different question concerns prescribing gaussian and geodesic curvatures simultaneously, for example, in [1] the author applies the technique of solving the Neumann problem on a compact manifold with boundary to the problem of finding a metric, pointwise conformal to a given metric with prescribed gaussian curvature and with the prescribed geodesic curvature on the boundary when $\chi(M) \leq 0$. But here our approach is completely different, indeed our objective is to extend functions and transfer problems to a closed manifold to avoid difficulties on boundary. Fortunately, appropriate extensions always exist

Assume f is a smooth function defined on M the Whitney extension theorem assures that f can be extended so as to be smooth throughout $\widetilde{M}$.

Lemma 2. *Let f be a smooth function defined on M then there exists an extension $\widetilde{f}$ such that satisfies the sign condition.*

Proof. let $\bar{f}$ be an arbitrary extension which is not zero everywhere, suppose $\chi(M) > 0$ if there exists a point x_0 at which $f(x_0) > 0$ there is nothing to do otherwise multiply f to a function g, where

$$g = \begin{cases} 1, & \text{in an open neighborhood of the boundary,} \\ \text{negative}, & \text{at some point,} \end{cases}$$

fg is a desired extension. If $\chi(M) < 0$ we can modify the extension likewise. If $\chi(M) = 0$ and f does not vanish identically and does not change sign, it is strictly positive or negative thus we just need to multiply it to a function which is equal to the identity in an open neighborhood of the boundary of D^2 and changes sign elsewhere. □

Theorem 2. *Let M be a compact, connected and oriented* 2-*manifold with smooth boundary then any smooth function f is the gaussian curvature of some Riemmanian metric on M.*

Proof. By lemma 2 there exists an extension $\widetilde{f}$ of f such that satisfies the sign condition then by the theorem of Kazdan and warner [2] there exists a metric on $\widetilde{M}$ possesses $\widetilde{f}$ as its gaussian curvature, restriction of the metric to M is an expected metric. □

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