INTEGRAL REPRESENTATIONS FOR SPECTRAL FUNCTIONS OF SOME NONSELF-ADJOINT JACOBI MATRICES

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Abstract. We study a Jacobi matrix \( J \) with complex numbers \( a_n, n \in \mathbb{Z}_+ \), in the main diagonal such that \( r_0 \leq \text{Im} a_n \leq r_1, r_0, r_1 \in \mathbb{R} \). We obtain an integral representation for the (generalized) spectral function of the matrix \( J \). The method of our study is similar to Marchenko’s method for nonself-adjoint differential operators.

1. Introduction

The main object of our present investigation will be a three-diagonal semi-infinite complex number matrix of the following form:

\[
J = \begin{pmatrix}
a_0 & b_0 & 0 & 0 & \cdots \\
b_0 & a_1 & b_1 & 0 & \cdots \\
0 & b_1 & a_2 & b_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where \( b_n > 0 \), and

\[
a_n \in \mathbb{C}: r_0 \leq \text{Im} a_n \leq r_1,
\]

for some \( r_0, r_1 \in \mathbb{R}, n \in \mathbb{Z}_+ \).

Thus, in the case \( r_0 = r_1 = 0 \) we obtain the classical Jacobi matrix. The spectral theory of Jacobi matrices is classic, see [1], [2], [3]. For the Jacobi matrix \( J \), there is a corresponding non-decreasing function \( \sigma(x), x \in \mathbb{R} \), which is called a spectral function. The procedure of a construction of \( \sigma(x) \) provides a solution of the direct spectral problem for \( J \). The inverse spectral problem is to reconstruct \( J \) from \( \sigma \). The corresponding procedure is well-known and simple.

Recently, we have introduced a notion of a spectral function for some nonself-adjoint semi-infinite banded matrices, see [4], [5]. The spectral function is a bilinear (that means linear with respect to the both arguments) functional \( \sigma(u, v), u, v \in \mathbb{P} \), defined on a set of complex polynomials \( \mathbb{P} \). We will use methods which were applied by Marchenko to some nonself-adjoint Sturm-Liouville operators (see [6]) and obtain an integral representation for the spectral function \( \sigma(u, v) \) of the matrix \( J \) from (1).

Notations. As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \) the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively. By \( \mathbb{P} \) we denote the set of all polynomials with complex coefficients. By \( l^2 \) we denote a space of vectors \( x = (x_0, x_1, x_2, \ldots) \), \( x_n \in \mathbb{C}, n \in \mathbb{Z}_+ \), such that \( \|x\| := (\sum_{n=0}^{\infty} |x_n|^2)^{\frac{1}{2}} < \infty \). By \( l^2_{\text{fin}} \) we denote a subset of \( l^2 \) which consists of finite vectors, i.e., vectors \( x = (x_0, x_1, x_2, \ldots) \), \( x_n \in \mathbb{C}, n \in \mathbb{Z}_+ \), with only a finite number of nonzero elements \( x_n \).

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2. Polynomials of the first and of the second kinds

Let \( J \) be the semi-infinite matrix from (1), (2). Consider the following difference equations:

\[(3) \quad a_0 y_0 + b_0 y_1 = \lambda y_0,\]

\[(4) \quad b_{n-1} y_{n-1} + a_n y_n + b_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N},\]

where \( y_n \) are unknowns and \( \lambda \) is a complex parameter.

By \( P_n(\lambda), \quad n \in \mathbb{Z}_+ \), we denote a solution of (3), (4) with the initial condition \( P_0 = 1 \).

Polynomials \( P_n(\lambda) \) we will call polynomials of the first kind. Denote by \( Q_n(\lambda), \quad n \in \mathbb{Z}_+ \), a solution of (4) with the initial conditions \( Q_0 = 0, Q_1 = \frac{1}{b_0} \). Polynomials \( Q_n(\lambda) \) we will call polynomials of the second kind.

If we write relation (4) for \( P_n \) and then multiply it by \( Q_n \), we will get

\[(5) \quad b_{n-1} P_{n-1} Q_n + a_n P_n Q_n + b_n P_{n+1} Q_n = \lambda P_n Q_n, \quad n \in \mathbb{N}.\]

In a similar manner we will get

\[(6) \quad b_{n-1} Q_{n-1} P_n + a_n Q_n P_n + b_n Q_{n+1} P_n = \lambda Q_n P_n, \quad n \in \mathbb{N}.\]

Subtract (6) from (5) to get

\[(7) \quad b_{n-1} (P_{n-1} Q_n - P_n Q_{n-1}) = b_n (P_n Q_{n+1} - P_{n+1} Q_n), \quad n \in \mathbb{N}.\]

Using the initial conditions we obtain

\[(8) \quad P_{n-1}(\lambda) Q_n(\lambda) - P_n(\lambda) Q_{n-1}(\lambda) = \frac{1}{b_{n-1}}, \quad n \in \mathbb{N}.\]

We will use relation (7) in the sequel.

**Proposition 1.** Let \( y_n = y_n(\lambda), \quad n \in \mathbb{Z}_+ \), be an arbitrary solution of difference equation (4). The following relation holds true:

\[(9) \quad \sum_{j=1}^{n-1} (\text{Im} a_j - \text{Im} \lambda) |y_j(\lambda)|^2 = b_{n-1} \text{Im}(y_{n-1}(\lambda)|y_n(\lambda)|^2) - b_0 \text{Im}(y_0(\lambda)|y_1(\lambda)|^2), \quad n = 2, 3, \ldots\]

**Proof.** Set \( \tilde{a}_n = \tilde{a}_n(\lambda) = a_n - \lambda, \quad n \in \mathbb{N}, \) and rewrite relation (4) in the following form:

\[(10) \quad b_{n-1} y_{n-1} + \tilde{a}_n y_n + b_n y_{n+1} = 0, \quad n \in \mathbb{N}.\]

Apply the complex conjugation to the both sides of (9) to get

\[(11) \quad b_{n-1} \bar{y}_{n-1} + \bar{\tilde{a}}_n \bar{y}_n + b_n \bar{y}_{n+1} = 0, \quad n \in \mathbb{N}.\]

Multiply relation (9) by \( \bar{y}_n \), relation (10) by \( y_n \), and then subtract to obtain

\[(12) \quad (\tilde{a}_n - \bar{\tilde{a}}_n) y_n \bar{y}_n = A_n = A_n(\lambda) = b_n (y_0 \bar{y}_{n+1} - \bar{y}_0 y_{n+1}) = b_n 2i \text{Im}(y_0 \bar{y}_{n+1}), \quad n \in \mathbb{Z}_+.\]

Then we can write

\[(13) \quad \sum_{j=1}^{n-1} (\tilde{a}_j - \bar{\tilde{a}}_j) y_j \bar{y}_j = A_n - A_0 = 2ib_{n-1} \text{Im}(y_{n-1} \bar{y}_n) - 2ib_0 \text{Im}(y_0 \bar{y}_n), \quad n = 2, 3, \ldots\]

Therefore relation (8) is true. \( \square \)
Corollary 1. Let $P_n(\lambda)$ and $Q_n(\lambda)$, $n \in \mathbb{Z}_+$, be polynomials of the first and of the second kinds for difference equations $(3)$, $(4)$, respectively. Polynomials $P_n(\lambda)$ satisfy the following relation:

\begin{equation}
(14) \quad \sum_{j=0}^{n-1} (\text{Im} a_j - \text{Im} \lambda)|p_j(\lambda)|^2 = b_{n-1} \text{Im}(P_{n-1}(\lambda)\overline{P_n(\lambda)}), \quad n \in \mathbb{N}.
\end{equation}

Choose an arbitrary $w \in \mathbb{C}$ and consider the polynomials

\begin{equation}
(15) \quad \Psi_n(\lambda, w) = wP_n(\lambda) + Q_n(\lambda), \quad n \in \mathbb{Z}_+.
\end{equation}

The polynomials $\Psi_n(\lambda, w)$ satisfy the following relation:

\begin{equation}
(16) \quad \sum_{j=0}^{n-1} (\text{Im} a_j - \text{Im} \lambda)|\Psi_j(\lambda, w)|^2 = b_{n-1} \text{Im}(\Psi_{n-1}(\lambda, w)\overline{\Psi_n(\lambda, w)}) - \text{Im} w, \quad n \in \mathbb{N}.
\end{equation}

Proof. To obtain relations $(14)$, $(16)$ for $n \geq 2$, it is sufficient to write relation $(8)$ for the polynomials $P_n(\lambda)$ and $\Psi_n(\lambda, w)$, respectively, and to use the initial conditions. For the case $n = 1$ relations $(14)$, $(16)$ can be verified using the initial conditions.

Set

\begin{equation}
(17) \quad \Pi = \Pi(r_0, r_1) = \{ \lambda \in \mathbb{C} : r_0 \leq \text{Im} \lambda \leq r_1 \}.
\end{equation}

Corollary 2. Let $P_n(\lambda)$, $n \in \mathbb{Z}_+$, be polynomials of the first kind for difference equations $(3)$, $(4)$. The roots of polynomials $P_n(\lambda)$ lie in the strip $\Pi(r_0, r_1)$.

Proof. For an arbitrary root $\lambda_0 \in \mathbb{C}$ of $P_{n-1}(\lambda)$, $n = 2, 3, \ldots$, by $(14)$ we obtain

\begin{equation}
(18) \quad \sum_{j=0}^{n-1} (\text{Im} a_j - \text{Im} \lambda_0)|p_j(\lambda_0)|^2 = 0.
\end{equation}

Suppose that $\text{Im} \lambda_0 > r_1$. By $(2)$ we obtain

\[ \text{Im} a_j - \text{Im} \lambda_0 < 0, \quad j \in \mathbb{Z}_+. \]

Then $(18)$ leads to a contradiction since $P_0 = 1$. If we suppose that $\text{Im} \lambda_0 < r_0$, we will get

\[ \text{Im} a_j - \text{Im} \lambda_0 > 0, \quad j \in \mathbb{Z}_+. \]

That contradicts relation $(18)$ as well. \qed

3. Weyl’s discs

Like in the classical case (see [1]), an important role in our further considerations will play the following function:

\begin{equation}
(19) \quad w_n(\lambda, \tau) = -\frac{Q_n(\lambda) - \tau Q_{n-1}(\lambda)}{P_n(\lambda) - \tau P_{n-1}(\lambda)},
\end{equation}

where $\lambda, \tau \in \mathbb{C}$, $n \in \mathbb{N}$ ($P_n, Q_n$ are polynomials of the first and of the second kinds for difference equations $(3)$, $(4)$). We set

\begin{equation}
(20) \quad \Pi_+ = \Pi_+(r_1) = \{ \lambda \in \mathbb{C} : \text{Im} \lambda > r_1 \}, \quad \Pi_- = \Pi_-(r_0) = \{ \lambda \in \mathbb{C} : \text{Im} \lambda < r_0 \},
\end{equation}

\begin{equation}
(21) \quad \Pi_0 = \Pi_0(r_0, r_1) = \Pi_+(r_1) \cup \Pi_-(r_0).
\end{equation}

1) Choose an arbitrary $\lambda \in \Pi_+(r_1)$ and $n \in \mathbb{N}$. By virtue of Corollary 2, relations $(14)$ and $(2)$ we get

\[ b_{n-1} \text{Im}(P_{n-1}(\lambda)\overline{P_n(\lambda)}) = b_{n-1}|P_{n-1}(\lambda)|^2 \text{Im} \left( \frac{P_n(\lambda)}{P_{n-1}(\lambda)} \right) < 0. \]
Thus, we have
\begin{equation}
\Im\left(\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right) > 0.
\end{equation}

So, a pole of the map \(w_n(\lambda, \tau)\) (for the fixed \(\lambda \in \Pi_+\), \(n \in \mathbb{N}\)) lies in the upper half-plane \(\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \Im \lambda > 0\}\). In particular, this means that the real line \(\mathbb{R}\) is mapped on a circle \(C_\mathcal{N}(\lambda)\) in the \(w\)-plane (the complex plane of the variable \(w\)). The lower half-plane \(\mathbb{C}_- = \{\tau \in \mathbb{C} : \Im \tau \leq 0\}\) is mapped on a disc \(D_n(\lambda)\). The inverse map for \(w_n(\lambda, \tau)\) has the following form:
\begin{equation}
\tau_n(\lambda, w) = \frac{wP_n(\lambda) + Q_n(\lambda)}{wP_{n-1}(\lambda) + Q_{n-1}(\lambda)} = \frac{\Psi_n(\lambda, w)}{\Psi_{n-1}(\lambda, w)}.
\end{equation}

For an arbitrary \(w \in \mathbb{C} : \Psi_{n-1}(\lambda, w) \neq 0\) (this means that \(w \neq -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}\) by virtue of relation (16) we can write
\begin{equation}
\sum_{j=0}^{n-1} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 = -b_{n-1}|\Psi_{n-1}(\lambda)|^2 \Im(\frac{\Psi_n(\lambda, w)}{\Psi_{n-1}(\lambda)}) - \Im w.
\end{equation}

In our case we have \(\Im a_j - \Im \lambda < 0, \ j \in \mathbb{Z}_+\). Therefore
\begin{equation}
\sum_{j=0}^{n-1} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 = \Im w + b_{n-1}|\Psi_{n-1}(\lambda)|^2 \Im \tau_n(\lambda, w).
\end{equation}

From the last relation and relation (16) for the case \(w = -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}\), we see that the disc \(D_n(\lambda)\) consists of \(w \in \mathbb{C}\) such that
\begin{equation}
\sum_{j=0}^{n-1} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 \leq \Im w.
\end{equation}

From relation (26) it follows that
\begin{equation}
D_{n+1}(\lambda) \subseteq D_n(\lambda), \quad n \in \mathbb{N}, \quad \lambda \in \Pi_+.
\end{equation}

Hence, there exists a non-empty intersection \(D_\infty(\lambda) = \bigcap_{j \in \mathbb{N}} D_j(\lambda)\). From relation (26) it follows that \(D_\infty(\lambda)\) consists of \(w \in \mathbb{C}\) such that
\begin{equation}
\sum_{j=0}^{\infty} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 \leq \Im w.
\end{equation}

2) Choose an arbitrary \(\lambda \in \Pi_-(r_0)\) and \(n \in \mathbb{N}\). Reasoning similarly, we obtain that
\begin{equation}
\Im\left(\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right) < 0.
\end{equation}

A pole of the map \(w_n(\lambda, \tau)\) lies in the lower half-plane \(\mathbb{C}^- = \{\lambda \in \mathbb{C} : \Im \lambda < 0\}\). The real line is mapped on a circle \(C_\mathcal{N}(\lambda)\) and the upper half-plane \(\mathbb{C}_+ = \{\tau \in \mathbb{C} : \Im \tau \geq 0\}\) is mapped on a disc \(D_n(\lambda)\). For \(w \in \mathbb{C} : \Psi_{n-1}(\lambda, w) \neq 0\), by virtue of relation (16) we can write relation (24). In our case we have \(\Im a_j - \Im \lambda > 0, \ j \in \mathbb{Z}_+\), therefore
\begin{equation}
\sum_{j=0}^{n-1} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 = -\Im w - b_{n-1}|\Psi_{n-1}(\lambda)|^2 \Im \tau_n(\lambda, w).
\end{equation}

From relation (15) and relation (16) for the case \(w = -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}\), we see that the disc \(D_n(\lambda)\) consists of \(w \in \mathbb{C}\) such that
\begin{equation}
\sum_{j=0}^{n-1} (\Im a_j - \Im \lambda)|\Psi_j(\lambda, w)|^2 \leq -\Im w.
\end{equation}
Proposition 2. From relation (31) it follows that

\[ D_{n+1}(\lambda) \subseteq D_n(\lambda), \quad n \in \mathbb{N}, \quad \lambda \in \Pi_r. \]

Thus, there exists a non-empty intersection \( D_\infty(\lambda) = \bigcap_{j \in \mathbb{N}} D_j(\lambda) \). From relation (31) it follows that \( D_\infty(\lambda) \) consists of \( w \in \mathbb{C} \) such that

\[ \sum_{j=0}^{\infty} |\text{Im} \, a_j - \text{Im} \lambda||\Psi_j(\lambda, w)|^2 \leq -\text{Im} \, w. \]

The radius of \( C_n(\lambda) \) is denoted by \( r_n(\lambda), \lambda \in \Pi_0 \). We will need an analytic expression for \( r_n(\lambda) \).

**Proposition 3.** Let \( \lambda \in \Pi_0 \) and \( n \in \mathbb{N} \). The radius of the circle \( D_n(\lambda) \) is equal to

\[ r_n(\lambda) = \frac{1}{b_n-1|P_n(\lambda)P_{n-1}(\lambda) - P_{n-1}(\lambda)P_n(\lambda)|} = \frac{1}{2 \sum_{j=0}^{n-1} |\text{Im} \, a_j - \text{Im} \lambda||P_j(\lambda)|^2}. \]

**Proof.** To obtain the first equality in (34), one repeats the standard arguments from the proof of Theorem 1.2.3 in [1]. The second equality follows from relation (14). \( \square \)

Consider a sequence of functions,

\[ \hat{w}_n(\lambda) := w_n(\lambda, 0) = -\frac{Q_n(\lambda)}{P_n(\lambda)}, \quad \lambda \in \Pi_0(r_0, r_1), \quad n \in \mathbb{N}. \]

Notice that \( \hat{w}_n(\lambda) \in D_n(\lambda), \ n \in \mathbb{N}, \ \lambda \in \Pi_0 \). Hence, using relations (26), (31) we can write

\[ |\hat{w}_n(\lambda)|^2 = |\hat{w}_n(\lambda)P_0(\lambda) + Q_0(\lambda)|^2 \leq \sum_{j=0}^{n-1} \frac{|\text{Im} \, a_j - \text{Im} \lambda|}{|\text{Im} \, a_0 - \text{Im} \lambda|} |\hat{w}_n(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 \]

\[ \leq \frac{|\text{Im} \, \hat{w}_n(\lambda)|}{|\text{Im} \, a_0 - \text{Im} \lambda|} \leq \frac{|\hat{w}_n(\lambda)|}{|\text{Im} \, a_0 - \text{Im} \lambda|}. \]

Consequently, we obtain

\[ |\hat{w}_n(\lambda)| \leq \frac{1}{|\text{Im} \, a_0 - \text{Im} \lambda|}, \quad \lambda \in \Pi_0, \quad n \in \mathbb{N}. \]

Thus, in any compact subset of \( \Pi_0 \), the sequence of functions \( \hat{w}_n(\lambda) \) is uniformly bounded. The functions \( \hat{w}_n(\lambda) \) are analytic in \( \Pi_0 \) as it follows from Corollary 2. By virtue of Montel’s theorem (see [7]) we can assert that there exists a subsequence \( \hat{w}_{n_k}(\lambda), k \in \mathbb{N} \), which is uniformly convergent to a function \( m(\lambda) \) in \( \Pi_0 \). The function \( m(\lambda) \) is analytic by Weierstrass’s theorem. Passing to the limit in (35) with \( n = n_k, k \to \infty \), we obtain

\[ |m(\lambda)| \leq \frac{1}{|\text{Im} \, a_0 - \text{Im} \lambda|}, \quad \lambda \in \Pi_0. \]

Observe that \( m(\lambda) \in D_n(\lambda) \) for any \( n \in \mathbb{N} \), and therefore

\[ m(\lambda) \in D_\infty(\lambda), \quad \lambda \in \Pi_0. \]

For an arbitrary \( \varepsilon > 0 \) we set

\[ \Pi_{0,\varepsilon} = \Pi_{0,\varepsilon}(r_0, r_1) = \{ \lambda \in \mathbb{C} : \text{Im} \, \lambda \leq r_0 - \varepsilon \} \cup \{ \lambda \in \mathbb{C} : \text{Im} \, \lambda \geq r_1 + \varepsilon \}. \]

**Proposition 4.** For the function \( m(\lambda) \) the following relation holds true:

\[ m(\lambda) \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}, \quad \varepsilon > 0. \]
Proof. Since \( m(\lambda) \in D_\infty(\lambda) \), \( \hat{w}_n(\lambda) \in D_\infty(\lambda) \), \( n \in \mathbb{N} \), \( \lambda \in \Pi_0 \), we can write
\[
(39) \quad |m(\lambda) + \frac{Q_n(\lambda)}{P_n(\lambda)}| \leq 2r_n(\lambda), \quad \lambda \in \Pi_{0,\varepsilon}, \quad \varepsilon > 0, \quad n \in \mathbb{N}.
\]
From relation (34) we see that
\[
|r_n(\lambda)| \leq \frac{1}{2|\text{Im} a_1 - \text{Im} \lambda||P_1(\lambda)|^2} \leq \frac{\nu_0^2}{2\varepsilon^2 |\lambda - a_0|^2}, \quad \lambda \in \Pi_{0,\varepsilon}, \quad n = 2, 3, \ldots
\]
Thus, for any fixed \( n, n = 2, 3, \ldots \), we obtain
\[
r_n(\lambda) \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}.
\]
Passing to the limit in relation (39) we see that
\[
m(\lambda) + \frac{Q_n(\lambda)}{P_n(\lambda)} \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}.
\]
It remains to notice that
\[
\frac{Q_n(\lambda)}{P_n(\lambda)} \to 0, \quad \lambda \to \infty,
\]
since \( \deg Q_n = n - 1 \), \( \deg P_n = n \). \( \square \)

The following theorem is valid.

**Theorem 1.** Difference equation (4) has a solution \( y_n = m(\lambda)P_n(\lambda) + Q_n(\lambda) \), \( n \in \mathbb{Z}_+ \), which belongs to \( l^2 \) for any \( \lambda \in \Pi_0 \).

Proof. Since the function \( m(\lambda), \lambda \in \Pi_0 \), belongs to the disc \( D_\infty(\lambda) \), from relations (28), (33) it follows that
\[
(40) \quad \sum_{j=0}^{\infty} |\text{Im} a_j - \text{Im} \lambda||m(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 < \infty.
\]
Since \( |\text{Im} a_j - \text{Im} \lambda| \geq |\text{Im} \lambda - r_1| > 0, \lambda \in \Pi_+ \), and \( |\text{Im} a_j - \text{Im} \lambda| \geq r_0 - |\text{Im} \lambda| > 0, \lambda \in \Pi_- \), the result follows. \( \square \)

4. The spectral function

Let \( J \) be the semi-infinite matrix from (1), (2). Observe that it is a matrix that is complex symmetric (with respect to the transposition). Let \( \{P_n(\lambda)\}_{n \in \mathbb{Z}_+}, \{Q_n(\lambda)\}_{n \in \mathbb{Z}_+} \) be the defined above solutions of the corresponding difference equations (3),(4). Recall (see [4, p. 474]) that a linear with respect to the both arguments functional \( \sigma(u, v) \), \( u, v \in \mathbb{P} \), is called a spectral function of difference equations (3),(4) if it satisfies relations
\[
(41) \quad \sigma(P_n, P_m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+.
\]
For the given difference equations (3), (4) it is not hard to obtain the spectral function using (41) as a definition and then extending this definition by the linearity. Namely, if \( P(\lambda) = \sum_{j=0}^{\infty} \xi_j P_j(\lambda), \xi_j \in \mathbb{C} \), and \( R(\lambda) = \sum_{j=0}^{\infty} \nu_j P_j(\lambda), \nu_j \in \mathbb{C} \), we set
\[
(42) \quad \sigma(P, R) = \sum_{j=0}^{\infty} \xi_j \nu_j.
\]
Here all sums are finite. However, representation (42) is not very convenient. It requires the knowledge of all coefficients of resolutions of the polynomials \( P, R \) via the polynomials \( \{P_n(\lambda)\}_{n \in \mathbb{Z}_+} \). We are going to derive an analytic representation for the spectral function \( \sigma \).
Note that according to Theorem 1 in [4] we have

\begin{equation}
\sigma(P, R) = \sigma(PR, 1), \quad P, R \in \mathbb{P}.
\end{equation}

That means that it is enough to obtain an analytic representation for \( \sigma(u, 1) \), \( u \in \mathbb{P} \). If

\begin{equation}
\psi(u) = \lim_{j \to \infty} u_j P_k(\lambda), \quad u_k \in \mathbb{C},
\end{equation}

then by (41) we will get

\begin{equation}
\sigma(u, 1) = u_0.
\end{equation}

Let \( \Psi_n(\lambda, w) \) and \( m(\lambda) \) be defined as in the previous Section. We set

\begin{equation}
\Psi_n(\lambda) := m(\lambda)P_n(\lambda) + Q_n(\lambda), \quad \lambda \in \Pi_0,
\end{equation}

and

\begin{equation}
\Psi_f(\lambda) := \sum_{j=0}^{\infty} \Psi_j(\lambda)f_j, \quad f = (f_0, f_1, f_2, \ldots) \in l^2_{\Pi_0}, \quad \lambda \in \Pi_0.
\end{equation}

**Proposition 4.** Let \( f = (f_0, f_1, f_2, \ldots) \in l^2_{\Pi_0} \) and \( \varepsilon > 0 \). For the function \( \Psi_f(\lambda) \) the following relation holds:

\begin{equation}
\Psi_f(\lambda) = -\frac{1}{\lambda} \left( \lambda f_0 + \varepsilon(1) \right),
\end{equation}

where \( \varepsilon(1) \to 0 \) as \( \lambda \to \infty \) in a strip \( \Pi_{0, \varepsilon} \).

**Proof.** Let \( f \) and \( \varepsilon \) be from the statement of the Proposition. Set \( g = (g_0, g_1, g_2, \ldots) \), where

\begin{equation}
g_0 = a_0 f_0 + b_0 f_1, \quad g_n = b_{n-1} f_{n-1} + a_n f_n + b_n f_{n+1}, \quad n \in \mathbb{N}.
\end{equation}

Observe that \( g \in l^2_{\Pi_0} \). We can write

\begin{align*}
\Psi_g(\lambda) &= \sum_{j=0}^{\infty} \Psi_j(\lambda)g_j = \Psi_0(\lambda)(a_0 f_0 + b_0 f_1) + \sum_{j=1}^{\infty} \Psi_j(\lambda)(b_{j-1} f_{j-1} + a_j f_j + b_j f_{j+1}) \\
&= \Psi_0(\lambda)(a_0 f_0 + b_0 f_1) + \sum_{k=0}^{\infty} \Psi_{k+1}(\lambda)b_k f_k + \sum_{j=1}^{\infty} \Psi_j(\lambda)a_j f_j + \sum_{l=2}^{\infty} \Psi_{l-1}(\lambda)b_{l-1} f_1 \\
&= \Psi_0(\lambda)a_0 f_0 + \Psi_1(\lambda)b_0 f_0 + \sum_{j=1}^{\infty} (b_{j-1} \Psi_{j-1}(\lambda) + a_j \Psi_j(\lambda) + b_j \Psi_{j+1}(\lambda)) f_j \\
&= \Psi_0(\lambda)a_0 f_0 + \Psi_1(\lambda)b_0 f_0 + \lambda \sum_{j=1}^{\infty} \Psi_j(\lambda)f_j, \quad \lambda \in \Pi_0,
\end{align*}

where we have used the fact that \( \Psi_j(\lambda) \) is a solution of difference equation (4).

Since \( \Psi_0(\lambda) = m(\lambda) \), and \( b_0 \Psi_1(\lambda) = \lambda m(\lambda) - a_0 m(\lambda) + 1 \), we get

\begin{equation}
\Psi_g(\lambda) = \lambda m(\lambda) f_0 + f_0 + \lambda \sum_{j=1}^{\infty} \Psi_j(\lambda)f_j = f_0 + \lambda \sum_{j=0}^{\infty} \Psi_j(\lambda)f_j = f_0 + \lambda \Psi_f(\lambda).
\end{equation}

Therefore

\begin{equation}
\Psi_f(\lambda) = \frac{1}{\lambda} \left( \lambda f_0 + \Psi_g(\lambda) \right), \quad \lambda \in \Pi_0.
\end{equation}
By virtue of the Cauchy-Buniakovskiy inequality we can write
\[
|Ψ_g(λ)| \leq \left( \sum_{j=0}^{∞} |Ψ_j(λ)|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{∞} |g_j(λ)|^2 \right)^{\frac{1}{2}}, \quad λ ∈ Π_0.
\]

If λ ∈ Π_{0,ε} then |Im a_j − Im λ| > ε. Since the function m(λ), λ ∈ Π_{0,ε}, belongs to the disc D_∞(λ), by virtue of relations (28), (33) we can write
\[
\sum_{j=0}^{∞} |m(λ) P_j(λ) + Q_j(λ)|^2 ≤ \sum_{j=0}^{∞} |Im a_j − Im λ||m(λ) P_j(λ) + Q_j(λ)|^2 ≤ |Im m(λ)|.
\]

Hence, we get
\[
|Ψ_g(λ)| ≤ \frac{|m(λ)|}{ε} \left( \sum_{j=0}^{∞} |g_j(λ)|^2 \right)^{\frac{1}{2}}, \quad λ ∈ Π_{0,ε}.
\]

Applying Proposition 3 we complete the proof. □

**Theorem 2.** The spectral function σ of difference equations (3), (4) has the following representation:
\[
σ(P, R) = \frac{1}{2πi} \lim_{δ→0} \left\{ \int_{−∞+i(r_1+δ)}^{∞+i(r_1+δ)} P(λ)R(λ)e^{−δλ^2}m(λ) dλ + \int_{−∞+(r_0−δ)}^{∞+(r_0−δ)} P(λ)R(λ)e^{−δλ^2}m(λ) dλ \right\}, \quad P, R \in P,
\]
where
\[
ε > 0 : \; ε > r_1, \; ε > r_0.
\]

**Proof.** We first note that the function Ψ_f(λ) from (47) is analytic in Π_0. Choose an arbitrary ε > 0 which satisfies (54) and consider points a_N^- = −N + i(r_1 + ε), c^+ = i(r_1 + ε), b_N^+ = N + i(r_1 + ε), and a_N^- = −N + i(r_0 − ε), c^- = i(r_0 − ε), b_N^- = N + i(r_0 − ε) in the complex λ-plane. We also denote
\[
C_N^+ = \{ λ ∈ C : |λ − c^+| = N, \; Im λ ≥ r_1 + ε \},
\]
\[
C_N^- = \{ λ ∈ C : |λ − c^-| = N, \; Im λ ≤ r_0 − ε \}.
\]

Condition (54) ensures that the points a_N^+, c^+, b_N^+ and the half of the circle, C_N^+, lie in the open upper half-plane C_N^+. The points a_N^-, c^-, b_N^- and the half of the circle, C_N^-, lie in the open lower half-plane C_N^-. Using the analyticity we can write
\[
\int_{a_N^-}^{b_N^+} Ψ_f(λ) dλ + \int_{C_N^+} Ψ_f(λ) dλ = 0,
\]
\[
\int_{b_N^-}^{a_N^-} Ψ_f(λ) dλ + \int_{C_N^-} Ψ_f(λ) dλ = 0.
\]

By virtue of Proposition 4 we can write
\[
\int_{C_N} Ψ_f(λ) dλ = −f_0 \int_{C_N^+} \frac{1}{λ} dλ − \int_{C_N^-} \frac{1}{λ} Ψ(1) dλ,
\]
where Ψ(1) = −Ψ_g(λ) (see (49)) is an analytic function in Π_0. Since |λ| ≥ N − |r_1 + ε| in C_N^+, we get
\[
\left| \frac{Ψ(1)}{λ} \right| ≤ \frac{|Ψ(1)|}{N − |r_1 + ε|},
\]
and the second term in the right-hand side of (57) tends to zero as $N \to \infty$. For the first term in the right-hand side of (57), we can write

$$-f_0(\ln a_N^+ - \ln b_N^+) = -f_0 i(\arg a_N^+ - \arg b_N^+) \to -\pi i f_0,$$

as $N \to \infty$. Here we have used an arbitrary analytic branch of the logarithm in $\mathbb{C}\setminus[0, +\infty)$. Calculating arguments we used that points $a_N^+, b_N^+$ lie in $\mathbb{C}_+$.

Passing to the limit in (55) we get

$$\lim_{N \to \infty} \int_{a_N^+}^{b_N^+} \Psi_f(\lambda) \, d\lambda = \pi i f_0. \tag{58}$$

Proceeding in an analogous manner with relation (56) we obtain

$$\lim_{N \to \infty} \int_{a_N^-}^{b_N^-} \Psi_f(\lambda) \, d\lambda = \pi i f_0. \tag{59}$$

Summing up relations (58) and (59) we get

$$\lim_{N \to \infty} \left\{ \int_{a_N^+}^{b_N^+} \Psi_f(\lambda) \, d\lambda + \int_{a_N^-}^{b_N^-} \Psi_f(\lambda) \, d\lambda \right\} = 2\pi i f_0. \tag{60}$$

Let us show that

$$\lim_{N \to \infty} \int_{a_N}^{b_N} \Psi_f(\lambda) \, d\lambda = \lim_{\delta \to 0} \int_{-\infty+i(\hat{r}_0+\epsilon)}^{\infty+i(\hat{r}_0+\epsilon)} e^{-\delta \lambda^2} \Psi_f(\lambda) \, d\lambda. \tag{61}$$

We first note that the integral in the right-hand side of (61) exists, since $\Psi_f(\lambda)$ is bounded (see (48)). For an arbitrary $\epsilon > 0$ we can write

$$\left| \int_{-\infty+i(\hat{r}_0+\epsilon)}^{\infty+i(\hat{r}_0+\epsilon)} e^{-\delta \lambda^2} \Psi_f(\lambda) \, d\lambda - \lim_{N \to \infty} \int_{a_N}^{b_N} \Psi_f(\lambda) \, d\lambda \right| \leq \int_{a_N}^{b_N} (e^{-\delta \lambda^2} - 1) \Psi_f(\lambda) \, d\lambda \tag{62}$$

for $N \geq N_0$, $N_0 \in \mathbb{N}$. On the finite segment $[a_N^+, b_N^-]$, the function $(e^{-\delta \lambda^2} - 1)\Psi_f(\lambda)$ uniformly tends to zero as $\delta \to 0$. Therefore,

$$\int_{a_N}^{b_N} (e^{-\delta \lambda^2} - 1) \Psi_f(\lambda) \, d\lambda \to 0, \quad \delta \to 0.$$

Hence, we can choose $\hat{\delta} > 0$ such that $|\delta| < \hat{\delta}$ implies

$$\left| \int_{a_N}^{b_N} (e^{-\delta \lambda^2} - 1) \Psi_f(\lambda) \, d\lambda \right| \leq \frac{\epsilon}{2}. \tag{63}$$

From relations (62), (63) it follows that (61) holds. In an analogous manner we obtain

$$\lim_{N \to \infty} \int_{b_N^-}^{a_N^-} \Psi_f(\lambda) \, d\lambda = \lim_{\delta \to 0} \int_{-\infty+i(\hat{r}_0-\epsilon)}^{\infty+i(\hat{r}_0-\epsilon)} e^{-\delta \lambda^2} \Psi_f(\lambda) \, d\lambda. \tag{64}$$

From (60), (61), (64) we obtain

$$\lim_{\delta \to 0} \left\{ \int_{-\infty+i(\hat{r}_0+\epsilon)}^{\infty+i(\hat{r}_0+\epsilon)} e^{-\delta \lambda^2} \Psi_f(\lambda) \, d\lambda + \int_{-\infty+i(\hat{r}_0-\epsilon)}^{\infty+i(\hat{r}_0-\epsilon)} e^{-\delta \lambda^2} \Psi_f(\lambda) \, d\lambda \right\} = 2\pi i f_0. \tag{65}$$
Let $u(\lambda) \in \mathcal{P}$ be an arbitrary complex polynomial which has resolution (44). A vector of coefficients $u = (u_0, u_1, u_2, \ldots)$ belongs to $l_2^d$. For $\lambda \in \Pi_0$ we can write

$$\Psi_u(\lambda) = \sum_{j=0}^{\infty} \Psi_j(\lambda) u_j = \sum_{j=0}^{\infty} (m(\lambda) P_j(\lambda) + Q_j(\lambda)) u_j = m(\lambda) u(\lambda) + \sum_{j=0}^{\infty} Q_j(\lambda) u_j.$$  \hspace{1cm} (66)

Let us show that

$$\lim_{\delta \to 0} \left\{ \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda + \int_{-\infty+i(r_0-\varepsilon)}^{\infty+i(r_0-\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda \right\} = 0, \quad j \in \mathbb{Z}_+.$$  \hspace{1cm} (67)

Since the function $e^{-\delta \lambda^2} Q_j(\lambda)$ is analytic in $\mathbb{C}$, we have

$$\int_{-N+i(r_1+\varepsilon)}^{N+i(r_1+\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda + \int_{N+i(r_0-\varepsilon)}^{-N+i(r_0-\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda$$

$$+ \int_{N+i(r_1+\varepsilon)}^{-N+i(r_1+\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda + \int_{-N+i(r_0-\varepsilon)}^{N+i(r_0-\varepsilon)} e^{-\delta \lambda^2} Q_j(\lambda) \, d\lambda = 0.$$  \hspace{1cm} (68)

The last two terms in the left-hand side of (68) tend to zero as $N \to \infty$. In fact, the length of the path of integration is constant and the function under the integral tends to zero as $N \to \infty$, in the both cases. So, proceeding to the limit in (68) we obtain (67).

If we write relation (65) for the function $\Psi_u(\lambda)$ from (66) and use (67), we will get

$$\lim_{\delta \to 0} \left\{ \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta \lambda^2} m(\lambda) u(\lambda) \, d\lambda + \int_{-\infty+i(r_0-\varepsilon)}^{\infty+i(r_0-\varepsilon)} e^{-\delta \lambda^2} m(\lambda) u(\lambda) \, d\lambda \right\}$$

$$= 2\pi i u_0 = 2\pi i \sigma(u(\lambda), 1).$$  \hspace{1cm} (69)

If we take into account relation (43), we will obtain relation (53). The proof is complete. \hspace{1cm} \square

References