Збірник праць
Ін-ту математики НАН України
2009, т.6, №2, 276-287

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Geodesic Webs and PDE Systems of Euler Equations

We find necessary and sufficient conditions for the foliation defined by level sets of a function $f(x_1, \dots, x_n)$ to be totally geodesic in a torsion-free connection and apply them to find the conditions for d-webs of hypersurfaces to be geodesic, and in the case of flat connections, for d-webs $(d \geq n+1)$ of hypersurfaces to be hyperplanar webs. These conditions are systems of generalized Euler equations, and for flat connections we give an explicit construction of their solutions.

Keywords: *web, hyperplanar web, Euler equation, foliation, connection*

1. Introduction

In this paper we study necessary and sufficient conditions for the foliation defined by level sets of a function to be totally geodesic in a torsion-free connection on a manifold and find necessary and sufficient conditions for webs of hypersurfaces to be geodesic. These conditions has the form of a second-order PDE system for web functions. The system has an infinite pseudogroup of symmetries and the factorization of the system with respect to the pseudogroup leads us to a first-order PDE system. In the planar case (cf. [1]), the system coincides with the classical Euler equation and therefore can be solved in a constructive way. We provide a method to solve the system in arbitrary dimension and flat connection.

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2. Geodesic Foliations and Flex Equations

Let M^n be a smooth manifold of dimension n. Let vector fields $\partial_1, \ldots, \partial_n$ form a basis in the tangent bundle, and let $\omega^1, \ldots, \omega^n$ be the dual basis. Then

$$[\partial_i, \partial_j] = \sum_k c_{ij}^k \partial_k$$

for some functions $c_{ij}^k \in C^\infty(M)$, and

$$d\omega^k + \sum_{i<j} c_{ij}^k \omega^i \wedge \omega^j = 0.$$

Let ∇ be a linear connection in the tangent bundle, and let Γ_{ij}^k be the Christoffel symbols of second type. Then

$$\nabla_i(\partial_j) = \sum_k \Gamma_{ij}^k \partial_k,$$

where $\nabla_i \overset{\text{def}}{=} \nabla_{\partial_i}$, and

$$\nabla_i\left(\omega^k\right) = -\sum_j \Gamma_{ij}^k \omega^j.$$

In [1] we proved the following result.

Theorem 1. *The foliation defined by the level sets of a function $f(x_1, \ldots, x_n)$ is totally geodesic in a torsion-free connection ∇ if and only if the function f satisfies the following system of PDEs:*

$$\frac{\partial_i(f_i)}{f_i f_i} - \frac{\partial_i(f_j) + \partial_j(f_i)}{f_i f_j} + \frac{\partial_j(f_j)}{f_j f_j} =$$

$$= \sum_k \left(\Gamma_{ii}^k \frac{f_k}{f_i f_i} + \Gamma_{jj}^k \frac{f_k}{f_j f_j} - (\Gamma_{ij}^k + \Gamma_{ji}^k)\frac{f_k}{f_i f_j}\right) \tag{1}$$

for all $i < j, i, j = 1, \ldots, n$; here $f_i = \dfrac{\partial f}{\partial x_i}$.

We call such a system a *flex system.*

Note that conditions (1) can be used to obtain necessary and sufficient conditions for a d-web formed by the level sets of the functions $f_\alpha(x_1, \dots, x_n), \alpha = 1, \dots, d$, to be a *geodesic d-web,* i.e., to have the leaves of all its foliations to be totally geodesic: one should apply conditions (1) to the all web functions $f_\alpha, \alpha = 1, \dots, d$

2.1. **Geodesic Webs on Manifolds of Constant Curvature.** In what follows, we shall use the following definition.

Definition 1. *We call by* $(\text{Flex } f)_{ij}$ *the following function:*

$$(\text{Flex } f)_{ij} = f_j^2 f_{ii} - 2 f_i f_j f_{ij} + f_i^2 f_{jj},$$

where $i, j = 1, \dots, n,\ f_i = \dfrac{\partial f}{\partial x_i}$ *and* $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$.

It is easy to see that $(\text{Flex } f)_{ij} = (\text{Flex } f)_{ji}$, and $(\text{Flex } f)_{ii} = 0$.

Proposition 1. *Let* $(\mathbb{R}^n, g)$ *be a manifold of constant curvature with the metric tensor*

$$g = \frac{dx_1^2 + \cdots + dx_n^2}{\left(1 + \kappa\left(x_1^2 + \cdots + x_n^2\right)\right)^2},$$

where κ *is a constant. Then the level sets of a function*

$$f(x_1, \dots, x_n)$$

are geodesics of the metric g *if and only if the function* f *satisfies the following PDE system:*

$$(\text{Flex } f)_{ij} = \frac{2\kappa\left(f_i^2 + f_j^2\right)}{1 + \kappa\left(x_1^2 + \cdots + x_n^2\right)} \sum_k x_k f_k \tag{2}$$

for all i, j.

Proof. To prove formula (2), first note that the components of the metric tensor g are

$$g_{ii} = b^2,\ g_{ij} = 0, \quad i \neq j,$$

where

$$b = \frac{1}{1 + \kappa\left(x_1^2 + \cdots + x_n^2\right)}.$$

It follows that

$$g^{ii} = g_{ii}^{-1},\ g^{ij} = 0,\quad i \neq j.$$

We compute Γ^i_{jk} using the classical formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) \tag{3}$$

and get

$$\Gamma^k_{ii} = 2\kappa x_k b,\ k \neq i;\ \Gamma^i_{ii} = -2\kappa x_i b;\ \Gamma^k_{ij} = 0,\ i, j \neq k,\ i \neq j;$$
$$\Gamma^i_{ij} = -2\kappa x_j b,\ i \neq j;\ \Gamma^j_{ij} = -2\kappa x_i b,\quad i \neq j.$$

Substituting these values of Γ^i_{jk} into the right-hand side of formula (1), we get formula (2). □

Note that if $n = 2$, then PDE system (2) reduces to the single equation

$$\text{Flex } f = \frac{2\kappa\left(x_1 f_1 + x_2 f_2\right)\left(f_1^2 + f_2^2\right)}{1 + \kappa\left(x_1^2 + x_2^2\right)},$$

where Flex $f = (\text{Flex } f)_{12}$.

This formula coincides with the corresponding formula in [1].

We rewrite formula (2) as follows:

$$\frac{(\text{Flex } f)_{ij}}{f_i^2 + f_j^2} = 2\kappa b \sum_k x_k f_k. \tag{4}$$

The left-hand side of equation (4) does not depend on i and j. Thus we have

$$\frac{(\text{Flex } f)_{ij}}{f_i^2 + f_j^2} = \frac{(\text{Flex } f)_{kl}}{f_k^2 + f_l^2}$$

for any i, j, k, and l.

It follows that *if*

$$(\text{Flex } f)_{ij} = 0 \tag{5}$$

for some fixed i and j, then (5) *holds for any i and j.*

In other words, one has the following result.

Theorem 2. *Let W be a geodesic d-web on the manifold $(\mathbb{R}^n, g)$ given by web-functions $\{f^1, \dots, f^d\}$ such that $(f^a_k)^2 + (f^a_l)^2 \neq 0$ for all $a = 1, \dots, d$ and $k, l = 1, 2, \dots, n$. Assume that the intersection of W with the plane (x_{i_0}, x_{j_0}), for given i_0 and j_0, is a linear planar d-web. Then the intersection of W with arbitrary planes (x_i, x_j) are linear webs too.*

2.2. **Geodesic Webs on Hypersurfaces in $\mathbb{R}^n$.**

Proposition 2. *Let $(M, g) \subset \mathbb{R}^n$ be a hypersurface defined by an equation $x_n = u(x_1, \dots, x_{n-1})$ with the induced metric g and the Levi-Civita connection ∇. Then the foliation defined by the level sets of a function $f(x_1, \dots, x_{n-1})$ is totally geodesic in the connection ∇ if and only if the function f satisfies the following system of PDEs:*

$$(\text{Flex } f)_{ij} = \frac{u_1 f_1 + \cdots + u_{n-1} f_{n-1}}{1 + u_1^2 + \cdots + u_{n-1}^2}(f_j^2 u_{ii} - 2 f_i f_j u_{ij} + f_i^2 u_{jj}). \tag{6}$$

Proof. To prove formula (6), note that the metric induced by a surface $x_n = u(x_1, \dots, x_{n-1})$ is

$$g = ds^2 = \sum_{k=1}^{n-1} (1 + u_k^2) dx_k^2 + 2 \sum_{i,j=1 (i \neq j)}^{n-1} u_i u_j dx_i dx_j.$$

Thus the metric tensor g has the following matrix:

$$(g_{ij}) = \begin{pmatrix} 1 + u_1^2 & u_1 u_2 & \dots & u_1 u_{n-1} \\ u_2 u_1 & 1 + u_2^2 & \dots & u_2 u_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_{n-1} u_2 & \dots & 1 + u_{n-1}^2 \end{pmatrix},$$

and the inverse tensor g^{-1} has the matrix

$$(g^{ij}) = \frac{1}{1+\sum\limits_{k=1}^{n-1}(1+u_k^2)} \times$$

$$\times \begin{pmatrix} \sum\limits_{k=2}^{n-1}(1+u_k^2) & -u_1u_2 & \dots & -u_1u_{n-1} \\ -u_2u_1 & \sum\limits_{k=1 (k\neq 2)}^{n-1}(1+u_k^2) & \dots & -u_2u_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -u_{n-1}u_1 & -u_{n-1}u_2 & \dots & \sum\limits_{k=1}^{n-2}(1+u_k^2) \end{pmatrix}.$$

Computing Γ^i_{jk} by formula (3), we find that

$$\Gamma^k_{ij} = \frac{u_k u_{ij}}{1+\sum\limits_{k=1}^{n-1}(1+u_k^2)}.$$

Applying these formulas to the right-hand side of (1), we get formula (6). □

We rewrite equation (6) in the form

$$\frac{(\text{Flex } f)_{ij}}{f_j^2 u_{ii} - 2f_i f_j u_{ij} + f_i^2 u_{jj}} = \frac{u_1 f_1 + \dots + u_n f_n}{1 + u_1^2 + \dots + u_n^2}. \tag{7}$$

It follows that the left-hand side of (7) does not depend on i and j, i.e., we have

$$\frac{(\text{Flex } f)_{ij}}{f_j^2 u_{ii} - 2f_i f_j u_{ij} + f_i^2 u_{jj}} = \frac{(\text{Flex } f)_{kl}}{f_l^2 u_{kk} - 2f_k f_l u_{kl} + f_k^2 u_{ll}}$$

for any i, j, k and l. This means that *if*

$$(\text{Flex } f)_{ij} = 0$$

for some fixed i *and* j, *then*

$$(\text{Flex } f)_{kl} = 0$$

for any k *and* l.

In other words, we have a result similar to the result in Theorem 2.

Theorem 3. *Let* W *be a geodesic* d*-web on the hypersurface* (M, g) *given by web functions* $\{f^1, \ldots, f^d\}$ *such that*

$$\left(f_j^a\right)^2 u_{ii} - 2 f_i^a f_j^a u_{ij} + \left(f_i^a\right)^2 u_{jj} \neq 0,$$

for all $a = 1, \ldots, d$ *and* $k, l = 1, 2, \ldots, n$. *Assume that the intersection of* W *with the plane* (x_{i_0}, x_{j_0}), *for given* i_0 *and* j_0, *is a linear planar* d*-web. Then the intersection of* W *with arbitrary planes* (x_i, x_j) *are linear webs too.*

3. Hyperplanar Webs

In this section we consider hyperplanar geodesic webs in $\mathbb{R}^n$ endowed with a flat linear connection ∇.

In what follows, we shall use coordinates $x_1, \ldots, x_n$ in which the Christoffel symbols Γ^i_{jk} of ∇ vanish.

The following theorem gives us a criterion for a web of hypersurfaces to be hyperplanar.

Theorem 4. *Suppose that a* d*-web of hypersurfaces,* $d \geq n+1$, *is given locally by web functions* $f_\alpha(x_1, \ldots, x_n), \alpha = 1, ..., d$. *Then the web is hyperplanar if and only if the web functions satisfy the following PDE system:*

$$(\text{Flex } f)_{st} = 0, \tag{8}$$

for all $s < t = 1, \ldots, n$.

Proof. For the proof, one should apply Theorem 1 to all foliations of the web. □

In order to integrate the above PDEs system, we introduce the functions

$$A_s = \frac{f_s}{f_{s+1}}, \quad s = 1, \ldots, n-1,$$

and the vector fields

$$X_s = \frac{\partial}{\partial x_s} - A_s \frac{\partial}{\partial x_{s+1}}, \; s = 1, \ldots, n-1.$$

Then the system can be written as

$$X_s \left(A_t\right) = 0,$$

where $s, t = 1, \ldots, n-1$.

Note that

$$\left[X_s, X_t\right] = 0$$

if the function f is a solution of (8).

Hence, the vector fields $X_1, \ldots, X_{n-1}$ generate a completely integrable $(n-1)$-dimensional distribution, and the functions

$$A_1, \ldots, A_{n-1}$$

are the first integrals of this distribution.

Moreover, the definition of the functions A_s shows that

$$X_s(f) = 0, \; s = 1, \ldots, n-1,$$

also.

As a result, we get that

$$A_s = \Phi_s\left(f\right), \quad s = 1, \ldots, n-1,$$

for some functions Φ_s.

In these terms, we get the following system of equations for f:

$$\frac{\partial f}{\partial x_s} = \Phi_s\left(f\right) \frac{\partial f}{\partial x_{s+1}}, \quad s = 1, \ldots, n-1,$$

or

$$\frac{\partial f}{\partial x_s} = \Psi_s\left(f\right) \frac{\partial f}{\partial x_n}, \; s = 1, \ldots, n-1, \tag{9}$$

where $\Psi_{n-1} = \Phi_{n-1}$, and

$$\Psi_s = \Phi_{n-1} \cdots \Phi_s$$

for $s = 1, \dots, n-2$.

This system is a sequence of the Euler-type equations and therefore can be integrated. Keeping in mind that a solution of the single Euler-type equation

$$\frac{\partial f}{\partial x_s} = \Psi_s(f) \frac{\partial f}{\partial x_n}$$

is given by the implicit equation

$$f = u_0 \left(x_n + \Psi_s(f)\, x_s \right),$$

where $u_0(x_n)$ is an initial condition, when $x_s = 0$, and Ψ_s is an arbitrary nonvanishing function, we get solutions f of system (8) in the form:

$$f = u_0 \left(x_n + \Psi_{n-1}(f)\, x_{n-1} + \cdots + \Psi_1(f)\, x_1 \right),$$

where $u_0(x_n)$ is an initial condition, when

$$x_1 = \cdots = x_{n-1} = 0,$$

and Ψ_s are arbitrary nonvanishing functions.

Thus, we have proved the following result.

Theorem 5. *Web functions of hyperplanar webs have the form*

$$f = u_0 \left(x_n + \Psi_{n-1}(f)\, x_{n-1} + \cdots + \Psi_1(f)\, x_1 \right), \tag{10}$$

where $u_0(x_n)$ are initial conditions, when $x_1 = \cdots = x_{n-1} =$ 0, and Ψ_s are arbitrary nonvanishing functions.

Example 15. *Assume that $n = 3$,*

$$f_1(x_1, x_2, x_3) = x_1, \quad f_2(x_1, x_2, x_3) = x_2, \quad f_3(x_1, x_2, x_3) = x_3,$$

and take $u_0 = x_3$, $\Psi_1(f_4) = f_4^2$, $\Psi_2(f_4) = f_4$ in (10). Then we get the hyperplanar 4-web with the remaining web function

$$f_4 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1 x_3}}{2x_1}.$$

It follows that the level surfaces $f_4 = C$ of this function are defined by the equation

$$x_1(C^2 x_1 - C x_2 + x_3 + C) = 0,$$

i.e., they form a one-parameter family of 2*-planes*

$$C^2 x_1 - C x_2 + x_3 + C = 0.$$

Differentiating the last equation with respect to C and excluding C, we find that the envelope of this family is defined by the equation

$$(x_2)^2 - 4x_1 x_3 - 2x_2 + 1 = 0.$$

Therefore, the envelope is the second-degree cone.

Example 16. *Assume that $n = 3$,*

$$f_1(x_1, x_2, x_3) = x_1, \quad f_2(x_1, x_2, x_3) = x_2, \quad f_3(x_1, x_2, x_3) = x_3,$$

and take $u_0 = x_3, \Psi_1(f_4) = 1, \Psi_2(f_4) = f_4^2$ in (10)*. Then we get the linear* 4*-web with the remaining web function*

$$f_4 = \left(\frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2} \right)^2.$$

The level surfaces $f_4 = C^2$ of this function are defined by the equation

$$x_2(x_1 + C^2 x_2 + x_3 - C) = 0,$$

i.e., they form a one-parameter family of 2*-planes*

$$x_1 + C^2 x_2 + x_3 - C = 0.$$

Differentiating the last equation with respect to C and excluding C, we find that the envelope of this family is defined by the equation

$$4x_1 x_2 + 4x_2 x_3 - 1 = 0.$$

Therefore, the envelope is the hyperbolic cylinder.

In the next example no one foliation of a web W_3 coincides with a foliation of coordinate lines, i.e., all four web functions are unknown.

Example 17. *Assume that $n = 3$ and take*

Description 1.

(i) $u_{01} = x_3, \quad \Psi_1(f_1) = f_1^2, \quad \Psi_2(f_1) = f_1;$
(ii) $u_{02} = x_3, \quad \Psi_1(f_2) = 1, \quad \Psi_2(f_2) = f_2^2;$
(iii) $u_{03} = x_3^2, \quad \Psi_1(f_3) = f_3, \quad \Psi_2(f_3) = 1;$
(iv) $u_{04} = x_3, \quad \Psi_1(f_4) = \Psi_2(f_4) = f_4.$

in (10). *Then we get the linear* 4*-web with the web functions*

$$f_1 = \frac{x_2 - 1 \pm \sqrt{(x_2 - 1)^2 - 4x_1x_3}}{2x_1},$$
$$f_2 = \left(\frac{1 \pm \sqrt{1 - 4x_2(x_1 + x_3)}}{2x_2}\right)^2$$

(*see Examples* 15 *and* 16) *and*

$$f_3 = \left(\frac{1 \pm \sqrt{1 - 4x_1(x_2 + x_3)}}{2x_1}\right)^2,$$
$$f_4 = \frac{x_3}{1 - x_1 - x_2}.$$

It follows that the leaves of the foliation X_1 are tangent 2*-planes to the second-degree cone*

$$(x_2)^2 - 4x_1x_3 - 2x_2 + 1 = 0$$

(*cf. Example* 15 *and* 16), *the leaves of the foliation X_2 and X_3 are tangent* 2*-planes to the hyperbolic cylinders*

$$4x_1x_2 + 4x_2x_3 - 1 = 0 \text{ and } 4x_1x_2 + 4x_1x_3 - 1 = 0$$

(*cf. Example* 16), *and the leaves of the foliation X_4 are* 2*-planes of the one-parameter family of parallel* 2*-planes*

$$Cx_1 + Cx_2 + x_3 = 1,$$

where C is an arbitrary constant.

References

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