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DIRECT SPECTRAL PROBLEM FOR THE GENERALIZED JACOBI HERMITIAN MATRICES

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To the memory of A. Ya. Povzner

Abstract. In this article we will introduce and investigate some generalized Jacobi matrices. These matrices have three-diagonal block structure and they are Hermitian. We will give necessary and sufficient conditions for selfadjointness of the operator which is generated by the matrix of such a type, and consider its generalized eigenvector expansion.

1. Introduction

At first we will recall the direct spectral problem for the classical Jacobi matrix (see, e.g. [1, 3, 6]). In the classical theory, one considers the Hermitian Jacobi matrix

$$(1)\quad J = \begin{bmatrix} b_0 & c_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \quad b_n \in \mathbb{R}, \quad a_n = c_n > 0, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\},$$

on the space ℓ_2 of sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathbb{C}$.

This matrix, defined on finite sequences $f \in \ell_{\text{fin}}$, gives rise to an operator on ℓ_2, which is Hermitian with equal deficiency numbers and, therefore, has a selfadjoint extension on ℓ_2. Under some conditions imposed on J, e.g. $\sum_{n=0}^{\infty} \frac{1}{a_n} = \infty$, the closure $\widetilde{J}$ of J is selfadjoint.

The direct spectral problem, i.e., the eigenfunction expansion for $\widetilde{J}$ (or for some selfadjoint extension of J), is constructed in the following way (for simplicity we will assume that $\widetilde{J}$ is selfadjoint).

We introduce $\forall \lambda \in \mathbb{R}$ a sequence of polynomials,

$$P(\lambda) = \big(P_n(\lambda)\big)_{n=0}^{\infty},$$

as a solution of the equation

$$(2)\quad \begin{aligned} &JP(\lambda) = \lambda P(\lambda), \quad P_0(\lambda) = 1, \quad \text{i.e.,} \quad \forall n \in \mathbb{N}_0 \\ &a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \\ &P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1. \end{aligned}$$

This recurrence has a solution. It is constructed inductively, starting with $P_0(\lambda)$, which can be done, since $a_n > 0$ for all n.

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The sequence of polynomials $P(\lambda)$ is a generalized eigenvector for $\widetilde{J}$ with eigenvalue λ; we use some quasinuclear rigging of the space $H = \ell_2$,

$$H_- \supset H_0 \supset H_+, \quad P(\lambda) \in H_-. \tag{3}$$

The corresponding Fourier transformation $F =\widehat{}$ is given by

$$\ell_2 \supset \ell_{\text{fin}} \ni f = (f_n)_{n=0}^\infty \mapsto \widehat{f}(\lambda) = \sum_{n=0}^\infty f_n P_n(\lambda) \in L^2(\mathbb{R}, d\rho(\lambda)) =: L^2. \tag{4}$$

This mapping is a unitary operator (after taking the closure) between ℓ_2 and L^2. The image of $\widetilde{J}$ is the operator of multiplication by λ on the space L^2. The polynomials $P_n(\lambda)$ are orthonormal w.r.t. the spectral measure $d\rho(\lambda)$,

$$\int_{\mathbb{R}} P_j(\lambda) P_k(\lambda) d\rho(\lambda) = \delta_{j,k}, \quad j,k \in \mathbb{N}_0. \tag{5}$$

Note that (5) is a consequence of the Parseval equality that holds true for mapping (4),

$$\forall f, g \in \ell_{\text{fin}} \quad (f,g)_{\ell_2} = \int_{\mathbb{R}} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)} d\rho(\lambda). \tag{6}$$

In this paper we will deal with the following situation. The matrix under consideration has the same structure as in (1) but $a_i, b_i, c_i, i \in \mathbb{N}_0$, are matrices (with complex elements) of dimensions $(i+2)\times(i+1)$, $(i+1)\times(i+1)$, $(i+1)\times(i+2)$, respectively. We assume that this matrix is Hermitian. The detailed form of such matrix is given in Section 2. Also in this section, we formulate a criterion and a sufficient condition for selfadjointness of the operator generated by such a matrix on the space $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \cdots$. These results have analogs in the classical theory (see [3], Ch. 7). It is necessary to say that matrices of such type appear in [4], but they are normal matrices and are connected with a complex moment problem. Also truncated Jacobi matrices of similar structure appear in papers of Yuan Xu (see, e.g., [7]).

In Section 3 we will introduce an analog of the first order polynomials for the matrix under consideration. An essential difference in our case, as compared with the the classical situation, is that the matrices a_i, c_i are not invertible, so we have to assume that $\forall i \in \mathbb{N}_0$ rank $c_i = i+1$. It is natural that these matrices are not invertible, since they are not square. Using these polynomials we will construct generalized eigenvector expansion. It is necessary to note that because the a_i, c_i are not invertible, there is no a scalar spectral measure $d\rho(\lambda)$. The obtained measure is infinite dimensional matrix-valued measure that is similar to the one in the case of partial difference equations (compare with [2]).

2. Hermitian block Jacobi-type matrices and selfadjointness of the corresponding operators

Let us consider the complex Hilbert space

$$\mathbf{l}_2 = H_0 \oplus H_1 \oplus H_2 \oplus \cdots, \quad H_i = \mathbb{C}^{i+1}, \quad i \in \mathbb{N}_0, \tag{7}$$

of vectors $\mathbf{l}_2 \ni f = (f_n)_{n=0}^\infty$, where $f_n = (f_{n;j})_{j=0}^n \in H_n$; $f = \sum_{n=0}^\infty \sum_{j=0}^n f_{n;j} e_{n;j}$, (here $e_{n;j}, n = 0,1,\ldots, j = 0,1,\ldots,n$, are elements of the standard basis in $\mathbf{l}_2$) with the scalar product $(f,g)_{\mathbf{l}_2} = \sum_{n=0}^\infty (f_n, g_n)_{H_n}$; $f, g \in \mathbf{l}_2$. Consider the Hilbert space rigging

$$\mathbf{l} = (\mathbf{l}_{\text{fin}})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\text{fin}}, \tag{8}$$

where $\mathbf{l}_{\text{fin}}$ is the space of finite vectors, $\mathbf{l}$ is the space of arbitrary vectors, $\mathbf{l}_2(p)$ is the space of infinite vectors with the scalar product $(f,g)_{\mathbf{l}_2(p)} = \sum_{n=0}^\infty (f_n, g_n)_{H_n} p_n$; $f, g \in \mathbf{l}_2(p)$ (here $p = (p_n)_{n=0}^\infty, p_n > 0$, is a given weight). In what follows, $p_n \geq 1$ and $\sum_{n=0}^\infty p_n^{-1} < \infty$; therefore the embedding of the positive space $\mathbf{l}_2(p) \subset \mathbf{l}_2$ is quasinuclear. So, the rigging (8) is quasinuclear.

Let us consider, in the space (7), the Hermitian matrix $J = (J_{j,k})_{j,k=0}^{\infty}$ with operator-valued complex elements $J_{j,k}\colon H_k \to H_j$, $J_{j,k} = (J_{j,k;\alpha,\beta})_{\alpha=0\beta=0}^{j\quad k}$ of the following block Jacobi structure:

$$J = \begin{bmatrix} b_0 & c_0 & 0 & 0 & \dots \\ a_0 & b_1 & c_1 & 0 & \dots \\ 0 & a_1 & b_2 & c_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \text{where} \quad \begin{aligned} a_i &= J_{i+1,i} : H_i \to H_{i+1}, \\ b_i &= J_{i,i} : H_i \to H_i, \\ c_i &= J_{i,i+1} : H_{i+1} \to H_i. \end{aligned} \tag{9}$$

For the matrix J to be Hermitian, it is necessary and sufficient that $b_i = b_i^*, a_i = c_i^*$, where $*$ denotes the adjoint to the matrix. Also we suppose that $\forall i \in \mathbb{N}_0$ rank $c_i = i+1$.

Remark 1. Actually $b_i = b_i^* \Leftrightarrow (b_i u_i, v_i)_{H_i} = (u_i, b_i v_i)_{H_i} \forall u_i, v_i \in H_i$ and $a_i = c_i^* \Leftrightarrow (a_i \xi_i, \zeta_{i+1})_{H_{i+1}} = (\xi_i, c_i \zeta_{i+1})_{H_i} \forall \xi_i \in H_i, \zeta_{i+1} \in H_{i+1}, i \in \mathbb{N}_0$.

Let $u \in \mathbf{l}_2$. Then matrix J acts on u in following way:

$$(Ju)_j = a_{j-1} u_{j-1} + b_j u_j + c_j u_{j+1}, \quad \text{where} \quad u_{-1} = 0. \tag{10}$$

It is easy to show that $\forall k, l \in \mathbb{N}_0, k \leq l$, the following analogue of Green's formula takes place:

$$\begin{aligned} \sum_{j=k}^{l} \left[\left((Ju)_j, v_j\right)_{H_j} - \left(u_j, (Jv)_j\right)_{H_j} \right] &= \left[(c_l u_{l+1}, v_l)_{H_l} - (a_l u_l, v_{l+1})_{H_{l+1}} \right] \\ &\quad - \left[(c_{k-1} u_k, v_{k-1})_{H_{k-1}} - (a_{k-1} u_{k-1}, v_k)_{H_k} \right], \quad \forall u, v \in \mathbf{l}_2. \end{aligned} \tag{11}$$

Using rigging (8) and formula (11) we can construct, from the matrix J, an operator $\mathbf{J}$ that acts on $\mathbf{l}_2$, (see, e.g., a similar scheme of construction in [3], Ch. 7, § 1). The construction is following. Consider some operator J' on finite vectors in $\mathbf{l}_2$, that acts by formula (10), i.e., $(J'u)_j = (Ju)_j, u \in \mathbf{l}_{\text{fin}}$, and $u_{-1} = 0$. Using (11) we can conclude that J' is Hermitian. By $\mathbf{J}$ we denote the closure of the operator J'. It is easy to see that domain of the operator $\mathbf{J}^*$ consists of $v \in \mathbf{l}_2$ for which $Jv \in \mathbf{l}_2$.

Let us consider equation which gives possibility to find eigenvectors for operator $\mathbf{J}$.

$$(J\varphi(z))_j = a_{j-1}\varphi_{j-1}(z) + b_j \varphi_j(z) + c_j \varphi_{j+1}(z) = z\varphi_j(z), \quad z \in \mathbb{C}, \quad j \in \mathbb{N}_0, \tag{12}$$

where $\varphi \in \mathbf{l}$; $\varphi_{-1}(z) = 0$.

Remark 2. This equation can be considered as a recurrence relation for finding φ_{j+1} by using φ_j and φ_{j-1}. In Section 3 we will make a few assumptions. They guarantee that (12) is solvable.

Proposition 1. *The operator* $\mathbf{J}$ *is selfadjoint if and only if any non-zero solution of system (12) satisfies* $\sum_{j=0}^{\infty} \|\varphi_j(z)\|_{H_j}^2 = \infty$, *where* $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Let $z \in \mathbb{C}$, $Im\, z \neq 0$, be some fixed number. Consider the deficiency subspace $N_{\bar{z}}$ of the operator $\mathbf{J}$ orthogonal to $R(\mathbf{J} - \bar{z}E)$ (here $R(A)$ is the range of the operator A). It coincides with the subspace of solutions of the equation $\mathbf{J}^*\varphi = z\varphi$. Due to the construction of $\mathbf{J}^*$, we have $(\mathbf{J}^*\varphi(z))_j = (J\varphi(z))_j = z\varphi(z), \varphi_{-1}(z) = 0, \varphi \in D(\mathbf{J}^*)$. So, the dimension of $N_{\bar{z}}$ is not equal to zero if and only if $\sum_{j=0}^{\infty} \|\varphi_j(z)\|_{H_j}^2 < \infty$, where $\varphi(z)$ is some non-zero solution of (12). $\square$

Theorem 1. *Let the matrix* J *be such that* $\sum_{j=0}^{\infty}(\|a_j\|_{j;j+1} + \|c_j\|_{j+1;j})^{-1} = \infty$, *where* $\|\cdot\|_{k;\,l}$ *denotes the norm of a* $(l+1) \times (k+1)$*–matrix or the respective operator that acts from* H_k *to* H_l. *Then the operator* $\mathbf{J}$ *is selfadjoint.*

Proof. Let $z \in \mathbb{C}$, $Im\, z \neq 0$, be some fixed number. According to Proposition 1, it is sufficient to show that for any non-zero solution of (12), $\sum_{j=0}^{\infty} \|\varphi_j(z)\|_{H_j}^2 = \infty$. Consider some non-zero solution $\varphi(z) = (\varphi_0(z), \varphi_1(z), \ldots)$ of equation (12). Let $n_0 \in \mathbb{N}_0$ be the index of the first non-zero element of $\varphi(z)$, i.e., $H_{n_0} \ni \varphi_{n_0}(z) \neq \overline{0}$. Since $\forall j = 0, \ldots, n_0 - 1\, \varphi_j(z) \equiv \overline{0}$, it follows from (12) that $\varphi_{n_0}(z)$ does not depend on z, i.e., $\varphi_{n_0}(z) = \varphi_{n_0}$. Consider the identity

$$(z - \bar{z}) \sum_{i=0}^{j} \big(\varphi_i(z), \varphi_i(z)\big)_{H_i} = \sum_{i=0}^{j} \big[\big((J\varphi(z))_i, \varphi_i(z)\big)_{H_i} - \big(\varphi_i(z), (J\varphi(z))_i\big)_{H_i}\big].$$

Using formula (11) we have

$$(z - \bar{z}) \sum_{i=0}^{j} \big(\varphi_i(z), \varphi_i(z)\big)_{H_i} = \big(c_j \varphi_{j+1}(z), \varphi_j(z)\big)_{H_j} - \big(a_j \varphi_j(z), \varphi_{j+1}(z)\big)_{H_{j+1}}.$$

Let $j \geq n_0$. Then $c \|\varphi_{n_0}\|_{H_{n_0}}^2 \leq \big(\|c_j\|_{j+1;j} + \|a_j\|_{j;j+1} \big) \|\varphi_j(z)\|_{H_j} \|\varphi_{j+1}(z)\|_{H_{j+1}}$, where $c > 0$ is some constant. So,

$$\infty = \sum_{j=0}^{\infty} \big(\|c_j\|_{j+1;j} + \|a_j\|_{j;j+1} \big)^{-1} < \frac{1}{c} \frac{1}{\|\varphi_{n_0}\|_{H_{n_0}}^2} \sum_{j=0}^{\infty} \|\varphi_j(z)\|_{H_j} \|\varphi_{j+1}(z)\|_{H_{j+1}}$$
$$\leq \frac{1}{c} \frac{1}{\|\varphi_{n_0}\|_{H_{n_0}}^2} \sum_{j=0}^{\infty} \|\varphi_j(z)\|_{H_j}^2 .$$

□

In what follows, the operator $\mathbf{J}$ is assumed to be selfadjoint.

3. The direct spectral problem

Let us consider equation (12). Since eigenvalues of a selfadjoint operator are real, we have the following:

$$\begin{aligned}
&b_0\varphi_0 + c_0\varphi_1(\lambda) = \lambda\varphi_0,\\
&a_0\varphi_0 + b_1\varphi_1(\lambda) + c_1\varphi_2(\lambda) = \lambda\varphi_1(\lambda),\\
&\ldots\\
&a_{j-1}\varphi_{j-1}(\lambda) + b_j\varphi_j(\lambda) + c_j\varphi_{j+1}(\lambda) = \lambda\varphi_j(\lambda),\\
&\ldots,\\
&j = 1, 2, 3, \ldots, \quad \lambda \in \mathbb{R}.
\end{aligned} \tag{13}$$

As one can see, none of the equation in system (13) defines $\varphi_{j+1}(\lambda)$ in unique way from $\varphi_j(\lambda)$ and $\varphi_{j-1}(\lambda)$. Let us find a solution in following manner.

Assume that rank $c_j = j+1$ and the matrix $c_j = \{c_{j;\alpha,\beta}\}_{\alpha=0}^{j}{}_{\beta=0}^{j+1}$ is as follows: for the matrix $\widetilde{c}_j := \{c_{j;\alpha,\beta}\}_{\alpha=0}^{j}{}_{\beta=1}^{j+1}$ there exists an inverse, $\widetilde{c}_j^{-1}$, and $c_{j;\cdot,0}$ is linearly dependent on the columns of $\widetilde{c}_j$, where $c_{j;\cdot,i} = \{c_{j;\alpha,i}\}_{\alpha=0}^{j}$, $i = 0, 1, \ldots, j+1$, is the ith column of matrix c_j. We will use the notation of such type for other matrices. Also, let $\varphi_{j;0}(\lambda) = \varphi_{j;0} \in \mathbb{C}$, $j = 0, 1, \ldots$, be some complex constants, where $\varphi_{0,0} := \varphi_0$, and all of the above indicated $\varphi_{j;0}$ generate a vector of "boundary conditions", $\varphi_{\cdot;0} := \begin{pmatrix} \varphi_0 \\ \varphi_{1;0} \\ \varphi_{2;0} \\ \cdots \end{pmatrix}$.

Remark 3. We make this assumption for simplification of the subsequent construction. In fact, it is easy to make the following calculations in the same way in the general case.

Since $\operatorname{rank} c_j = j+1$, there are $j+1$ columns of this matrix which make a linearly independent system and then one more column that is a linear combination of the independent ones.

In accordance with the assumption, system (13) can be rewritten in the following way:

$$\begin{aligned}
&\varphi_{1;1}(\lambda) = \frac{1}{c_{0;0,1}}(\lambda - b_0)\varphi_0 - \frac{c_{0;0,0}}{c_{0;0,1}}\varphi_{1;0},\\
&\begin{pmatrix}\varphi_{j+1;1}(\lambda)\\ \dots \\ \varphi_{j+1;j+1}(\lambda)\end{pmatrix} = \widetilde{c}_j^{\,-1}(\lambda I_j - b_j)\varphi_j(\lambda) - \widetilde{c}_j^{\,-1}a_{j-1}\varphi_{j-1}(\lambda)\\
&\qquad\qquad - \widetilde{c}_j^{\,-1}c_{j;\cdot,0}\varphi_{j+1;0}, \quad j\in\mathbb{N},
\end{aligned}\tag{14}$$

where I_j is identity matrix on H_j.

Denote by $P_{\alpha;(j,\cdot)}(\lambda) := \big(P_{\alpha;(j;k)}(\lambda)\big)_{k=0}^{j}$, $\alpha = 0,1,\dots$, $j = 0,1,\dots$, a solution of equation (14) satisfying the boundary conditions $P_{\alpha;(j,0)} = \delta_{j,\alpha}, j = 0,1,\dots$ (the vector that has its αth coordinate equal to 1 and all the others are zeros will be denoted by δ_α). Let us consider a procedure of its construction. For a better understanding we will describe this procedure in the simplest case and then give it in general.

0^0. Let $\varphi_{\cdot,0} = \delta_0$. Then the corresponding solutions $\varphi_{j,k}(\lambda), j = 0,1,\dots, k = 0,1,\dots,j$, of (14) are given by the polynomials $P_{0;(j,k)}(\lambda), j = 0,1,\dots, k = 0,1,\dots,j$. We obtain a set of complex-valued polynomials, which is convenient to represent in following order:

$$\begin{matrix}
P_{0;(0,0)}(\lambda) & P_{0;(1,0)}(\lambda) & P_{0;(2,0)}(\lambda) & P_{0;(3,0)}(\lambda) & \dots & P_{0;(j,0)}(\lambda) & \dots\\
 & P_{0;(1,1)}(\lambda) & P_{0;(2,1)}(\lambda) & P_{0;(3,1)}(\lambda) & \dots & P_{0;(j,1)}(\lambda) & \dots\\
 & & P_{0;(2,2)}(\lambda) & P_{0;(3,2)}(\lambda) & \dots & P_{0;(j,2)}(\lambda) & \dots\\
 & & & P_{0;(3,3)}(\lambda) & \dots & P_{0;(j,3)}(\lambda) & \dots\\
 & & & & & \dots & \\
 & & & & & P_{0;(j,j)}(\lambda) & \dots
\end{matrix}\tag{15}$$

The construction of these polynomials is as following:
a) Since $\varphi_{\cdot,0} = \delta_0$, we have $P_{0;(0,0)}(\lambda) = 1$.
b) From the boundary conditions, $P_{0;(1,0)}(\lambda) = 0$. From (14), we have $P_{0;(1,1)}(\lambda) = \frac{1}{c_{0;0,1}}(\lambda - b_0)P_{0;(0,0)}(\lambda) - \frac{c_{0;0,0}}{c_{0;0,1}}P_{0;(1,0)}(\lambda) = \frac{1}{c_{0;0,1}}(\lambda - b_0)$.
c) Since $P_{0;(2,0)}(\lambda) = 0$, from (14) we obtain $\begin{pmatrix}P_{0;(2,1)}(\lambda)\\ P_{0;(2,2)}(\lambda)\end{pmatrix} = \widetilde{c}_1^{\,-1}(\lambda I_1 - b_1)\begin{pmatrix}0\\ P_{0;(1,1)}(\lambda)\end{pmatrix} - \widetilde{c}_1^{\,-1}a_0$. Thus

$$\begin{aligned}
P_{0;(2,1)}(\lambda) &= \frac{1}{c_{0;0,1}}\left(\widetilde{c}_1^{\,-1}(\lambda I_1 - b_1)\right)_{0,1}(\lambda - b_0) - (\widetilde{c}_1^{\,-1}a_0)_0,\\
P_{0;(2,2)}(\lambda) &= \frac{1}{c_{0;0,1}}\left(\widetilde{c}_1^{\,-1}(\lambda I_1 - b_1)\right)_{1,1}(\lambda - b_0) - (\widetilde{c}_1^{\,-1}a_0)_1,
\end{aligned}$$

where $\left(\widetilde{c}_1^{\,-1}(\lambda I_1 - b_1)\right)_{m,n}$ is the matrix element in the mth row and nth column, and $(\widetilde{c}_1^{\,-1}a_0)_m$ is the mth coordinate of this vector.
d) Since $\varphi_{\cdot,0} = \delta_0$, we have $P_{0;(j,0)}(\lambda) = 0$, $j = 2,3,\dots$, and from (14) it follows that

$$\begin{pmatrix}P_{0;(j,1)}(\lambda)\\ P_{0;(j,2)}(\lambda)\\ \dots\\ P_{0;(j,j)}(\lambda)\end{pmatrix} = \widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\begin{pmatrix}0\\ P_{0;(j-1,1)}(\lambda)\\ \dots\\ P_{0;(j-1,j-1)}(\lambda)\end{pmatrix} - \widetilde{c}_{j-1}^{\,-1}a_{j-2}\begin{pmatrix}0\\ P_{0;(j-2,1)}(\lambda)\\ \dots\\ P_{0;(j-2,j-2)}(\lambda)\end{pmatrix}.$$

Then $\forall k=1,2,\dots,j$,

$$\begin{aligned}P_{0;(j,k)}(\lambda)&=(\widetilde{c}_{j-1}^{-1}(\lambda I_{j-1}-b_{j-1}))_{k-1,\cdot}P_{0;(j-1,\cdot)}(\lambda)-(\widetilde{c}_{j-1}^{-1}a_{j-2})_{k-1,\cdot}P_{0;(j-2,\cdot)}(\lambda)\\&=\sum_{i=0}^{j-1}(\widetilde{c}_{j-1}^{-1}(\lambda I_{j-1}-b_{j-1}))_{k-1,i}P_{0;(j-1,i)}(\lambda)-\sum_{i=0}^{j-2}(\widetilde{c}_{j-1}^{-1}a_{j-2})_{k-1,i}P_{0;(j-2,i)}(\lambda).\end{aligned}$$

α^0. Let us consider the general situation. Let $\varphi_{\cdot;0}=\delta_\alpha$, where $\alpha=1,2,\dots$ is some fixed number. Then the solutions of (14) with this boundary conditions gives polynomials $P_{\alpha;(j,k)}(\lambda)$, i.e., $\varphi_{j;k}(\lambda)=P_{\alpha;(j,k)}(\lambda)$, $j=0,1,\dots,k=0,1,\dots,j$. It is convenient to represent these polynomials in the order similar to (15),

$$\begin{array}{ccccccc}P_{\alpha;(0,0)}(\lambda) & \dots & P_{\alpha;(\alpha-1,0)}(\lambda) & P_{\alpha;(\alpha,0)}(\lambda) & \dots & P_{\alpha;(j,0)}(\lambda) & \dots\\ & & P_{\alpha;(\alpha-1,1)}(\lambda) & P_{\alpha;(\alpha,1)}(\lambda) & \dots & P_{\alpha;(j,1)}(\lambda) & \dots\\ & & \dots & \dots & & \dots & \dots\\ & & P_{\alpha;(\alpha-1,\alpha-1)}(\lambda) & P_{\alpha;(\alpha,\alpha-1)}(\lambda) & \dots & P_{\alpha;(j,\alpha-1)}(\lambda) & \dots\\ & & & P_{\alpha;(\alpha,\alpha)}(\lambda) & \dots & P_{\alpha;(j,\alpha)}(\lambda) & \dots\\ & & & & & \dots & \\ & & & & & P_{\alpha;(j,j)}(\lambda) & \dots\end{array}\tag{16}$$

Because $\varphi_{\cdot;0}=\delta_\alpha$,

$$P_{\alpha;(j,0)}(\lambda)=\varphi_{j;0}=\delta_{j,\alpha},\quad j=0,1,\dots\tag{17}$$

a) From (14) and (17) we obtain $P_{\alpha;(j,k)}(\lambda)=0$ if $j=0,1,\dots,\alpha-1,k=0,1,\dots,j$.
b) From (17) $P_{\alpha;(\alpha,0)}(\lambda)=1$, and according to (14) we have

$$\begin{pmatrix}P_{\alpha;(\alpha,1)}(\lambda)\\P_{\alpha;(\alpha,2)}(\lambda)\\\dots\\P_{\alpha;(\alpha,\alpha)}(\lambda)\end{pmatrix}=-\widetilde{c}_{\alpha-1}^{-1}c_{\alpha-1;\cdot,0}P_{\alpha;(\alpha,0)}(\lambda).$$

Then $\forall k=1,2,\dots,\alpha$ we get

$$P_{\alpha;(\alpha,k)}(\lambda)=-(\widetilde{c}_{\alpha-1}^{-1})_{k-1,\cdot}c_{\alpha-1;\cdot,0}=-\sum_{i=0}^{\alpha}(\widetilde{c}_{\alpha-1}^{-1})_{k-1,i}c_{\alpha-1;i,0}.\tag{18}$$

c) From (17) it follows that $P_{\alpha;(\alpha+1,0)}(\lambda)=0$. So (14) gives

$$\begin{pmatrix}P_{\alpha;(\alpha+1,1)}(\lambda)\\\dots\\P_{\alpha;(\alpha+1,\alpha+1)}(\lambda)\end{pmatrix}=\widetilde{c}_{\alpha}^{-1}(\lambda I_\alpha-b_\alpha)P_{\alpha;(\alpha,\cdot)}(\lambda),$$

and then

$$\begin{aligned}P_{\alpha;(\alpha+1,k)}(\lambda)&=\big(\widetilde{c}_{\alpha}^{-1}(\lambda I_\alpha-b_\alpha)\big)_{k-1,\cdot}P_{\alpha;(\alpha,\cdot)}(\lambda)\\&=\sum_{i=0}^{\alpha+1}\big(\widetilde{c}_{\alpha}^{-1}(\lambda I_\alpha-b_\alpha)\big)_{k-1,i}P_{\alpha;(\alpha,i)}(\lambda),\quad k=1,2,\dots,\alpha+1.\end{aligned}\tag{19}$$

d) Let us consider the general situation. From (14) and (17) $\forall j=\alpha+1,\alpha+2,\dots$ we get

$$\begin{pmatrix}P_{\alpha;(j,1)}(\lambda)\\\dots\\P_{\alpha;(j,j)}(\lambda)\end{pmatrix}=\widetilde{c}_{j-1}^{-1}(\lambda I_{j-1}-b_{j-1})P_{\alpha;(j-1,\cdot)}(\lambda)-\widetilde{c}_{j-1}^{-1}a_{j-2}P_{\alpha;(j-2,\cdot)}(\lambda)$$

or

$$
\begin{aligned}
P_{\alpha;(j,k)}(\lambda) &= \left(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\right)_{k-1,\cdot} P_{\alpha;(j-1,\cdot)}(\lambda) - \left(\widetilde{c}_{j-1}^{\,-1} a_{j-2}\right)_{k-1,\cdot} P_{\alpha;(j-2,\cdot)}(\lambda) \\
&= \sum_{i=0}^{j-1} \left(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\right)_{k-1,i} P_{\alpha;(j-1,i)}(\lambda) \\
&\quad - \sum_{i=0}^{j-2} \left(\widetilde{c}_{j-1}^{\,-1} a_{j-2}\right)_{k-1,i} P_{\alpha;(j-2,i)}(\lambda), \quad k = 1, 2, \ldots, j.
\end{aligned} \tag{20}
$$

So, from 0^0 and α^0 (mainly, from (17)–(20)) we can summarize the following. Let $\alpha = 0, 1, \ldots$ Then for any fixed α and $\forall k = 1, 2, \ldots, j$ we get

$$
P_{\alpha;(j,k)}(\lambda) = \begin{cases}
0, & j = 0, \ldots, \alpha - 1, \\
-(\widetilde{c}_{j-1}^{\,-1})_{k-1,\cdot}\, c_{j-1;\cdot,0}, & j = \alpha, \\
\left(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\right)_{k-1,\cdot} P_{\alpha;(j-1,\cdot)}(\lambda), & j = \alpha + 1, \\
\left(\widetilde{c}_{j-1}^{\,-1}(\lambda I_{j-1} - b_{j-1})\right)_{k-1,\cdot} P_{\alpha;(j-1,\cdot)}(\lambda) - & \\
\quad - \left(\widetilde{c}_{j-1}^{\,-1} a_{j-2}\right)_{k-1,\cdot} P_{\alpha;(j-2,\cdot)}(\lambda), & j = \alpha + 2, \alpha + 3, \ldots
\end{cases} \tag{21}
$$

Using formulae (21) we can define and calculate step by step all the polynomials $P_{\alpha;(j,k)}(\lambda)$ for all permitted α, j, k.

Also, using formulae (21) it is easy to calculate the degree of the constructed polynomials. Now we will give the distribution of the degrees of $P_{\alpha;(j,k)}(\lambda)$ for any fixed α according to the order which was considered in (16).

$$
\begin{array}{c|cccccccccc}
 & 0 & 1 & \ldots & \alpha-1 & \alpha & \alpha+1 & \alpha+2 & \ldots & j \to \\
\hline
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & \\
1 & & 0 & \ldots & 0 & [1] & [\lambda] & [\lambda^2] & \ldots & \\
\vdots & & & & \vdots & \vdots & \vdots & \vdots & & \\
\alpha-1 & & & & 0 & [1] & [\lambda] & [\lambda^2] & \ldots & \\
\alpha & & & & & [1] & [\lambda] & [\lambda^2] & \ldots & \\
\alpha+1 & & & & & & [\lambda] & [\lambda^2] & \ldots & \\
\alpha+2 & & & & & & & [\lambda^2] & \ldots & \\
\vdots & & & & & & & & & \\
k \downarrow & & & & & & & & &
\end{array} \tag{22}
$$

Here $[\lambda^m]$, $m = 0, 1, \ldots$, means that there is a polynomial of degree m in the respective place. It is easy to see that the degree of the polynomial $P_{\alpha;(j,k)}(\lambda)$ equals $j - \alpha$, if $j \geq \alpha$.

Now we come back to solving equation (14) with some boundary conditions $\varphi_{\cdot,0}$. The procedure will be given in a constructive way by using induction. Consider the first equation in (14). Then the solution can be written in the form

$$
\begin{aligned}
\begin{pmatrix} \varphi_{1;0} \\ \varphi_{1;1}(\lambda) \end{pmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{1}{c_{0;0,1}}(\lambda - b_0) & -\frac{c_{0;0,0}}{c_{0;0,1}} \end{bmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_{1;0} \end{pmatrix} \\
&= \begin{pmatrix} P_{0;(1,0)}(\lambda) & P_{1;(1,0)}(\lambda) \\ P_{0;(1,1)}(\lambda) & P_{1;(1,1)}(\lambda) \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_{1;0} \end{pmatrix} = \varphi_0 P_{0;(1,\cdot)}(\lambda) + \varphi_{1;0} P_{1;(1,\cdot)}(\lambda).
\end{aligned} \tag{23}
$$

So, $\varphi_{1;1}(\lambda) = \varphi_0 P_{0;(1,1)}(\lambda) + \varphi_{1;0} P_{1;(1,1)}(\lambda)$.

Also we consider the construction of $\varphi_2(\lambda)$. It will be useful for understanding the procedure to solve (14) in general. From (14) and (23) we have

$$
\begin{aligned}
\begin{pmatrix}\varphi_{2;1}(\lambda)\\ \varphi_{2;2}(\lambda)\end{pmatrix} &= \widetilde{c}_1^{-1}(\lambda I_1 - b_1)\big(\varphi_0 P_{0;(1,\cdot)}(\lambda) + \varphi_{1;0}P_{1;(1,\cdot)}(\lambda)\big) - \widetilde{c}_1^{-1}a_0\varphi_0\\
&- \widetilde{c}_1^{-1}c_{1;\cdot,0}\varphi_{2;0}\\
&= \varphi_0\big(\widetilde{c}_1^{-1}(\lambda I_1 - b_1)P_{0;(1,\cdot)}(\lambda) - \widetilde{c}_1^{-1}a_0\big) + \varphi_{1;0}\big(\widetilde{c}_1^{-1}(\lambda I_1 - b_1)P_{1;(1,\cdot)}(\lambda)\big)\\
&+ \varphi_{2;0}\big(-\widetilde{c}_1^{-1}c_{1;\cdot,0}\big)\\
&= \varphi_0\begin{pmatrix}(\widetilde{c}_1^{-1}(\lambda I_1 - b_1))_{0,\cdot}P_{0;(1,\cdot)}(\lambda) - (\widetilde{c}_1^{-1})_{0,\cdot}a_0\\ (\widetilde{c}_1^{-1}(\lambda I_1 - b_1))_{1,\cdot}P_{0;(1,\cdot)}(\lambda) - (\widetilde{c}_1^{-1})_{1,\cdot}a_0\end{pmatrix}\\
&+ \varphi_{1;0}\begin{pmatrix}(\widetilde{c}_1^{-1}(\lambda I_1 - b_1))_{0,\cdot}P_{1;(1,\cdot)}(\lambda)\\ (\widetilde{c}_1^{-1}(\lambda I_1 - b_1))_{1,\cdot}P_{1;(1,\cdot)}(\lambda)\end{pmatrix} + \varphi_{2;0}\begin{pmatrix}-(\widetilde{c}_1^{-1})_{0,\cdot}c_{1;\cdot,0}\\ -(\widetilde{c}_1^{-1})_{1,\cdot}c_{1;\cdot,0}\end{pmatrix}.
\end{aligned}
\tag{24}
$$

Then from (21) and (24) we obtain

$$
\begin{aligned}
&\varphi_2(\lambda) = \varphi_0 P_{0;(2,\cdot)}(\lambda) + \varphi_{1;0}P_{1;(2,\cdot)}(\lambda) + \varphi_{2;0}P_{2;(2,\cdot)}(\lambda) = \sum_{\alpha=0}^{2}\varphi_{\alpha;0}P_{\alpha;(2,\cdot)}(\lambda) \quad \text{or}\\
&\varphi_{2;k}(\lambda) = \sum_{\alpha=0}^{2}\varphi_{\alpha;0}P_{\alpha;(2,k)}(\lambda), \quad k = 0,1,2.
\end{aligned}
\tag{25}
$$

Consider the general situation by using induction. Let us suppose that, for some fixed $j \in \mathbb{N}$,

$$
\varphi_m(\lambda) = \sum_{\alpha=0}^{m}\varphi_{\alpha;0}P_{\alpha;(m,\cdot)}(\lambda), \quad m = j-1, j.
\tag{26}
$$

From (14) and (26) we obtain

$$
\begin{aligned}
\begin{pmatrix}\varphi_{j+1;1}(\lambda)\\ \cdots\\ \varphi_{j+1;j+1}(\lambda)\end{pmatrix} &= \widetilde{c}_j^{-1}(\lambda I_j - b_j)\sum_{\alpha=0}^{j}\varphi_{\alpha;0}P_{\alpha;(j,\cdot)}(\lambda)\\
&- \widetilde{c}_j^{-1}a_{j-1}\sum_{\alpha=0}^{j-1}\varphi_{\alpha;0}P_{\alpha;(j-1,\cdot)}(\lambda) - \widetilde{c}_j^{-1}c_{j;\cdot,0}\varphi_{j+1;0}\\
&= \sum_{\alpha=0}^{j-1}\varphi_{\alpha;0}\left(\widetilde{c}_j^{-1}(\lambda I_j - b_j)P_{\alpha;(j,\cdot)}(\lambda) - \widetilde{c}_j^{-1}a_{j-1}P_{\alpha;(j-1,\cdot)}(\lambda)\right)\\
&+ \varphi_{j;0}\widetilde{c}_j^{-1}(\lambda I_j - b_j)P_{j;(j,\cdot)}(\lambda) + \varphi_{j+1;0}(-\widetilde{c}_j^{-1}c_{j;\cdot,0}).
\end{aligned}
\tag{27}
$$

So, from (21) we get that $\varphi_{j+1}(\lambda)$ has the form (26) for $m = j+1$.

Thus, for $\varphi_j(\lambda)$, the following formulae take place in the vector and coordinate forms:

$$
\begin{aligned}
&\varphi_j(\lambda) = \sum_{\alpha=0}^{j}\varphi_{\alpha;0}P_{\alpha;(j,\cdot)}(\lambda),\\
&\varphi_{j;k}(\lambda) = \sum_{\alpha=0}^{j}\varphi_{\alpha;0}P_{\alpha;(j,k)}(\lambda), \quad j = 0,1,\dots, \quad k = 0,1,\dots,j.
\end{aligned}
\tag{28}
$$

So, from our calculations we obtain the following Theorem.

Theorem 2. *All solutions of equation* (13) *can be represented in the form* (28)*, where* $P_{\alpha;(j,k)}(\lambda)$ *are polynomials that can be calculated by recursion formulas* (21) *with the initial conditions* $P_{\alpha;(j,0)}(\lambda) = \delta_{j,\alpha}$, $j, \alpha \in \mathbb{N}_0$.

Now we will use the result from [3], Ch. 5, and [5], Ch. 15, about the generalized eigenvector expansion for a selfadjoint operator connected with the chain (8) in a standard way. For our operator $\mathbf{J}$ we have the representation

$$\mathbf{J}f = \int_{\mathbb{R}} \lambda\Phi(\lambda)\, d\sigma(\lambda) f, \quad f \in \mathbf{l}_2(p), \tag{29}$$

where $d\sigma(\lambda)$ is a spectral measure, $\Phi(\lambda)\colon \mathbf{l}_2(p) \to \mathbf{l}_2(p^{-1})$ is a generalized projection operator and $\Phi(\lambda)$ is positive-definite kernel, i.e., $\forall f \in \mathbf{l}_2(p)\,(\Phi(\lambda)f, f)_{\mathbf{l}_2} \geq 0$. For all $f, g \in \mathbf{l}_2(p)$ we have the Parseval equality

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} (\Phi(\lambda)f, g)_{\mathbf{l}_2} d\sigma(\lambda). \tag{30}$$

Let us denote by π_n the operator of orthogonal projection on $H_n, n \in \mathbb{N}_0$, in $\mathbf{l}_2$. Hence $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously in the space $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$.

Let us consider the operator matrix $(\Phi_{j,k}(\lambda))_{j,k=0}^{\infty}$, where

$$\Phi_{j,k}(\lambda) = \pi_j \Phi(\lambda)\pi_k : \mathbf{l}_2 \to H_j \quad (\text{or, in fact,} \quad H_k \to H_j). \tag{31}$$

Remark 4. It is necessary to say that $\Phi_{j,k}(\lambda)$ is a $(j+1)\times(k+1)$-matrix which stands at the jth block of rows and the kth block of columns. Enumeration of the blocks starts with 0 and is carried in such a way that the ith block of rows (columns) consists of $i+1$ rows (columns). So $\Phi(\lambda)$ has the form

$$\left(\begin{array}{c|cc|c} * & * & * & \dots \\ \hline * & * & * & \dots \\ * & * & * & \dots \\ \hline \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

where $*$ denotes elements of the matrix.

The Parseval identity (30) can be rewritten as follows: $\forall f, g \in \mathbf{l}_{\text{fin}}$

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \sum_{j=0}^{\infty} (\pi_j \Phi(\lambda) f, \pi_j g)_{H_j} d\sigma(\lambda) = \int_{\mathbb{R}} \sum_{j,k=0}^{\infty} (\Phi_{j,k}(\lambda) f_k, g_j)_{H_j} d\sigma(\lambda). \tag{32}$$

Lemma 1. *For every fixed $j, k \in \mathbb{N}_0$, elements of the matrix* (31)*, $\Phi_{j,k}(\lambda)\colon H_k \to H_j$, have the following representation:*

(33)
$$\Phi_{j,k;l,m}(\lambda) = \sum_{\alpha=0}^{j}\sum_{\beta=0}^{k} \Phi_{\alpha,\beta;0,0}(\lambda)\overline{P_{\beta;(k,m)}(\lambda)}P_{\alpha;(j,l)}(\lambda), \quad l = 0,\dots,j, \quad m = 0,\dots,k.$$

Proof. Since $\Phi(\lambda)$ is a projection onto generalized eigenvectors of the selfadjoint operator $\mathbf{J}$ with the corresponding generalized eigenvalue λ, we see that the vector $\varphi(\lambda) = (\varphi_j(\lambda)_{j=0}^{\infty})$ such that $\varphi_{j;l}(\lambda) = \Phi_{j,k;l,m}(\lambda), l = 0,\dots,j$, is a solution of (13) for any fixed $k, m, k \in \mathbb{N}_0, m = 0,\dots,k$. Indeed, since $\varphi(\lambda) = \Phi(\lambda)e_{k;m}$, $k = 0, 1, \dots, m = 0, \dots, k$, we have $0 = \big(\varphi(\lambda), (J-\lambda I)f\big)_{\mathbf{l}_2}, f \in \mathbf{l}_{\text{fin}}, f_{-1} = 0$. Using Green's formula (11) we obtain that $0 = \big((J-\lambda I)\varphi(\lambda), f\big)_{\mathbf{l}_2} - a_{-1}\varphi_{-1}\overline{f_0}$. Since f is an arbitrary vector from $\mathbf{l}_{\text{fin}}$, we have $\big((J-\lambda I)\varphi(\lambda)\big)_j = 0, j = 1, 2, \dots$, and $(b_0-\lambda)\varphi_0 + c_0\varphi_1(\lambda) = 0$. Therefore, $\varphi(\lambda) \in \mathbf{l}_2(p^{-1})$ exists as a usual solution of the difference equation $J\varphi = \lambda\varphi$.

Consider a solution of (13) in form (28) with the same initial conditions. Then $\varphi_j(\lambda) = \sum_{\alpha=0}^{j} \varphi_{\alpha;0} P_{\alpha;(j,\cdot)}(\lambda)$, where $\varphi_0 = \Phi_{0,k;0,m}(\lambda), \varphi_{1;0} = \Phi_{1,k;0,m}(\lambda), \dots, \varphi_{j;0} = \Phi_{j,k;0,m}(\lambda)$.

Using Theorem 2 we see that

$$(34)\quad \Phi_{j,k;l,m}(\lambda)=\sum_{\alpha=0}^{j}\Phi_{\alpha,k;0,m}(\lambda)P_{\alpha;(j,l)}(\lambda);\quad j,k\in\mathbb{N}_0,\quad l=0,\dots,j,\quad m=0,\dots,k.$$

Since the operator $\mathbf{J}$ is selfadjoint, the operator $\Phi(\lambda)\colon \mathbf{l}_2(p)\to\mathbf{l}_2(p^{-1})$ is formally selfadjoint on $\mathbf{l}_2$. So, $\Phi_{j,k}(\lambda)=(\Phi_{k,j}(\lambda))^*$, $j,k\in\mathbb{N}_0$; therefore $\Phi_{j,k;l,m}(\lambda)=\overline{\Phi_{k,j;m,l}(\lambda)}$; $j,k\in\mathbb{N}_0, l=0,\dots,j, m=0,\dots,k$. Thus $\forall i,k\in\mathbb{N}_0$

$$(35)\quad \begin{aligned}\Phi_{i,k;0,m}(\lambda)=\overline{\Phi_{k,i;m,0}(\lambda)}&=\sum_{\beta=0}^{k}\overline{\Phi_{\beta,i;0,0}(\lambda)P_{\beta;(k,m)}(\lambda)}\\ &=\sum_{\beta=0}^{k}\Phi_{i,\beta;0,0}(\lambda)\overline{P_{\beta;(k,m)}(\lambda)};\quad m=0,\dots,k.\end{aligned}$$

Substituting (35) into (34) we obtain (33). □

It will be essential for us to rewrite the Parseval identities (30),(32) in the form which involves the polynomials $P_{\alpha;(j,k)}(\lambda)$ introduced above.

Using Parseval equality (32) and representation (33) we get: $\forall f,g\in\mathbf{l}_{\text{fin}}$

$$\begin{aligned}(f,g)_{\mathbf{l}_2}&=\int_{\mathbb{R}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{j}\sum_{m=0}^{k}\Phi_{j,k;l,m}(\lambda)f_{k;m}\overline{g_{j;l}}\,d\sigma(\lambda)\\ &=\int_{\mathbb{R}}\sum_{j,k=0}^{\infty}\sum_{l=0}^{j}\sum_{m=0}^{k}\sum_{\alpha=0}^{j}\sum_{\beta=0}^{k}\Phi_{\alpha,\beta;0,0}(\lambda)\overline{P_{\beta;(k,m)}(\lambda)}P_{\alpha;(j,l)}(\lambda)f_{k;m}\overline{g_{j;l}}\,d\sigma(\lambda)\\ &=\int_{\mathbb{R}}\sum_{j,k=0}^{\infty}\sum_{\alpha=0}^{j}\sum_{\beta=0}^{k}\Phi_{\alpha,\beta;0,0}(\lambda)\big(f_k,P_{\beta;(k,\cdot)}(\lambda)\big)_{H_k}\overline{\big(g_j,P_{\alpha;(j,\cdot)}(\lambda)\big)}_{H_j}d\sigma(\lambda).\end{aligned}$$

In the last expression the range of the indexes (j,α) is $j=0,\dots,\infty;\alpha=0,\dots,j$. We can also get all points in a given range by making $\alpha=0,\dots,\infty; j=\alpha,\dots,\infty$. The same situation takes place for (k,β). So, after changing the summing order in the last expression we obtain: $\forall f,g\in\mathbf{l}_{\text{fin}}$

$$(36)\quad (f,g)_{\mathbf{l}_2}=\int_{\mathbb{R}}\sum_{\alpha,\beta=0}^{\infty}\Phi_{\alpha,\beta;0,0}(\lambda)\sum_{k=\beta}^{\infty}\big(f_k,P_{\beta;(k,\cdot)}(\lambda)\big)_{H_k}\sum_{j=\alpha}^{\infty}\overline{\big(g_j,P_{\alpha;(j,\cdot)}(\lambda)\big)}_{H_j}d\sigma(\lambda).$$

Since $\Phi(\lambda)\geq 0$ and $\Phi_{\alpha,\beta;0,0}(\lambda)=(\Phi(\lambda)e_{\beta;0},e_{\alpha;0})_{\mathbf{l}_2}$, $\alpha,\beta=0,1,\dots$, it is easy to see that $\big(\Phi_{\alpha,\beta;0,0}(\lambda)\big)_{\alpha,\beta=0}^{\infty}$ is a positive-definite matrix.

Let us construct the matrix spectral measure $\Sigma(\cdot)$ by the formula

$$(37)\qquad d\Sigma(\lambda)=\begin{pmatrix}\Phi_{0,0;0,0}(\lambda) & \Phi_{0,1;0,0}(\lambda) & \dots\\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,1;0,0}(\lambda) & \dots\\ \vdots & \vdots & \ddots\end{pmatrix}d\sigma(\lambda)=\big(\Phi_{\alpha,\beta;0,0}(\lambda)\,d\sigma(\lambda)\big)_{\alpha,\beta=0}^{\infty}.$$

Consider the space of finite vectors $u(\lambda)=\big(u_0(\lambda),u_1(\lambda),\dots\big),\lambda\in\mathbb{R}$, ($u_i(\cdot), i=0,1,\dots,$ are complex-valued functions of real variable) with the scalar product

$$(u(\lambda),v(\lambda))_{L^2(\mathbb{R},d\Sigma(\lambda))}=\int_{\mathbb{R}}(d\Sigma(\lambda)u(\lambda),v(\lambda))_{\ell_2}.$$

Let us introduce the Hilbert space $L^2(\mathbb{R},d\Sigma(\lambda))$ as a completion of the given space of finite vectors with respect to the scalar product $(\cdot,\cdot)_{L^2(\mathbb{R},d\Sigma(\lambda))}$. Also, introduce for $f\in\mathbf{l}_{\text{fin}}$

a Fourier transform $\widehat{f}(\lambda) = \begin{pmatrix} \widehat{f}_0(\lambda) \\ \widehat{f}_1(\lambda) \\ \dots \end{pmatrix} \in L^2(\mathbb{R}, d\Sigma(\lambda))$ by the formula

$$\widehat{f}_\beta(\lambda) = \sum_{k=\beta}^{\infty} \big(f_k, P_{\beta;(k,\cdot)}(\lambda)\big)_{H_k}, \quad \beta \in \mathbb{N}_0. \tag{38}$$

According to (36) and (38), we have the following: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(f, g)_{\mathbf{l}_2} = (\widehat{f}(\lambda), \widehat{g}(\lambda))_{L^2(\mathbb{R}, d\Sigma(\lambda))}. \tag{39}$$

If we consider the vectors $f, g \in \mathbf{l}_{\mathrm{fin}}$ of the form $f = e_{N;\xi}, g = e_{M;\zeta}$ then the Parseval identity (36) gives: $\forall N, M \in \mathbb{N}_0, \xi = 0, \dots, N, \zeta = 0, \dots, M$

$$\begin{aligned} \delta_{N,M}\delta_{\xi,\zeta} &= \int_{\mathbb{R}} \sum_{\beta=0}^{N} \sum_{\alpha=0}^{M} \Phi_{\alpha,\beta;0,0}(\lambda)\overline{P_{\beta;(N,\xi)}(\lambda)} P_{\alpha;(M,\zeta)}(\lambda)\, d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \left(d\Sigma(\lambda) \begin{pmatrix} \overline{P_{0;(N,\xi)}(\lambda)} \\ \dots \\ \overline{P_{N;(N,\xi)}(\lambda)} \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} \overline{P_{0;(M,\zeta)}(\lambda)} \\ \dots \\ \overline{P_{M;(M,\zeta)}(\lambda)} \\ 0 \\ \dots \end{pmatrix} \right)_{\ell_2}. \end{aligned} \tag{40}$$

It is easy to find elements of the matrix J in terms of the polynomials $P_{\alpha;(j,k)}(\lambda)$. The representations (29), (30) and (31) give: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$\begin{aligned} (Jf, g)_{\mathbf{l}_2} = (\mathbf{J}f, g)_{\mathbf{l}_2} &= \int_{\mathbb{R}} \lambda (\Phi(\lambda) f, g)_{\mathbf{l}_2} d\sigma(\lambda) = \int_{\mathbb{R}} \lambda \sum_{j,k=0}^{\infty} (\Phi_{j,k}(\lambda) f_k, g_j)_{H_j} d\sigma(\lambda) \\ &= \big(\lambda \widehat{f}(\lambda), \widehat{g}(\lambda)\big)_{L^2(\mathbb{R}, d\Sigma(\lambda))}. \end{aligned} \tag{41}$$

The last equality in (41) is obtained from representation (33), by changing the order of summation (similar to (36)) and using the definition of the Fourier transform. If we consider in (41) $f = f_{(k;m)} = e_{k;m}, g = g_{(j;l)} = e_{j;l}$, we will get: $\forall j, k \in \mathbb{N}_0$ and $l = 0, \dots, j, m = 0, \dots, k$

$$\begin{aligned} J_{j,k;l,m} = (Jf_{(k;m)}, g_{(j;l)})_{\mathbf{l}_2} &= \int_{\mathbb{R}} \sum_{\beta=0}^{k} \sum_{\alpha=0}^{j} \lambda \Phi_{\alpha,\beta;0,0}(\lambda)\overline{P_{\beta;(k,m)}(\lambda)} P_{\alpha;(j,l)}(\lambda)\, d\sigma(\lambda) \\ &= \int_{\mathbb{R}} \lambda \left(d\Sigma(\lambda) \begin{pmatrix} \overline{P_{0;(k,m)}(\lambda)} \\ \dots \\ \overline{P_{k;(k,m)}(\lambda)} \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} \overline{P_{0;(j,l)}(\lambda)} \\ \dots \\ \overline{P_{j;(j,l)}(\lambda)} \\ 0 \\ \dots \end{pmatrix} \right)_{\ell_2}. \end{aligned} \tag{42}$$

Using the above-mentioned results we can formulate the following Theorem.

Theorem 3. *Let, in accordance with representation* (29), $\Phi(\cdot)$ *and* $\sigma(\cdot)$ *be the generalized projection operator and the spectral measure of the operator* $\mathbf{J}$. *Construct the spectral matrix* $d\Sigma(\cdot)$ *using* (37). *Then orthogonality relations* (40) *take place, and one has a generalized eigenvector expansion due to* (38), (39). *The operator* $\mathbf{J}$ *can be reconstructed from formulas* (41), (42).

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