SELF-CONSISTENT RENORMALIZATION AS AN EFFICIENT REALIZATION OF MAIN IDEAS OF THE BOGOLIUBOV–PARASIUK R-OPERATION

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Il libro della natura é scritto in lingua matematica.
Galileo Galilei, [*Il Saggiatore, 1623*].

"...At the present time, the intimate connection between causality and the analytic continuation is revealed. So, it is not improbable to develop a subtraction procedure even in the most general case by the use of analytic continuation techniques."
O.S. Parasiuk, [[7], *p.566, the last paragraph, 1956*].

This possibility is realized explicitly and efficiently in a body of our self-consistent renormalization (SCR). The self-consistency means that all formal relations between UV-divergent Feynman amplitudes are automatically retained as well as between their regular values obtained in the framework of the SCR. Self-consistent renormalization is efficiently applicable on equal grounds both to renormalizable and nonrenormalizable theories. The SCR furnishes new means for the constructive treatment of new subjects: i) UV-divergence problems associated with symmetries, Ward identities, and quantum anomalies; ii) new relations between finite bare and finite physical parameters of quantum field theories. The aim of this paper is to expose main ideas and properties of the SCR and to describe three mutually complementary algorithms of the SCR that are presented in the form maximally suited for practical applications.

1. Introduction

The keystone idea of a purely mathematical genesis of the ultraviolet (UV) divergencies of Feynman amplitudes (FAs) in quantum field theories is at the heart of the Bogoliubov–Parasiuk R-operation [1–7]. Using this idea along with related considerations of mathematicians of the 19th and 20th centuries,[1] the author has developed an universal, high-efficient, and self-consistent renormalization (SCR) technique which is applicable for any dimension $n = 2r_n + \delta_n, \delta_n = 0, 1,\ r_n \in \{0 \cup \mathbb{N}_+\}$ of a space-time that is endowed by a pseudo-Euclidean (p, q)

[1] It is appropriate to pointed out here that the first regularization recipe to subtract infinities for turning a divergent integral into a convergent one had been used in Cauchy's *"extraordinary integral"* [9–11] and in d'Adhémar's [12, 13] and Hadamard's [14–17] *"finite part of a divergent integral"*. These recipes are similar but not identical. But, in both cases, it was extended the validity of the usual rules of change of a variable, integration by parts, and differentiation with respect to the upper limit of integration to these new objects. The Cauchy's "extraordinary integral" has been used for an efficient analytic continuation of the $\Gamma(z)$-function to some noninteger real values $\mathrm{Re}\, z < 0$ firstly by Cauchy himself [10] in 1827, and then in the strips $(-n-1 < \mathrm{Re}\, z < -n)$ by Saalschütz [18, 19] in 1887-1888. The term "finite part of a divergent integral" was introduced by d'Adhémar in his thesis presented at the Sorbonne University in December 1903 and defended in April 1904 (see [[20], p.477]). Referring to Hadamard's article [14], d'Adhémar [[13], p.371] writes "...Independently of each other, we understood the role of these *finite parts*...". In d'Adhémar's thesis and articles, this notion was applied to the construction of solutions of the equation for cylindrical waves [12, 13], whereas Hadamard used finite parts for the solution of the Cauchy problem for second-order equations with variable coefficients [14–16] and an arbitrary number of independent variables [17]. On the applications of d'Adhemar's and Hadamard's "finite part of a divergent integral" in more details, see Hadamard's book [21]. 40 years later on, when analyzing the connections between the intuitive and logical ways of mathematical inventions, Hadamard [22] wrote: "...All mathematicians must consider themselves as logics. For example, I have been asked by what kind of guessing I thought of the device of the "finite part of a divergent integral", which I have used for the integration of partial differential equations. Certainly, considering in itself, it looks typically like "thinking aside". But, in fact, for a long while my mind refused to conceive that idea until positively compelled to, I was led to it step by step as the mathematical reader will easily verify if he takes the trouble to consult my researches on the subject, especially my *Recherches sur les solution fondamentales et l'int'egration des 'equations lin'eaires aux d'eriv'ees partielles*, 2nd Memoir, especially p.121 and so on (*Annales Scientifiques de l'Ecole Normale Superieure,* Vol.**XXII**, 1905) [16]. I could not avoid it any more than the prisoner in Poe's tale *The Pit and Pendulum* could avoid the hole at the center of his cell...", see [[22], p.110, and p.104 or p.86 in two identical Russian translations from French edition of 1959]. About further developments see M. Riesz [23, 24], F. Bureau [25], R. Courant [26], and S.G. Samko, A.A. Kilbas, and O.I. Marichev [27].

metric $g^{\mu\nu}$, where $p+q=n$, and for an arbitrary topology of Feynman graphs.

Algorithmically, the SCR is an efficient realization of the Bogoliubov–Parasiuk R-operation as some special analytical extension of the UV-divergent FAs in two parameters ω^G and ν^G by means of recurrence, compatibility, and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on the mathematical properties of FAs only. The parameters ω^G and ν^G depend on a space-time dimension n, a graph-topological invariant $|\mathcal{C}|$ determining a number of independent circuits of a graph G, and two FAs characteristics $\lambda_{\mathcal{L}}$ and d^G. The numbers $\lambda_{\mathcal{L}}$ and d^G determine the maximal degree of polynomials of the denominator, $d^{\text{den}}=2\lambda_{\mathcal{L}}$, and the numerator, $d^{\text{num}}=d^G$, respectively in the integrand. As a result, the SCR is efficiently applicable on equal grounds both to renormalizable and nonrenormalizable theories, which is very important for quantum gravity.

The self-consistency means that all formal relations between UV-divergent FAs are automatically retained, as well as between their regular values obtained in the framework of the SCR. The SCR furnishes new means for the constructive treatment of new subjects: i) UV-divergence problems associated with symmetries, Ward identities, reduction identities, and quantum anomalies; ii) new relations between *finite bare* and *finite physical* parameters of quantum field theories.

The aim of this article is to expose main ideas and properties of the SCR (see Sections 2 and 3) and to describe three mutually complementary algorithms of the SCR (see Sections 3–5) which are presented in the form maximally suited for practical applications.

2. The Bases and Possibilities of the SCR

2.1. The SCR is an efficient realization of the Bogoliubov-Parasiuk R-operation [1–8] which is supplemented with *recurrence, compatibility, and differential* relations *fixing a renormalization arbitrariness of the R-operation* in some universal way based on *mathematical properties* of Feynman amplitudes (FAs) only. In its turn, the Bogoliubov-Parasiuk approach is rested on an idea that the *nature of UV-divergences is purely mathematical* and, *per se*, the R-operation is a constructive form of the Hahn–Banach theorem on extensions of linear functionals (see, for e.g., [28–30]).

2.2. Elaborating this idea, the author [31–45] has obtained the high-efficiency realization of this renormalization scheme (renormscheme). In this realization:

• Properties of special functions of the hypergeometric type are essentially used.[2]

• Combinatorics is simplified considerably. Our investigations confirm the very important assertion by D.A. Slavnov [52] that the combinatorics of the R-operation is overcomplicated considerably and can be simplified essentially.

• Renormalization arbitrariness of the R-operation is fixed in such a way that the basic functions $(R_0^\nu\mathcal{F})_{sj}\equiv(R_0^\nu\mathcal{F})_{sj}(\omega;M_\epsilon,A)$ of renormalized FAs obey *the same recurrence relations* as the basic functions $\mathcal{F}_{sj}\equiv\mathcal{F}_{sj}(\omega;M_\epsilon,A)$ of convergent or dimensionally regularized FAs:

$$M_\epsilon\,\mathcal{F}_{s-2,j-1}-A\,\mathcal{F}_{s,j-1}+(\omega+j)\,\mathcal{F}_{sj}=0,$$

$$M_\epsilon\,(R_0^\nu\mathcal{F})_{s-2,j-1}-A\,(R_0^\nu\mathcal{F})_{s,j-1}+$$

$$+(\omega+j)\,(R_0^\nu\mathcal{F})_{sj}=0. \tag{2.1}$$

The explicit form of $\mathcal{F}_{sj}$ and $(R_0^\nu\mathcal{F})_{sj}$ are given below by Eqs. (3.30)–(3.31). On the self-consistent version of the Clifford aspect of the dimensional regularization which efficiently overcomes the known difficulties connected with n-dimensional generalization of the Dirac γ^5 matrix, see [39, 53, 54].

• *Compatibility relations* of *the first kind*:

$$(R_0^\nu\mathcal{F})_{sj}=\mathcal{F}_{sj},\quad \text{if } \nu_{sj}:=[(\nu-s)/2]+j\le-1,$$

$$(R_0^{\nu+1}\mathcal{F})_{s+1,j}=(R_0^\nu\mathcal{F})_{sj}, \tag{2.2}$$

and the *compatibility relations* of *the second kind*:

$$\mathcal{F}_{s-2,j-1}(\omega;M_\epsilon,A)=\mathcal{F}_{s,j-1}(\omega;M_\epsilon,A)=$$

$$=\mathcal{F}_{sj}(\omega-1;M_\epsilon,A),$$

[2] The connection of particular FAs with the hypergeometric functions are well known. See, for example, the investigations of analytic properties of convergent scalar FAs by using of algebraic topology methods [46–48], or calculations of some classes of FAs for needs of phenomenological physics, by using the differential equation method [49–51]. But, in our case, this connection is established for general divergent FAs in any space-time dimension n and the (p,q) pseudo-Euclidean metric, $p+q=n$. Apart from, this connection suggests some simple method of fixing a renormalization arbitrariness of the Bogoliubov–Parasiuk R-operation in some universal way based on the mathematical properties of FAs only. As a result, we obtain the self-consistent renormalization with new valuable properties and possibilities.

$$(R_0^\nu\mathcal{F})_{s-2,j-1}(\omega;M_\epsilon,A)=(R_0^\nu\mathcal{F})_{sj}(\omega-1;M_\epsilon,A),$$

$$(R_0^\nu\mathcal{F})_{s,j-1}(\omega;M_\epsilon,A)=(R_0^{\nu-2}\mathcal{F})_{sj}(\omega-1;M_\epsilon,A), \quad (2.3)$$

are satisfied *automatically.* From the first of Eqs. (2.2), it follows that the formulae for *regular values obtained in the framework of the SCR describe uniformly both divergent and convergent* FAs.

• *Differential relations* for $\mathcal{F}_{sj}$ and $(R_0^\nu\mathcal{F})_{sj}$ with respect to *mass-damping variables* $\mu_l := (m_l^2 - i\epsilon_l),\ l\in\mathcal{L}$,

$$\frac{\partial^m}{\partial\mu_{l_1}\cdots\partial\mu_{l_m}}\begin{bmatrix}\mathcal{F}_{sj}(\omega)\\(R_0^\nu\mathcal{F})_{sj}(\omega)\end{bmatrix}=$$

$$=(-1)^m\alpha_{l_1}\cdots\alpha_{l_m}\begin{bmatrix}\mathcal{F}_{sj}(\omega-m)\\(R_0^\nu\mathcal{F})_{sj}(\omega-m)\end{bmatrix} \quad (2.4)$$

are the same, and the *differential relations* for ones with respect to *external momenta* $k_e,\ e\in\mathcal{E}$,

$$\partial_{e_1}^{\sigma_1}\cdots\partial_{e_m}^{\sigma_m}\begin{bmatrix}\mathcal{F}_{sj}(\omega)\\(R_0^\nu\mathcal{F})_{sj}(\omega)\end{bmatrix}=2^m\sum_{\varkappa=0}^{[m/2]}\mathcal{A}_{e_1\cdots e_m}^{\sigma_1\cdots\sigma_m}(\varkappa)\times$$

$$\times\begin{bmatrix}\mathcal{F}_{sj}(\omega-m+\varkappa)\\(R_0^{\nu-2m+2\varkappa}\mathcal{F})_{sj}(\omega-m+\varkappa)\end{bmatrix} \quad (2.5)$$

are almost the same. Here, $\partial_{e_i}^{\sigma_i}\equiv\partial/\partial(k_{e_i})_{\sigma_i}$, and $\mathcal{A}_{e_1\cdots e_m}^{\sigma_1\cdots\sigma_m}(\varkappa)\equiv\mathcal{A}_{e_1\cdots e_m}^{\sigma_1\cdots\sigma_m}(\varkappa|\alpha,k)$ are special homogeneous polynomials of degree $m-2\varkappa$ in $A_{e_i}^{\sigma_i}\equiv A_{e_i}^{\sigma_i}(\alpha,k):=\sum_{e\in\mathcal{E}}A_{e_ie}(\alpha)k_e^{\sigma_i}$ and of degree $\varkappa$ in $\binom{\sigma_i\sigma_j}{e_i\,e_j}:=A_{e_ie_j}(\alpha)g^{\sigma_i\sigma_j}$, where $A_{ee'}(\alpha)$ are matrix elements of the quadratic Kirchhoff form in external momenta $k_e,\ e\in\mathcal{E}$. The polynomials $\mathcal{A}_{e_1\cdots e_m}^{\sigma_1\cdots\sigma_m}(\varkappa|\alpha,k)$ have an algebraic structure of quantities generated by the Wick formula, which represents a T-product of m boson fields in terms of some set of N-products of $m-2\varkappa$ boson fields with $\varkappa$ primitive contractions. Here, the quantities $A_{e_i}^{\sigma_i}$ and $\binom{\sigma_i\sigma_j}{e_i\,e_j}$ play the role of boson fields and their contractions, respectively.

• It is essential that $\mathcal{F}_{sj}$ and $(R_0^\nu\mathcal{F})_{sj}$ as *functions of two variables* M_ϵ and A are the *homogeneous functions of the same degree* $\omega+j$. From this, it follows that they are solutions to the same partial differential equations, namely to the *Euler equation* for the homogeneous functions

$$\big[M_\epsilon\partial_{M_\epsilon}+A\partial_A-(\omega+j)\big]\begin{bmatrix}\mathcal{F}_{sj}(\omega)\\(R_0^\nu\mathcal{F})_{sj}(\omega)\end{bmatrix}=0, \quad (2.6)$$

and to some family of second-order equations emerging from Eq.(2.6), for example,

$$\big[M_\epsilon\partial^2_{M_\epsilon M_\epsilon}\pm(M_\epsilon\pm A)\partial^2_{M_\epsilon A}\pm A\partial^2_{AA}-$$

$$-(\omega+j-1)(\partial_{M_\epsilon}\pm\partial_A)\big]\begin{bmatrix}\mathcal{F}_{sj}(\omega)\\(R_0^\nu\mathcal{F})_{sj}(\omega)\end{bmatrix}=0, \quad (2.7)$$

that can be again represented as the Euler equation

$$\big[M_\epsilon\partial_{M_\epsilon}+A\partial_A-(\omega+j-1)\big]\times$$

$$\times\begin{bmatrix}(\partial_{M_\epsilon}\pm\partial_A)\mathcal{F}_{sj}(\omega)\\(\partial_{M_\epsilon}\pm\partial_A)\,(R_0^\nu\mathcal{F})_{sj}(\omega)\end{bmatrix}=0. \quad (2.8)$$

So, an important role of the quantities $(\partial_{M_\epsilon}\pm\partial_A)\mathcal{F}_{sj}(\omega)$ and $(\partial_{M_\epsilon}\pm\partial_A)(R_0^\nu\mathcal{F})_{sj}(\omega)$ is revealed in our problem. After repeating this procedure $N+1$ times, one obtains

$$\big[M_\epsilon\partial_{M_\epsilon}+A\partial_A-(\omega+j-N-1)\big]\times$$

$$\times\begin{bmatrix}(\partial_{M_\epsilon}\pm\partial_A)\mathcal{F}_{sj}^{N\pm}(\omega-N)\\(\partial_{M_\epsilon}\pm\partial_A)\,(R_0^\nu\mathcal{F})_{sj}^{N\pm}(\omega-N)\end{bmatrix}=0, \quad (2.9)$$

where we define $\mathcal{F}_{sj}^{N\pm}(\omega-N):=(\partial_{M_\epsilon}\pm\partial_A)^N\mathcal{F}_{sj}(\omega)$ and $(R_0^\nu\mathcal{F})_{sj}^{N\pm}(\omega-N):=(\partial_{M_\epsilon}\pm\partial_A)^N(R_0^\nu\mathcal{F})_{sj}(\omega)$. If N such that $(\omega-N+j)\le-1$ then both $(\partial_{M_\epsilon}+\partial_A)\mathcal{F}_{sj}^{N\pm}(\omega-N)=0$ and $(\partial_{M_\epsilon}+\partial_A)(R_0^\nu\mathcal{F})_{sj}^{N\pm}(\omega-N)=0$. As a result, Eq. (2.9) with the *plus sign* is degenerated into the *identical zero*, and the equation with the *minus sign* is reduced to the *Euler–Poisson–Darboux equation*

$$\left[\frac{\partial^2}{\partial M_\epsilon\partial A}+\frac{(\omega+j-N-1)/2}{M_\epsilon-A}\Big(\frac{\partial}{\partial M_\epsilon}-\frac{\partial}{\partial A}\Big)\right]\times$$

$$\times\begin{bmatrix}\mathcal{F}_{sj}^{N\pm}(\omega-N)\\(R_0^\nu\mathcal{F})_{sj}^{N\pm}(\omega-N)\end{bmatrix}=0. \quad (2.10)$$

The consistency of solutions to Eqs. (2.9)–(2.10) for different preassigned asymptotics of $(R_0^\nu\mathcal{F})_{sj}$ at the vicinity of $A=0$ leads to the relations

$$\partial_{M_\epsilon}\mathcal{F}_{sj}(\omega)=-\mathcal{F}_{sj}(\omega-1),$$

$$\partial_A\mathcal{F}_{sj}(\omega)=\mathcal{F}_{sj}(\omega-1),$$

$$\partial_{M_\epsilon}(R_0^\nu\mathcal{F})_{sj}(\omega)=-(R_0^\nu\mathcal{F})_{sj}(\omega-1),$$

$$\partial_A(R_0^\nu\mathcal{F})_{sj}(\omega)=(R_0^{\nu-2}\mathcal{F})_{sj}(\omega-1) \quad (2.11)$$

which are also followed from the explicit form of the basic functions $\mathcal{F}_{sj}$ and $(R_0^\nu\mathcal{F})_{sj}$, see Eqs. (3.30)–(3.31) below.

2.3. Relations (2.1)–(2.11) manifest the mutual consistency of asymptotic properties of different terms of FAs with respect to external momenta and masses. It is precisely these *recurrence, compatibility, and differential relations* that are of great importance for investigating the *problems of symmetries and anomalies* and for turning the developed renormscheme into *a self-consistent one*.

In addition, there exist some obvious identities of the generic nature which are called as the *reduction identities* (RIs) [40, 41], which leads in another way to the recurrence relations (2.1). The simple idea of cancelling the equal factors in factorized polynomials in a numerator and a denominator of integrands is used in RIs. The RIs also are of great importance for applications as *an origin of new nontrivial identities*. Some of them have been used essentially in our investigations [39–41, 43–45, 55–58].

2.4. Equations (2.1)–(2.11) and the explicit form of the basic functions $\mathcal{F}_{sj}$, $(R_0^\nu \mathcal{F})_{sj}$ (see Eqs. (3.30)–(3.33) and (3.36)–(3.40)) imply the following important properties of the SCR:

Algorithmic universality. The SCR is a special analytic continuation of any FA firstly given by an UV-divergent integral. In so doing, the divergence indices ν of FAs may be as large as one needs. Hereafter, this continuation will be named as the regular (i.e., finite) value of a FA. As a result, the regular values of FAs respect certain recurrence, compatibility, and differential properties of an universal character and have already been realized efficiently as convergent integrals. Therefore, the calculations of FAs corresponding to renormalizable and nonrenormalizable theories do not differ from each other in the framework of this renormscheme. Actually, the problem is reduced to calculations of the characteristic numbers, ω, ν_{sj}, and λ_{sj} determining the basic functions $(R_0^\nu \mathcal{F})_{sj}$.

Separation of problems. The SCR clearly and efficiently separate the problem of evaluating the regular values of UV-divergent quantities of quantum field theories from that of the relations between bare and physical parameters of these theories, i.e., the SCR realizes, in practice, this very important potential possibility of the Bogoliubov–Parasiuk R-operation.

Conservation of relations. Any formal relation between UV-divergent quantities will be retained also between regular values of those if the regular values of all quantities involved in this relation are calculated by *the same renormalization index* ν (the maximum one since, otherwise, we cannot guarantee the finiteness of all terms in the relation). So, the SCR is automatically consistent with the correspondence principle. As a result, the regular values obtained in the framework of the SCR do satisfy the vector and axial-vector canonical Ward identities (CWIs) simultaneously.

Extraction of anomalies (quantum corrections). In the SCR, owing to the analytic continuation technique, quantum anomalies (i.e., quantum corrections (QCs) more exactly) are automatically accounted for in quantities satisfying the CWIs. More specifically, the quantum anomalies (i.e., QCs) reveal themselves either as the oversubtraction effect for a non-chiral case and for the chiral limit case (in these cases, the Schwinger terms contributions (STCs) of current commutators are zero) or as the nonzero STCs for the chiral case. If necessary, the explicit form of quantum anomalies (i.e., QCs) can be easily extracted as a difference between two regular values of the same UV-divergent quantity calculated for proper and improper divergence indices.

2.5. Algorithmically, the SCR is a union of three efficient algorithms of finding:

i) the convergent α-parametric integral representations of renormalized FAs with a compact domain of integration of the simplex type and with the self-consistent basic functions $(R_0^\nu \mathcal{F})_{sj}$, $s = 0, \ldots, d^G$, $j = 0, \ldots, [s/2]$;

ii) the homogeneous k-polynomials $\mathcal{P}_{sj}^G(m, \alpha, k)$, $j = 0, 1, \ldots, [s/2]$, of degree $(s - 2j)$ in external momenta k_e, $e \in \mathcal{E}$, being as α-parametric images of homogeneous p-polynomials $\mathcal{P}_s^G(m, p)$, $s = 0, \ldots, d^G$, of degree s in internal momenta p_l, $l \in \mathcal{L}$;

iii) the α-parametric functions $\Delta(\alpha)$, $A(\alpha, k)$, $Y_l(\alpha, k)$, $X_{ll'}(\alpha)$, $l, l' \in \mathcal{L}$.

3. Parametric Integral Representations and Basic Functions of FAs in the SCR

3.1. From the mathematical point of view, any Feynman amplitude associated with an oriented graph G,

$$G := \langle \mathcal{V}, \mathcal{L} \cup \mathcal{E} \mid e_{il} = 0, \pm 1, \; v_i \in \mathcal{V}, \; l \in \mathcal{L} \cup \mathcal{E} \rangle,$$

in which $\mathcal{V}$ is a set of vertices; $\mathcal{L}$ is a set of internal lines; $\mathcal{E}$ is a set of external lines; and e_{il} is an incidence matrix (i.e., a vertex-line incidence matrix) such that $e_{il} = 0$ if the line $l \in \mathcal{L} \cup \mathcal{E}$ is nonincident to the vertex $v_i \in \mathcal{V}$; $e_{il} = 1$ if the line $l \in \mathcal{L} \cup \mathcal{E}$ is outgoing from the vertex $v_i \in \mathcal{V}$; $e_{il} = -1$ if the line $l \in \mathcal{L} \cup \mathcal{E}$ is incoming to the vertex $v_i \in \mathcal{V}$, can be always represented by the integral

$$I^G(m, k)_\epsilon := c^G \int_{-\infty}^{\infty} (d^n p)^{\mathcal{L}} \delta^G(p, k) \frac{\mathcal{P}^G(m, p)}{Q^G(m, p)_\epsilon},$$

$$(d^n p)^{\mathcal{L}} := d^n p_1 \cdots d^n p_{|\mathcal{L}|}, \quad d^n p_l := \prod_{\sigma=1}^{n} dp_l^{\sigma},$$

$$l \in \mathcal{L}, \quad m := (m_1, \ldots, m_{|\mathcal{L}|}), \tag{3.1}$$

$$p := (p_1, \ldots, p_{|\mathcal{L}|}), \quad k := (k_1, \ldots, k_{|\mathcal{E}|}).$$

Here, $\mathcal{P}^G(m,p)$ and $Q^G(m,p)$ are polynomials in the numerator and the denominator,

$$\mathcal{P}^G(m,p) := \prod_{v_i \in \mathcal{V}} P_i(m,p) \prod_{l \in \mathcal{L}} P_l(m,p) =$$

$$= \sum_{s=0}^{d^G} \mathcal{P}_s^G(m,p), \tag{3.2}$$

$$Q^G(m,p)_\epsilon := \prod_{l \in \mathcal{L}} (\mu_{l\epsilon} - p_l^2)^{\lambda_l},$$

$$\mu_{l\epsilon} := m_l^2 - i\epsilon_l, \; m_l \geq 0, \; \epsilon_l > 0, \; \lambda_l \in \mathbb{N}_+, \; \forall l \in \mathcal{L},$$

$\delta^G(p,k)$ is a product of vertex δ-functions

$$\delta^G(p,k) := \prod_{v_i \in \mathcal{V}} \delta_i(p,k),$$

$$\delta_i(p,k) := \delta\Big(\sum_{l \in \mathcal{L}} e_{il} p_l + \sum_{e \in \mathcal{E}} e_{ie} k_e\Big); \tag{3.3}$$

$|\mathcal{A}|$ is a number of elements of some finite set $\mathcal{A}$; $\mathbb{N}_+$ is the set of positive integers; $\mathcal{P}_s^G(m,p)$, $s = 0, \ldots, d^G$, are s-degree homogeneous polynomials in internal momenta p_l, $l \in \mathcal{L}$; $P_i(m,p)$, and $P_l(m,p)$ are multiplicative generating polynomials of the numerator $\mathcal{P}^G(m,p)$ that correspond to the vertex v_i-contribution $V_i(m,p,k)$ and to the internal line l-contribution $\Delta_l(m,p)_\epsilon$, respectively:

$$V_i(m,p,k) := P_i(m,p)\delta_i(p,k),$$

$$deg_p P_i(m,p) =: d_i \geq 0, \quad \forall v_i \in \mathcal{V},$$

$$\Delta_l(m,p)_\epsilon := \frac{P_l(m,p)}{(\mu_{l\epsilon} - p_l^2)^{\lambda_l}}, \tag{3.4}$$

$$deg_p P_l(m,p) =: d_l \geq 0, \quad \forall l \in \mathcal{L}.$$

The non-degenerate metric form

$$\mathrm{diag}\, g^{\mu\nu} := (\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}), \tag{3.5}$$

$$p + q = n = 2r_n + \delta_n, \quad \delta_n = 0,1, \quad r_n \in \{0 \cup \mathbb{N}_+\},$$

is used for each n-dimensional p_l-integration in Eq. (3.1).

3.2. Two characteristics

$$\nu^G := 2\,\omega^G + d^G, \qquad \omega^G := (n/2)|\mathcal{C}| - \lambda_{\mathcal{L}},$$

$$|\mathcal{C}| = |\mathcal{L}| - |\mathcal{V}| + 1, \quad \lambda_{\mathcal{L}} := \sum_{l \in \mathcal{L}} \lambda_l,$$

$$d^G := d_{\mathcal{V}} + d_{\mathcal{L}} = \sum_{v_i \in \mathcal{V}} d_i + \sum_{l \in \mathcal{L}} d_l, \tag{3.6}$$

of integral (3.1) are especially important. Here, $|\mathcal{C}|$ is the number of independent circuits of the graph G. There exist analogous characteristics for all one-particle irreducible (1PI) subgraphs $\overline{G} \subset G$. If $\nu^G \geq 0$ or $\nu^{\overline{G}} \geq 0$ for some 1PI $\overline{G} \subset G$, the integral is UV-divergent and a renormalization is needed [28, 30].

While Eqs. (3.1)–(3.3) are identical to the well-known representation in terms of vertex-line contributions,

$$\delta^G(p,k)\frac{\mathcal{P}^G(m,p)}{Q^G(m,p)_\epsilon} = \prod_{v_i \in \mathcal{V}} V_i(m,p,k) \prod_{l \in \mathcal{L}} \Delta_l(m,p)_\epsilon,$$

they are more suited for practical calculations. The universal decomposition of $\mathcal{P}^G(m,p)$ in terms of s-degree homogeneous p-polynomials $\mathcal{P}_s^G(m,p)$ is very useful.

3.3. We use of the Fock–Schwinger exponential α-representation (see, for example, [59–61]) along with the Hepp regularization [30] that introduces parameters $r_l > 0$ in the vicinity of $\alpha_l = 0$, $\forall l \in \mathcal{L}$,

$$\frac{1}{(\mu_{l\epsilon} - p_l^2)^{\lambda_l}} = \lim_{r_l \to 0} \int_{r_l}^{\infty} \frac{d\alpha_l\, \alpha_l^{\lambda_l - 1} i^{\lambda_l}}{\Gamma(\lambda_l)} e^{-i\alpha_l(\mu_{l\epsilon} - p_l^2)},$$

$$p_l^\tau = (-i\partial/\partial q_{l\tau}) e^{i(p_l \cdot q_l)}|_{q_l = 0}, \tag{3.7}$$

$$(p_l \cdot q_l) := p_{l\tau} q_{l\sigma} g^{\tau\sigma}, \quad 0 < r_l \leq \alpha_l \leq \infty, \; \forall l \in \mathcal{L}.$$

Then the ratio of polynomials $\mathcal{P}^G(m,p)/Q^G(m,p)_\epsilon$ in Eqs. (3.1)–(3.2) can be represented in the form

$$\frac{\mathcal{P}^G(m,p)}{Q^G(m,p)_\epsilon} = \lim_{\substack{r_l \to 0 \\ \forall l \in \mathcal{L}}} \Bigg\{ \int_{R_+^{|\mathcal{L}|}(\mathbf{r})} dv^G(\alpha)\, i^{\lambda_{\mathcal{L}}} \times$$

$$\times \sum_{s=0}^{d^G} \mathcal{P}_s^G(m, -i\partial/\partial q_{\mathcal{L}})\, e^{-iM_\epsilon + iW_{p_{\mathcal{L}}}^{q_{\mathcal{L}}}} \Bigg\}\Bigg|_{\substack{q_l = 0 \\ \forall l \in \mathcal{L}}},$$

$$W_{p_{\mathcal{L}}}^{q_{\mathcal{L}}} := (p_{\mathcal{L}}^T \cdot \alpha_{\mathcal{L}\mathcal{L}} p_{\mathcal{L}}) + (p_{\mathcal{L}}^T \cdot q_{\mathcal{L}}) = \tag{3.8}$$

$$= \sum_{l\in\mathcal{L}} \alpha_l p_l^2 + \sum_{l\in\mathcal{L}} (p_l \cdot q_l), \quad [\alpha_{\mathcal{L}\mathcal{L}}]_{ll'} := \alpha_l\, \delta_{ll'}.$$

In Eq. (3.8), $p_{\mathcal{L}}$ and $q_{\mathcal{L}}$ are $(|\mathcal{L}|\times n)$-dimensional actual and auxiliary internal momenta column-vectors associated with the set of internal lines, $\mathcal{L}$, of a graph G; τ is the transpose sign, so $p_{\mathcal{L}}^T$ is the row-vector; $\alpha_{\mathcal{L}\mathcal{L}}$ is the $|\mathcal{L}|$-dimensional diagonal matrix of α-parameters; and $\lambda_{\mathcal{L}}$ is defined in Eq.(3.6). Here, the integration measure $dv^G(\alpha)$, the integration region $R_+^{|\mathcal{L}|}(\mathbf{r})$, and the α-parametric function $M_\epsilon \equiv M(m,\alpha)_\epsilon$ which is the linear form in the square of internal masses with $i\epsilon$-damping are defined as

$$dv^G(\alpha) := \prod_{l\in\mathcal{L}} \left(\frac{d\alpha_l\, \alpha_l^{\lambda_l - 1}}{\Gamma(\lambda_l)} \right),$$

$$R_+^{|\mathcal{L}|}(\mathbf{r}) := \{\alpha_l |\, 0 < r_l \leq \alpha_l \leq \infty,\ \forall l\in\mathcal{L},\},$$

$$M_\epsilon := \sum_{l\in\mathcal{L}} \alpha_l \mu_{l\epsilon}, \quad \mu_{l\epsilon} := (m_l^2 - i\epsilon_l). \tag{3.9}$$

Now, substituting Eq. (3.8) in Eq. (3.1) and interchanging the order of integration in p_l and α_l, $\forall l\in\mathcal{L}$, we obtain the very useful representation of the regularized-by-Hepp integral $I^G(m,k)_\epsilon^{\mathbf{r}}$. Its integrand is the $(|\mathcal{L}|\times n)$-dimensional pseudo-Euclidean Gaussian-like expression but in the mutually dependent variables $p_l, \forall l\in\mathcal{L}$,

$$I^G(m,k)_\epsilon^{\mathbf{r}} := c^G \int\limits_{R_+^{|\mathcal{L}|}(\mathbf{r})} dv^G(\alpha) \times$$

$$\times \sum_{s=0}^{d^G} \mathcal{P}_s^G(m, -i\partial/\partial q_{\mathcal{L}}) \int\limits_{-\infty}^{\infty} (d^n p)^{\mathcal{L}} \delta^G(p_{\mathcal{L}}, k_{\mathcal{E}})\, i^{\lambda_{\mathcal{L}}} \times$$

$$\times e^{-iM_\epsilon + iW_{p_{\mathcal{L}}}^{q_{\mathcal{L}}}} \Big|_{\substack{q_l = 0\\ \forall l\in\mathcal{L}}}. \tag{3.10}$$

The set of internal lines, $\mathcal{L}$, can be always decomposed (as a rule, in more than one way) into two mutually disjoint subsets, $\mathcal{L} = \mathcal{M}\cup\mathcal{N}$ and $\mathcal{M}\cap\mathcal{N} = \emptyset$ which determine some *skeleton tree*, i.e., *1-tree* subgraph $G(\mathcal{V}, \mathcal{M}\cup\mathcal{E})$, with $|\mathcal{M}| = |\mathcal{V}|-1$, and the corresponding *co-tree* subgraph $G(\mathcal{V}, \mathcal{N}\cup\mathcal{E})$, with $|\mathcal{N}| = |\mathcal{L}|-|\mathcal{V}|+1 = |\mathcal{C}|$ of the graph G. Supports of all $\delta_i(p_{\mathcal{L}}, k_{\mathcal{E}})$-functions, $\forall v_i \in \mathcal{V}$, (see Eq. (3.3)) are defined by Eqs. (3.11) and are equivalent to the matrix relations given in Eqs. (3.12) and Sec. 5,

$$\textstyle\sum_{l\in\mathcal{L}} e_{il} p_l + \sum_{e\in\mathcal{E}} e_{ie} k_e = 0,\ \forall v_i \in \mathcal{V}, \tag{3.11}$$

$$e_{\{\mathcal{V}/j\}\mathcal{M}} p_{\mathcal{M}} + e_{\{\mathcal{V}/j\}\mathcal{N}} p_{\mathcal{N}} + e_{\{\mathcal{V}/j\}\mathcal{E}} k_{\mathcal{E}} = 0_{\{\mathcal{V}/j\}},$$

$$e_{j\mathcal{M}} p_{\mathcal{M}} + e_{j\mathcal{N}} p_{\mathcal{N}} + e_{j\mathcal{E}} k_{\mathcal{E}} = 0,\ v_j - \text{the basis vertex},$$

$$p_{\mathcal{L}} = e_{\mathcal{L}\mathcal{N}} p_{\mathcal{N}} + e_{\mathcal{L}\mathcal{E}}(j) k_{\mathcal{E}},$$

$$p_{\mathcal{M}} = e_{\mathcal{M}\mathcal{N}} p_{\mathcal{N}} + e_{\mathcal{M}\mathcal{E}}(j) k_{\mathcal{E}}. \tag{3.12}$$

Thus, the $(|\mathcal{M}|\times n)$-dimensional integration by means of $\delta_i(p_{\mathcal{L}}, k_{\mathcal{E}})$-functions, $\forall v_i \in \mathcal{V}/j$ (this is equivalent to make use of Eqs. (3.12)), gives rise to the intermediate α-parametric representation

$$I^G(m,k)_\epsilon^{\mathbf{r}} := \delta^G(k_{\mathcal{E}}) c^G \int\limits_{R_+^{|\mathcal{L}|}(\mathbf{r})} dv^G(\alpha) \times$$

$$\times \sum_{s=0}^{d^G} \mathcal{P}_s^G(m, -i\partial/\partial q_{\mathcal{L}}) \int\limits_{-\infty}^{\infty} (d^n p)^{\mathcal{N}}\, i^{\lambda_{\mathcal{L}}} \times$$

$$\times e^{-iM_\epsilon + iW_{\mathcal{N},\mathcal{E}}^{q_{\mathcal{L}}}} \Big|_{\substack{q_l = 0\\ \forall l\in\mathcal{L}}}, \tag{3.13}$$

$$W_{\mathcal{N},\mathcal{E}}^{q_{\mathcal{L}}} := (p_{\mathcal{N}}^T \cdot C_{\mathcal{N}\mathcal{N}}(\alpha) p_{\mathcal{N}}) + 2(f_{\mathcal{N}}^T \cdot p_{\mathcal{N}}) +$$

$$+ (k_{\mathcal{E}}^T \cdot E_{\mathcal{E}\mathcal{E}}(j|\alpha) k_{\mathcal{E}}) + (q_{\mathcal{L}}^T \cdot e_{\mathcal{L}\mathcal{E}}(j) k_{\mathcal{E}}),$$

$$f_{\mathcal{N}} := \Pi_{\mathcal{E}\mathcal{N}}^T(j|\alpha) k_{\mathcal{E}} + \tfrac{1}{2} e_{\mathcal{L}\mathcal{N}}^T q_{\mathcal{L}},$$

$$\delta^G(k_{\mathcal{E}}) := \delta\big(\textstyle\sum_{e\in\mathcal{E}} e(v^*)_e k_e\big).$$

The explicit forms and some properties of the matrices $e_{\mathcal{L}\mathcal{N}}$, $e_{\mathcal{L}\mathcal{E}}(j)$, $C_{\mathcal{N}\mathcal{N}}(\alpha)$, $E_{\mathcal{E}\mathcal{E}}(j|\alpha)$, and $\Pi_{\mathcal{E}\mathcal{N}}(j|\alpha)$ are given in Eqs. (5.1)-(5.4).

The change of the variables $p_{\mathcal{N}}$ by means of a nondegenerate linear transformation such that

$$p_{\mathcal{N}} = B_{\mathcal{N}\mathcal{N}}(\alpha)\tilde{p}_{\mathcal{N}} - B_{\mathcal{N}\mathcal{N}}(\alpha) B_{\mathcal{N}\mathcal{N}}^T(\alpha) f_{\mathcal{N}},$$

$$B^T_{\mathcal{NN}}(\alpha)C_{\mathcal{NN}}(\alpha)B_{\mathcal{NN}}(\alpha) = 1_{\mathcal{NN}},$$

$$B_{\mathcal{NN}}(\alpha)B^T_{\mathcal{NN}}(\alpha) = C^{-1}_{\mathcal{NN}}(\alpha),$$

$$\det B_{\mathcal{NN}}(\alpha) = [\det C_{\mathcal{NN}}(\alpha)]^{-1/2} =: \Delta(\alpha)^{-1/2},$$

$$(d^n p)^{\mathcal{N}} = (d^n \tilde{p})^{\mathcal{N}} |\det B_{\mathcal{NN}}(\alpha)|^n \, |\det g|^{|\mathcal{N}|}$$

$$= (d^n \tilde{p})^{\mathcal{N}}/\Delta(\alpha)^{n/2}, \; \det g = (-1)^q, \tag{3.14}$$

reduces Eqs. (3.13)–(3.14) to the form

$$I^G(m,k)^{\mathbf{r}}_{\epsilon} := \delta^G(k_{\mathcal{E}})c^G \int\limits_{R_+^{|\mathcal{L}|}(\mathbf{r})} \frac{dv^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \mathcal{P}^G_s(m, -i\partial/\partial q_{\mathcal{L}}) \int\limits_{-\infty}^{\infty} (d^n \tilde{p})^{\mathcal{N}} e^{i(\tilde{p}^T_{\mathcal{N}} \cdot \tilde{p}_{\mathcal{N}})} \, i^{\lambda_{\mathcal{L}}} \times$$

$$\times e^{-iM_\epsilon + i\widetilde{W}^{q_{\mathcal{L}}}_{\mathcal{E}}}\Big|_{\substack{q_l=0\\ \forall l\in\mathcal{L}}}, \tag{3.15}$$

$$\widetilde{W}^{q_{\mathcal{L}}}_{\mathcal{E}} := -(f^T_{\mathcal{N}} \cdot C^{-1}_{\mathcal{NN}}(\alpha) f_{\mathcal{N}}) +$$

$$+(k^T_{\mathcal{E}} \cdot E_{\mathcal{EE}}(j|\alpha)k_{\mathcal{E}}) + (q^T_{\mathcal{L}} \cdot e_{\mathcal{LE}}(j)k_{\mathcal{E}}) =$$

$$= (k^T_{\mathcal{E}} \cdot A_{\mathcal{EE}}(j|\alpha)k_{\mathcal{E}}) +$$

$$+(q^T_{\mathcal{L}} \cdot Y_{\mathcal{LE}}(j|\alpha)k_{\mathcal{E}}) - \tfrac{1}{4}(q^T_{\mathcal{L}} \cdot X_{\mathcal{LL}}(\alpha)q_{\mathcal{L}}).$$

With regard for the formula

$$\int\limits_{-\infty}^{\infty} dt e^{\pm it^2} = \pi^{1/2} e^{\pm i\pi/4},$$

which is followed from [62], Ch. 1.5., Eqs. (31) and (32), we find

$$\int\limits_{-\infty}^{\infty} d^n \tilde{p}_l e^{i\tilde{p}_l^2} = \pi^{n/2} e^{i(p-q)\pi/4} = \pi^{n/2} i^{(p-n/2)},$$

$$\int\limits_{-\infty}^{\infty} (d^n \tilde{p})^{\mathcal{N}} e^{i(\tilde{p}^T_{\mathcal{N}} \cdot \tilde{p}_{\mathcal{N}})} i^{\lambda_{\mathcal{L}}} = (\pi^{n/2} i^p)^{|\mathcal{N}|} i^{-\omega}. \tag{3.16}$$

So, all the integrations over n-dimensional pseudo-Euclidean momenta in the (p,q)-metric are performed. Thus, any FA (3.1) leads to the α-parametric representation in the fully exponential form,

$$I^G(m,k)^{\mathbf{r}}_{\epsilon} := (2\pi)^n \delta^G(k_{\mathcal{E}}) a^G \int\limits_{R_+^{|\mathcal{L}|}(\mathbf{r})} \frac{dv^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \mathcal{P}^G_s(m, -i\partial/\partial q_{\mathcal{L}})\, e^{-iM_\epsilon + i\widetilde{W}^{q_{\mathcal{L}}}_{\mathcal{E}}}\Big|_{\substack{q_l=0\\ \forall l\in\mathcal{L}}}, \tag{3.17}$$

$$\widetilde{W}^{q_{\mathcal{L}}}_{\mathcal{E}} := (k^T_{\mathcal{E}} \cdot A_{\mathcal{EE}}(j|\alpha)k_{\mathcal{E}}) +$$

$$+(q^T_{\mathcal{L}} \cdot Y_{\mathcal{LE}}(j|\alpha)k_{\mathcal{E}}) - \tfrac{1}{4}(q^T_{\mathcal{L}} \cdot X_{\mathcal{LL}}(\alpha)q_{\mathcal{L}}),$$

$$a^G := c^G (\pi^{n/2} i^p)^{|\mathcal{N}|} (2\pi)^{-n} i^{-\omega}, \; |\mathcal{N}| = |\mathcal{C}|,$$

where the n-dimensional auxiliary momenta, $q_l, l \in \mathcal{L}$, are still used. The explicit form and important properties of matrices $A_{\mathcal{EE}}(j|\alpha)$, $Y_{\mathcal{LE}}(j|\alpha)$, and $X_{\mathcal{LL}}(\alpha)$ are given in Eqs. (5.5)–(5.13). Some properties of them are new.

Next, the following two operations must be carried out: i) to differentiate the exponential function $\exp\{i(q^T_{\mathcal{L}} \cdot Y_{\mathcal{LE}}(j|\alpha)k_{\mathcal{E}}) - i/4\,(q^T_{\mathcal{L}} \cdot X_{\mathcal{LL}}(\alpha)q_{\mathcal{L}})\}$ by means of the s-homogeneous differential polynomials $\mathcal{P}^G_s(m, -i\partial/\partial q_{\mathcal{L}})$ in $-i\partial/\partial q_{l\sigma}$, $l \in \mathcal{L}$, $\sigma \in \{1,\dots,n\}$, $1 \le s \le d^G$; ii) to put $q_{l\sigma} = 0, \forall l \in \mathcal{L}$, $\forall \sigma \in \{1,\dots,n\}$, and $\forall s \in \{0,1,\dots,d^G\}$. Finally, we obtain the important α-parametric representation for the general FA (3.1),

$$I^G(m,k)^{\mathbf{r}}_{\epsilon} := (2\pi)^n \delta^G(k_{\mathcal{E}}) b^G \int\limits_{R_+^{|\mathcal{L}|}(\mathbf{r})} dv^G(\alpha) \times$$

$$\times \frac{1}{\Delta^{n/2}} \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m,\alpha,k) i^{-\omega-j} e^{-iM_\epsilon + iA}, \tag{3.18}$$

$$A \equiv A(\alpha,k) := (k^T_{\mathcal{E}} \cdot A_{\mathcal{EE}}(j|\alpha)k_{\mathcal{E}}),$$

$$b^G := c^G (\pi^{n/2} i^p)^{|\mathcal{C}|} (2\pi)^{-n}, \; |\mathcal{C}| = |\mathcal{N}|,$$

$$\mathcal{P}^G_{sj}(m,\rho\alpha,\tau k) = \rho^{-j} \tau^{s-2j}\, \mathcal{P}^G_{sj}(m,\alpha,k).$$

Here, $[s/2]$ means the largest integer $\leq s/2$, i.e., the integer part of $s/2$; the quadratic Kirchhoff form $A(\alpha, k)$ in external momenta k_e, $e \in \mathcal{E}$, and the Kirchhoff determinant $\Delta(\alpha) := \det C_{\mathcal{N}\mathcal{N}}(\alpha)$ are defined by the topological structure of a graph G and are homogeneous functions of the first and $|\mathcal{C}|$th degrees in α, respectively, see Sec. 5 for more details. The quantities $\mathcal{P}^G_{sj}(m, \alpha, k)$ are homogeneous k-polynomials in external momenta k_e, $e \in \mathcal{E}$, of the degree $s - 2j$, $j = 0, 1, \ldots, [s/2]$. They are α-parametric images of homogeneous polynomials $\mathcal{P}^G_s(m, p)$. Each monomial of $\mathcal{P}^G_{sj}(m, \alpha, k)$ is a product of $s - 2j$ linear Kirchhoff forms $Y_l(\alpha, k) := \sum_{e\in\mathcal{E}} Y_{le}(\alpha)k_e$ and j line-correlator functions $X_{ll'}(\alpha)$, $l, l' \in \mathcal{L}$, of a graph G. The parametric functions $Y_l(\alpha, k)$ and $X_{ll'}(\alpha)$ are homogeneous functions of the 0th and (-1)st degree in α, respectively (see Sections 4 and 5 for more details).

By introducing the new variables,

$$\alpha_l = \rho\alpha'_l, \quad \forall l \in \mathcal{L}/j, \quad \alpha_j = \rho\big(1 - \textstyle\sum_{l\in\mathcal{L}/j}\alpha'_l\big),$$

$$\sum_{l\in\mathcal{L}} \alpha'_l = 1, \quad \prod_{l\in\mathcal{L}} d\alpha_l = \rho^{|\mathcal{L}|-1} d\rho \prod_{l\in\mathcal{L}/j} d\alpha'_l,$$

$$R^{|\mathcal{L}|}_{+}(\mathbf{r}\,) \to R^1_{+}(r) \times \Sigma^{|\mathcal{L}|-1}, \ r > 0, \tag{3.19}$$

and assuming that $r_l = r > 0$, $\forall l \in \mathcal{L}$, we can perform the integration over the variable ρ, $0 < r \leq \rho \leq \infty$ by using [63], see Ch. 6.3., eq.(3) or Ch. 6.5., Eq. (29),

$$I^G(m, k)^r_\epsilon := (2\pi)^n \delta^G(k_{\mathcal{E}}) b^G \int\limits_{\Sigma^{|\mathcal{L}|-1}} \frac{d\mu^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m, \alpha, k) \mathcal{F}^r_{sj}(\omega; V_\epsilon),$$

$$\mathcal{F}^r_{sj}(\omega; V_\epsilon) := i^{-\omega-j} \int\limits_r^\infty d\rho\, \rho^{-\omega-j-1} e^{-i\rho V_\epsilon} =$$

$$= V_\epsilon^{\omega+j}\Gamma(-\omega - j; irV_\epsilon), \quad V_\epsilon := M_\epsilon - A, \tag{3.20}$$

$$\omega := (n/2)|\mathcal{C}| - \lambda_{\mathcal{L}}, \quad b^G := c^G(\pi^{n/2} i^p)^{|\mathcal{C}|}(2\pi)^{-n}.$$

Here, p involved in b^G is the number of positive squares in the space-time metric $g^{\mu\nu}$. The integration measure $d\mu^G(\alpha)$ and the integration domain $\Sigma^{|\mathcal{L}|-1}$ of the simplex type are defined as

$$d\mu^G(\alpha) := \delta\big(1 - \sum_{l\in\mathcal{L}} \alpha_l\big) \prod_{l\in\mathcal{L}} \Big(\frac{d\alpha_l\, \alpha_l^{\lambda_l - 1}}{\Gamma(\lambda_l)}\Big),$$

$$\Sigma^{|\mathcal{L}|-1} := \{\alpha_l |\, \alpha_l \geq 0, \ \forall l \in \mathcal{L}, \ \sum_{l\in\mathcal{L}} \alpha_l = 1\}. \tag{3.21}$$

In Eq. (3.20), $\Gamma(\alpha; x)$ is one of the two incomplete gamma-functions appearing in the decomposition, $\Gamma(\alpha) = \Gamma(\alpha; x) + \gamma(\alpha; x)$, see [64], Ch. 9.1., Eqs. (1)–(2), such that, at $\mathrm{Re}\,\alpha > 0$, $\Gamma(\alpha; 0) = \Gamma(\alpha)$, $\gamma(\alpha; 0) = 0$, where $\Gamma(\alpha)$ is an ordinary gamma-function.

It is worth noting that we actually have the regularization which combines three ones: i) the Hepp regularization [30], (due to a change in the region of integration over the auxiliary variable ρ); ii) the analytic regularization [80], (due to the complexification of the parameter $\lambda_{\mathcal{L}}$, the half-degree of the denominator polynomial); iii) the dimensional regularization [39,53,54] and other suitable references in them, (due to the complexification of the parameter n, the space-time dimension). Recall that $\lambda_{\mathcal{L}}$ and n are constituents of ω.

For convergent FAs, the quantities $\omega + j < 0$, $\forall j \in \{0, 1, \ldots, [d^G/2\}$, and there exists the limit $r \to 0$. After passing to the limit $r \to 0$ in Eq. (3.20), we obtain

$$I^G(m, k)_\epsilon := (2\pi)^n \delta^G(k_{\mathcal{E}}) b^G \int\limits_{\Sigma^{|\mathcal{L}|-1}} \frac{d\mu^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m, \alpha, k) \mathcal{F}_{sj}(\omega; M_\epsilon, A),$$

$$\mathcal{F}_{sj}(\omega; M_\epsilon, A) := i^{-\omega-j} \int\limits_0^\infty d\rho\, \rho^{-\omega-j-1} e^{-i\rho\,(M_\epsilon - A)} =$$

$$= M_\epsilon^{\omega+j}(1 - Z_\epsilon)^{\omega+j}\Gamma(-\omega - j) \tag{3.22}$$

$$= M_\epsilon^{\omega+j} \sum_{k=0}^\infty \Gamma(-\omega - j + k)\, \frac{Z^k_\epsilon}{k!}, \quad Z_\epsilon := A/M_\epsilon.$$

It is easy to verify that the basic functions $\mathcal{F}_{sj}(\omega; M_\epsilon, A)$ satisfy Eqs. (2.1), (2.3), (2.6)–(2.8), and (2.11).

In the case of divergent FAs, for which $\omega + j \geq 0$ at least for one $j \in \{0, 1, \ldots, [d^G/2\}$, the limit $r \to 0$ does not exist. In this case, expressions (3.18)–(3.21) strictly defined in the region $R_+^{|\mathcal{L}|}(r) := R_+^1(r) \times \Sigma^{|\mathcal{L}|-1}$ must be made meaningful in a wider region $R_+^{|\mathcal{L}|} := R_+^1 \times \Sigma^{|\mathcal{L}|-1}$, where $R_+^1 := R_+^1(r)|_{r=0}$.

The Bogoliubov–Parasiuk subtraction procedure used for this purpose replaces $I^G(m,k)^r_\epsilon$ by

$$(R_0^\nu I)^G(m,k)_\epsilon = (2\pi)^n \delta^G(k_\mathcal{E}) b^G \times$$

$$\times \int\limits_{R_+^{|\mathcal{L}|}} dv^G(\alpha) (R_0^\nu \mathcal{I})^G(m,\alpha,k)_\epsilon, \tag{3.23}$$

$$(R_0^\nu \mathcal{I})^G(m,\alpha,k)_\epsilon :=$$

$$:= \mathcal{I}^G(m,\alpha,k)_\epsilon - \sum_{\beta=0}^{\nu} \frac{1}{\beta\,!} \frac{\partial^\beta}{\partial\tau^\beta} \mathcal{I}^G(m,\alpha,\tau k)_\epsilon \bigg|_{\tau=0} =$$

$$= \frac{1}{\nu\,!} \int\limits_0^1 d\tau (1-\tau)^\nu \frac{\partial^{\nu+1}}{\partial\tau^{\nu+1}} \mathcal{I}^G(m,\alpha,\tau k)_\epsilon, \tag{3.24}$$

where the subtraction operations under the integral sign are performed by using the Schlömilch integro-differential formula (see the 2nd line of Eq. (3.24)) for the remainder term of Maclaurin's series. First, this formula was applied explicitly to FAs in the Parasiuk paper [8]. Although this expression guarantees a compact representation of the subtraction procedure, it is, nevertheless, inconvenient for computational purposes, because it involves the additional integration and differentiations in the integrand. The expression in the 1st line of Eq. (3.24) is all the more inconvenient for these purposes, since every term on the right-hand side of it may be associated with a divergent integral.

At the same time, the algorithm proposed and applied in [31–39] is based on the observation (see [64], Ch. 9.2., Eqs. (16, 17, 18)) that,

$$e^x - \sum_{k=0}^{\nu_{sj}} \frac{x^k}{k\,!} = e^x \tilde{\gamma}(1+\nu_{sj};x),\ \tilde{\gamma}(\alpha;x) := \frac{\gamma(\alpha;x)}{\Gamma(\alpha)},$$

$$\sum_{k=0}^{\nu_{sj}} \frac{x^k}{k\,!} = e^x \tilde{\Gamma}(1+\nu_{sj};x),\ \tilde{\Gamma}(\alpha;x) := \frac{\Gamma(\alpha;x)}{\Gamma(\alpha)},$$

$$\tilde{\gamma}(\alpha;x) + \tilde{\Gamma}(\alpha;x) = 1. \tag{3.25}$$

Now, if we use: the explicit form of the integrand in Eq. (3.18); the homogeneous properties for parametric functions in $k_e, e \in \mathcal{E}$, see Eqs. (5.8); the 1st line of Eq. (3.24); the 1st line of Eq. (3.25); and the relation

$$\sum_{\beta=0}^{\nu} \frac{1}{\beta\,!} \frac{\partial^\beta}{\partial\tau^\beta} \{\tau^{s-2j} e^{i\tau^2 A}\} \big|_{\tau=0} = \sum_{k=0}^{\nu_{sj}} \frac{(iA)^k}{k\,!},$$

$$\nu_{sj} := [(\nu - s)/2] + j, \tag{3.26}$$

we arrive at the multiplicative realization of the subtraction procedure in the integrand of Eq. (3.23) for the regular value of the general FA (3.1),

$$(R_0^\nu \mathcal{I})^G(m,\alpha,k)_\epsilon = \frac{1}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m,\alpha,k) i^{-\omega-j} e^{-iV_\epsilon} \tilde{\gamma}(1+\nu_{sj}; iA). \tag{3.27}$$

The integral in Eq. (3.23) with integrand (3.27) at $\nu \geq \nu^G$ is now well-defined in the domain $R_+^{|\mathcal{L}|}$. The substitution of (3.27) into integral (3.23) and the change of integration variables according to Eq. (3.19) give rise to the expression

$$(R_0^\nu I)^G(m,k)_\epsilon = (2\pi)^n \delta^G(k)\, b^G \int\limits_{\Sigma^{|\mathcal{L}|-1}} \frac{d\mu^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m,\alpha,k) (R_0^\nu \mathcal{F})_{sj}(\omega; M_\epsilon, A),$$

$$(R_0^\nu \mathcal{F})_{sj}(\omega; M_\epsilon, A) :=$$

$$:= i^{-\omega-j} \int\limits_0^\infty d\rho\, \rho^{-\omega-j-1} e^{-i\rho(M_\epsilon - A)} \tilde{\gamma}(1+\nu_{sj}; i\rho\, A) =$$

$$= M_\epsilon^{\omega+j} \frac{\Gamma(\lambda_{sj})}{\Gamma(2+\nu_{sj})} Z_\epsilon^{1+\nu_{sj}} {}_2F_1(1, \lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon),$$

$$\nu_{sj} := [(\nu - s)/2] + j,\ \lambda_{sj} := -\,\omega - j + 1 + \nu_{sj}. \tag{3.28}$$

The integration over ρ in (3.28) is performed with the use of the formula (see [65], Ch. 17.3., Eq. (15))

$$\int\limits_0^\infty dx\, x^{\mu-1} e^{-vx} \tilde{\gamma}(\nu; a\,x) =$$

$$= \frac{a^{\nu}\Gamma(\mu+\nu)}{(a+v)^{\mu+\nu}\Gamma(1+\nu)}\, {}_2F_1\big(1, \mu+\nu;\, 1+\nu;\, \frac{a}{a+v}\big),$$

$\mathrm{Re}\,(a+v) > 0,\ \mathrm{Re}\,v > 0,\ \mathrm{Re}\,(\mu+\nu) > 0.$

3.4. So, using the properties of special functions substantially, the author has obtained [31, 32, 34, 36, 37, 39–41, 44, 45] high-efficiency formulas which realize an analytical continuation (in the variables ω^G and ν^G) of the FAs which are represented first in Eqs. (3.1)–(3.3) by UV-divergent integrals and are given finally in Eqs. (3.28)–(3.30) as convergent ones. As a result, we have the following α-parametric integral representation:

$$\begin{bmatrix} I^G(m,k)_\epsilon \\ (R_0^\nu I)^G(m,k)_\epsilon \end{bmatrix} = (2\pi)^n \delta^G(k)\, b^G \int\limits_{\Sigma^{|\mathcal{L}|-1}} \frac{d\mu^G(\alpha)}{\Delta^{n/2}} \times$$

$$\times \sum_{s=0}^{d^G} \sum_{j=0}^{[s/2]} \mathcal{P}^G_{sj}(m,\alpha,k) \begin{bmatrix} \mathcal{F}_{sj}(\omega; M_\epsilon, A) \\ (R_0^\nu \mathcal{F})_{sj}(\omega; M_\epsilon, A) \end{bmatrix}, \qquad (3.29)$$

for the convergent or dimensionally regularized value $I^G(m,k)_\epsilon$ and for the regular value $(R_0^\nu I)^G(m,k)_\epsilon$ of integral (3.1). The subscripts 0 and superscript ν on R indicate that $(R_0^\nu I)^G(m,k)_\epsilon$ is the regular function in a vicinity of zero values of the external momenta k_e, $e \in \mathcal{E}$, and is evaluated for an renormalization index $\nu = \nu^G$.

The explicit forms of the basic functions $\mathcal{F}_{sj}(\omega; M_\epsilon, A)$ and $(R_0^\nu \mathcal{F})_{sj}(\omega; M_\epsilon, A)$ are as follows:

$$\mathcal{F}_{sj}(\omega; M_\epsilon, A) := M_\epsilon^{\,\omega+j} (1 - Z_\epsilon)^{\,\omega+j}\, \Gamma(-\,\omega - j) =$$

$$= M_\epsilon^{\,\omega+j} \sum_{k=0}^{\infty} \Gamma(-\,\omega - j + k)\, \frac{Z_\epsilon^k}{k!}, \quad Z_\epsilon := A/M_\epsilon,$$

$$(R_0^\nu \mathcal{F})_{sj}(\omega; M_\epsilon, A) := M_\epsilon^{\,\omega+j}\, \Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj}) \times$$

$$\times Z_\epsilon^{\,1+\nu_{sj}}\, {}_2F_1(1, \lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon) =$$

$$= M_\epsilon^{\,\omega+j} \sum_{k=1+\nu_{sj}}^{\infty} \Gamma(-\,\omega - j + k)\, \frac{Z_\epsilon^k}{k!}, \qquad (3.30)$$

$$\nu_{sj} := [(\nu - s)/2] + j = [\omega] + j + \sigma_s,$$

$$\lambda_{sj} := -\,\omega - j + 1 + \nu_{sj} = 1 - \delta_n \delta_{|\mathcal{C}|}/2 + \sigma_s,$$

$$[\omega] := r_n |\mathcal{C}| + \delta_n r_{|\mathcal{C}|} - \lambda_{\mathcal{L}}, \quad \omega = [\omega] + \delta_n \delta_{|\mathcal{C}|}/2,$$

$$\sigma_s := [(\delta_n \delta_{|\mathcal{C}|} + d - s)/2],$$

$$|\mathcal{C}| = 2 r_{|\mathcal{C}|} + \delta_{|\mathcal{C}|},$$

$$\nu = \nu^G, \quad \omega = \omega^G, \quad d = d^G.$$

The quantities $[(\nu - s)/2]$, $[(\nu + 1 - s)/2]$, and $[\omega]$ in Eqs. (3.28)–(3.31) are the integer parts of $(\nu - s)/2$, $(\nu + 1 - s)/2$, and ω, respectively. The subscripts (s,j) on $\mathcal{F}_{sj}$ and $(R_0^\nu \mathcal{F})_{sj}$ just mean that these functions are attached to the homogeneous k-polynomials $\mathcal{P}^G_{sj}(\alpha,m,k)$ of the degree $s - 2j$, $j = 0,\ldots,[s/2]$, in the external momenta k_e, $e \in \mathcal{E}$. The latter are α-images of the homogeneous p-polynomials $\mathcal{P}^G_s(m,p)$ of the degree s appearing in $\mathcal{P}^G(m,p)$, see Eqs. (3.2). The k-polynomials $\mathcal{P}^G_{sj}(\alpha,m,k)$ are constructed by means of the α-parametric functions $Y_l(\alpha,k)$ and $X_{ll'}(\alpha)$, $l,l' \in \mathcal{L}$. The efficient and universal algorithm of building $\mathcal{P}^G_{sj}(\alpha,m,k)$ is presented in Sec. 4. The α-parametric functions $M_\epsilon \equiv M(m,\alpha)_\epsilon$ and $A \equiv A(\alpha,k)$, incoming in Eqs. (3.30) are defined in Eqs. (3.9) and (3.18), respectively. The quantity $M(m,\alpha)_\epsilon$ is the linear form in the square of internal masses with $i\epsilon$-damping. The functions $A(\alpha,k)$ and $Y_l(\alpha,k)$ are known as the quadratic and linear Kirchhoff forms in the external momenta, k_e, $e \in \mathcal{E}$. The function $\Delta \equiv \Delta(\alpha)$ is the Kirchhoff determinant, and the $X_{ll'}(\alpha)$ are the line-correlator functions. The high-efficiency and universal algorithm of finding the α-parametric functions $A(\alpha,k)$, $Y_l(\alpha,k)$, $X_{ll'}(\alpha)$, and $\Delta(\alpha)$ is given in Sec. 5.

3.5. The investigation of a complicated tangle of problems associated, on the one hand, with renormalization methods and, on the other hand, with conserved and broken symmetries, the Ward identities behavior, the Schwinger terms contributions, and quantum anomalies requires finding the renormalized FAs for different divergence indices. For example, the amplitudes involved in the Ward identities have divergence indices ν^G and $\nu^G + 1$.

The regular values $(R_0^{\nu+1} I)^G(m,k)_\epsilon$ calculated for the renormalization index $\nu^G + 1$ once again have form of Eq. (3.29), but with another basic functions $(R_0^{\nu+1}\mathcal{F})_{sj}$:

$$(R_0^{\nu+1}\mathcal{F})_{sj} := M_\epsilon^{\,\omega+j}\, \Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj}) \times$$

$$\times Z_\epsilon^{\,1+\nu^1_{sj}}\, {}_2F_1(1, \lambda^1_{sj};\, 2+\nu^1_{sj};\, Z_\epsilon), \qquad (3.31)$$

$$\nu_{sj}^1 := [(\nu+1-s)/2] + j = [\omega] + \sigma_s^1 + j,$$

$$\lambda_{sj}^1 := -\,\omega - j + 1 + \nu_{sj}^1 = 1 + \sigma_s^1 - \delta_n \delta_{|\mathcal{C}|}/2,$$

$$\sigma_s^1 := [(\delta_n \delta_{|\mathcal{C}|} + d + 1 - s)/2].$$

In general, $(R_0^{\nu+1}\mathcal{F})_{sj} \neq (R_0^{\nu}\mathcal{F})_{sj}$, as far as $\nu_{sj}^1 \neq \nu_{sj}$. The difference between them is the important quantity

$$(\Delta_0^{\nu+1,\nu)}\mathcal{F})_{sj} := (R_0^{\nu+1}\mathcal{F})_{sj} - (R_0^{\nu}\mathcal{F})_{sj} =$$

$$= -\,\Theta_{sj}^{(\nu+1,\nu)} \frac{\Gamma(\lambda_{sj})}{\Gamma(2+\nu_{sj})} M_\epsilon^{\omega+j} Z_\epsilon^{1+\nu_{sj}}, \qquad (3.32)$$

$$\Theta_{sj}^{(\nu+1,\nu)} := H_+(\nu_{sj}^1)\theta_s^{(\nu+1,\nu)},$$

$$\theta_s^{(\nu+1,\nu)} := \nu_{sj}^1 - \nu_{sj} = \sigma_s^1 - \sigma_s = |\, \delta_\nu - \delta_s \,|,$$

$$\nu = 2r_\nu + \delta_\nu, \quad s = 2r_s + \delta_s, \quad \nu, s \in \{0 \cup \mathbb{N}_+\},$$

where $H_+(x)$ is the Heaviside step function such that $H_+(x) = 0,\ x < 0,\ H_+(x) = 1,\ x \geq 0$, and $\delta_\nu, \delta_s := \nu(\mathrm{mod}\,2), s(\mathrm{mod}\,2) = 0, 1$. It is this quantity that allows one to obtain some efficient formulas for calculating the *quantum corrections* (QCs) (i.e., quantum anomalies) to the canonical Ward identities (CWIs) of the most general kind, for example, to those involving canonically non-conserved vector and (or) axial-vector currents for nondegenerate fermion systems (i.e., for systems with different fermion masses). Another very useful quantity that is produced by differences

$$(\Delta_0^{(\nu+2,\nu)}\mathcal{F})_{sj} := (R_0^{\nu+2}\mathcal{F})_{sj} - (R_0^{\nu}\mathcal{F})_{sj} =$$

$$= (R_0^{\nu}\mathcal{F})_{s-2,j} - (R_0^{\nu-2}\mathcal{F})_{s-2,j} = (R_0^{\nu}\mathcal{F})_{s-2,j} - (R_0^{\nu}\mathcal{F})_{sj} =$$

$$= -\,H_+(1+\nu_{sj}) \frac{\Gamma(\lambda_{sj})}{\Gamma(2+\nu_{sj})} M_\epsilon^{\omega+j} Z_\epsilon^{1+\nu_{sj}} \qquad (3.33)$$

is closely related to $(\Delta_0^{(\nu+1,\nu)}\mathcal{F})_{sj}$.

3.6. The expressions given by Eqs. (3.28)–(3.30) have two very important properties.

First, they describe both divergent and convergent FAs in the unified manner. Really, due to the properties [62] Ch. 2.8, eqs.(4,19), i.e., ${}_2F_1(\alpha, \beta; \alpha;\, z) = (1-z)^{-\beta}$ and

$$\lim_{c \to 2-l,\, l=1,2,\ldots} {}_2F_1(a, b;\, c;\, z)/\Gamma(c) =$$

$$= \frac{(a)_{l-1}(b)_{l-1}}{(l-1)!} z^{l-1}\, {}_2F_1(a+l-1, b+l-1;\, l;\, z), \qquad (3.34)$$

in the case $a = 1$, $b = \lambda_{sj} = -\,\omega - j + 1 - l$, $c = 2 - l$, it follows from Eqs. (3.30) and (3.34) that

$$(R_0^{\nu}\mathcal{F})_{sj} = M_\epsilon^{\omega+j}\, \Gamma(-\,\omega - j)\, {}_2F_1(l, -\,\omega - j;\, l;\, Z_\epsilon) =$$

$$= \mathcal{F}_{sj}, \text{ if } \nu_{sj} = -\,l,\ l \in \mathbb{N}_+, \qquad (3.35)$$

i.e., the first relation in Eqs. (2.2).

Second, the basic functions $(R_0^{\nu}\mathcal{F})_{sj} \equiv (R_0^{\nu}\mathcal{F})_{sj}(\omega; M_\epsilon, A)$ of the self-consistently renormalized FAs obey *the same recurrence relations* as the basic functions $\mathcal{F}_{sj} \equiv \mathcal{F}_{sj}(\omega; M_\epsilon, A)$ of convergent or dimensionally regularized FAs. Really, let us multiply the recurrence relation (see [62] Ch. 2.8, Eq. (42))

$$(c-b-1)\, {}_2F_1(a, b;\, c;\, z) + b\, {}_2F_1(a, b+1;\, c;\, z) -$$

$$-(c-1)\, {}_2F_1(a, b;\, c-1;\, z) = 0, \qquad (3.36)$$

between the contiguous Gauss hypergeometric functions ${}_2F_1$ in the case $a = 1$, $b = \lambda_{sj}$, $c = 2 + \nu_{sj}$, by the quantity $M_\epsilon^{\omega+j} Z_\epsilon^{1+\nu_{sj}} \Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj})$. By using the relations $\nu_{s-2,j-1} = \nu_{sj}$, $\lambda_{s-2,j-1}(\omega) = \lambda_{sj}(\omega) + 1$, and $\nu_{s,j-1} = \nu_{sj} - 1$, $\lambda_{s,j-1}(\omega) = \lambda_{sj}(\omega)$, we obtain the recurrence relations

$$M_\epsilon\, (R_0^{\nu}\mathcal{F})_{s-2,j-1} - A\, (R_0^{\nu}\mathcal{F})_{s,j-1} +$$

$$+(\omega + j)\, (R_0^{\nu}\mathcal{F})_{sj} = 0, \qquad (3.37)$$

between the basic functions $(R_0^{\nu}\mathcal{F})_{sj} \equiv (R_0^{\nu}\mathcal{F})_{sj}(\omega; M_\epsilon, A)$, i.e., the second relation in Eqs. (2.1).

3.7. Transformation formulae (see [62] Ch. 2.1.4, Eqs. (22) and (23)) of ${}_2F_1$ give rise to the representations

$$(R_0^{\nu}\mathcal{F})_{sj} = \frac{(-1)\Gamma(\lambda_{sj}) A^{\nu_{sj}}}{\Gamma(2+\nu_{sj}) M_\epsilon^{\lambda_{sj}-1}} \left(\frac{Z_\epsilon}{Z_\epsilon - 1}\right) \times$$

$$\times\, {}_2F_1\left(1, \omega + j + 1;\, 2 + \nu_{sj};\, \frac{Z_\epsilon}{Z_\epsilon - 1}\right), \qquad (3.38)$$

$$(R_0^{\nu}\mathcal{F})_{sj} = (M_\epsilon - A)^{\omega+j} \frac{\Gamma(\lambda_{sj})}{\Gamma(2+\nu_{sj})} Z_\epsilon^{1+\nu_{sj}} \times$$

$$\times\, {}_2F_1(1 + \nu_{sj}, \omega + j + 1;\, 2 + \nu_{sj};\, Z_\epsilon). \qquad (3.39)$$

Equation (3.38) and the behavior of ${}_2F_1(a,b;\,c;\,z)$ in a vicinity $z \to 1_-$ determine completely the asymptotics of the basic functions $(R_0^\nu \mathcal{F})_{sj}$ for $A < 0$ in a vicinity $M_\epsilon \to 0$, i.e., the chiral limit

$$(R_0^\nu \mathcal{F})_{sj} \overset{M_\epsilon \to 0}{\simeq} \frac{(-1)\Gamma(\lambda_{sj}-1)A^{\nu_{sj}}}{\Gamma(1+\nu_{sj})\, M_\epsilon^{\lambda_{sj}-1}},$$

if $\nu_{sj} \geq 0$ and $\lambda_{sj} - 1 > 0$;

$$(R_0^\nu \mathcal{F})_{sj} \overset{M_\epsilon \to 0}{\simeq} \frac{(-1)A^{\nu_{sj}}}{\Gamma(1+\nu_{sj})} \ln\left(1 - A/M_\epsilon\right), \qquad (3.40)$$

if $\nu_{sj} \geq 0$ and $\lambda_{sj} - 1 = 0$;

$$(R_0^\nu \mathcal{F})_{sj} \overset{M_\epsilon \to 0}{\simeq} \Gamma(-\omega - j)(-A)^{\omega+j},$$

if $\nu_{sj} \geq 0$ and $\lambda_{sj} - 1 < 0$ or $\nu_{sj} \leq -1$,

which is equivalent also to the asymptotic behavior of the basic functions in the case $A \to -\infty$, $M_\epsilon \neq 0$. Equations (3.30) yield four different series of values for $\lambda_{sj} - 1$:

$$\lambda_{sj} - 1 = -\delta_n \delta_{|\mathcal{C}|}/2 +$$

$$+(r_d - r_s) + [(\delta_n \delta_{|\mathcal{C}|} + \delta_d - \delta_s)/2], \qquad (3.41)$$

$$\begin{aligned} \lambda_{sj} - 1 \;&=\; (r_d - r_s) - 1/2, \;\; \delta_n\delta_{|\mathcal{C}|} = 1 \,\&\, \delta_s \geq \delta_d; \\ &=\; (r_d - r_s) + 1/2, \;\; \delta_n\delta_{|\mathcal{C}|} = 1 \,\&\, \delta_d > \delta_s; \\ &=\; (r_d - r_s), \qquad\;\; \delta_n\delta_{|\mathcal{C}|} = 0 \,\&\, \delta_d \geq \delta_s; \\ &=\; (r_d - r_s) - 1, \quad\; \delta_n\delta_{|\mathcal{C}|} = 0 \,\&\, \delta_s > \delta_d; \end{aligned} \qquad (3.42)$$

$$d = 2r_d + \delta_d,\;\; s = 2r_s + \delta_s,\;\;\; \delta_n,\; \delta_{|\mathcal{C}|},\; \delta_d,\; \delta_s = 0, 1.$$

It is evident that Eq. (3.39) presents a multiplicative realization of the subtraction procedure explicitly,

$$(R_0^\nu \mathcal{F})_{sj} := \mathcal{F}_{sj} - (S_0^\nu \mathcal{F})_{sj} =$$

$$= \mathcal{F}_{sj}(\omega; M_\epsilon, A)\, (\Pi_0^\nu \mathcal{F})_{sj}(\omega; Z_\epsilon),$$

$$(\Pi_0^\nu \mathcal{F})_{sj}(\omega; Z_\epsilon) := \frac{(-\omega - j)_{1+\nu_{sj}}}{\Gamma(2+\nu_{sj})} Z_\epsilon^{1+\nu_{sj}} \times$$

$$\times {}_2F_1(1+\nu_{sj}, \omega + j + 1; 2 + \nu_{sj}; Z_\epsilon),$$

$$(S_0^\nu \mathcal{F})_{sj} := M_\epsilon^{\omega+j} \sum_{k=0}^{\nu_{sj}} \Gamma(-\omega - j + k)\, \frac{Z_\epsilon^k}{k!}. \qquad (3.43)$$

4. Homogeneous k-Polynomials $\mathcal{P}^G_{sj}(m, \alpha, k)$ of α-Parametric Representation of FAs

4.1. It is evident from Eq. (3.29) that the basic functions $(R_0^\nu \mathcal{F})_{sj}$ and the homogeneous k-polynomials $\mathcal{P}^G_{sj}(m, \alpha, k)$ in external momenta k_e, $e \in \mathcal{E}$, of degree $s - 2j$, $j = 0, 1, \ldots, [s/2]$, are two closely coupled important universal ingredients of the SCR representation of FAs. The latter are α-images of the homogeneous p-polynomials $\mathcal{P}^G_s(m, p)$ in the internal momenta p_l, $l \in \mathcal{L}$, of degree s, $s = 0, 1, \ldots, d^G$, appearing in the numerator polynomial $\mathcal{P}^G(m, p)$ (see Eqs. (3.1)–(3.2)).

Each monomial of $\mathcal{P}^G_{sj}(m, \alpha, k)$ is a product of $s - 2j$ linear Kirchhoff forms $Y_l(\alpha, k) := \sum_{e \in \mathcal{E}} Y_{le}(\alpha) k_e$ and j line-correlator functions $X_{ll'}(\alpha)$, $l, l' \in \mathcal{L}$, of a graph G. The efficient algorithm of finding these expressions from the initial homogeneous p-polynomials $\mathcal{P}^G_s(m, p)$ in the internal momenta p_l, $l \in \mathcal{L}$, of degree $s = 0, 1, \ldots, d^G$, has been elaborated in [31–34]. It resembles Wick relations between time-ordered and normal products of boson fields in quantum field theory. The main steps of this algorithm are as follows.

- The polynomials $\mathcal{P}^G_{s0}(m, \alpha, k)$ are determined as

$$\mathcal{P}^G_{s0}(m, \alpha, k) := \mathcal{P}^G_s(m, p)|_{p_l = Y_l(\alpha, k)},$$

$$j = 0,\;\; s = 0, 1, \ldots, d^G, \qquad (4.1)$$

i.e., by the straightforward substitution $p_l \to Y_l(\alpha, k)$, $\forall l \in \mathcal{L}$, in the polynomials $\mathcal{P}^G_s(m, p)$.

- The polynomials $\mathcal{P}^G_{sj}(m, \alpha, k)$, $j = 1, \ldots, [s/2]$, have the algebraic structure of quantities generated by the Wick formula which represents a T-product of s boson fields in terms of some set of N-products of $s - 2j$ boson fields with j primitive contractions. In this case, the linear Kirchhoff forms $Y^\sigma_l(\alpha, k)$ and their primitive correlators

$$\underbrace{Y^{\sigma_1}_{l_1} \cdots Y^{\sigma_2}_{l_2}} := (-1/2) X_{l_1 l_2}(\alpha) g^{\sigma_1 \sigma_2} \equiv$$

$$\equiv (-1/2) \left(\begin{smallmatrix} \sigma_1 \sigma_2 \\ l_1\, l_2 \end{smallmatrix}\right). \qquad (4.2)$$

play a role of boson fields and contractions, respectively.

4.2. As a result, we come to the following general formulae. As far as the homogeneous p-polynomials $\mathcal{P}^G_s(m, p)$ can be always represented as

$$\mathcal{P}^G_s(m, p) = \sum_{(i) \in G} a^{(i)}_s(m)\, p^{\sigma^{(i)}_1}_{l^{(i)}_1} p^{\sigma^{(i)}_2}_{l^{(i)}_2} \cdots\; p^{\sigma^{(i)}_s}_{l^{(i)}_s},$$

$$l_a^{(i)} \in \mathcal{L},\ a = 1, \dots, s, \tag{4.3}$$

where the coefficients $a_s^{(i)}(m)$ are functions of the masses m_l, $l \in \mathcal{L}$, it is sufficient to find the image of some general monomial entering into the sum over $(i) \in G$ in Eq. (4.3). The calculation according to the above-mentioned Wick-type rule yields

$$p_{l_1^{(i)}}^{\sigma_1^{(i)}} p_{l_2^{(i)}}^{\sigma_2^{(i)}} \cdots\ p_{l_s^{(i)}}^{\sigma_s^{(i)}} \to \sum_{j=0}^{[s/2]} \mathcal{P}^{\sigma_1^{(i)} \cdots\ \sigma_s^{(i)}}_{(l_1^{(i)} \cdots\ l_s^{(i)});j}(\alpha, k),$$

$$\mathcal{P}^{\sigma_1^{(i)} \cdots\ \sigma_s^{(i)}}_{(l_1^{(i)} \cdots\ l_s^{(i)});j}(\alpha, k) =$$

$$= (-2)^{-j} \sum_{d \in (1^{s-2j} 2^j)} \mathcal{P}^{\sigma_{d(1)}^{(i)} \cdots\ \sigma_{d(s)}^{(i)}}_{(l_{d(1)}^{(i)} \cdots\ l_{d(s)}^{(i)});j}(\alpha, k),$$

$$\mathcal{P}^{\sigma_{d(1)}^{(i)} \cdots\ \sigma_{d(s)}^{(i)}}_{(l_{d(1)}^{(i)} \cdots\ l_{d(s)}^{(i)});j}(\alpha, k) := \prod^{s-2j} Y_{l_{d(a)}^{(i)}}^{\sigma_{d(a)}^{(i)}}(\alpha, k) \times$$

$$\times \prod^{j} \big(X_{l_{d(b)}^{(i)} l_{d(c)}^{(i)}}(\alpha) g^{\sigma_{d(b)}^{(i)} \sigma_{d(c)}^{(i)}}\big), \tag{4.4}$$

where the summation in the second equation in (4.4) is extended over all partitions d of $(l_1^{(i)}, l_2^{(i)}, \dots, l_s^{(i)})$ according to the Young scheme $(1^{s-2j} 2^j)$. Then the image of homogeneous p-polynomials $\mathcal{P}_s^G(m, p)$ given by Eq. (4.3) is

$$\mathcal{P}_s^G(m, p) \to \sum_{j=0}^{[s/2]} \mathcal{P}_{sj}^G(m, \alpha, k),$$

$$\mathcal{P}_{sj}^G(m, \alpha, k) = \sum_{(i) \in G} a_s^{(i)}(m) \mathcal{P}^{\sigma_1^{(i)} \cdots\ \sigma_s^{(i)}}_{(l_1^{(i)} \cdots\ l_s^{(i)});j}(\alpha, k). \tag{4.5}$$

In so doing, we arrive at special j-degree homogeneous polynomials in the variables $\binom{\sigma_1 \sigma_2}{l_1\, l_2}$ involved in primitive correlators (see Eq. (4.2)). Polynomials of this type was introduced and named as hafnians by Caianiello [66, 67] in the course of his QED investigations. Hafnians are the counterparts of phaffians and closely connected with permanents. The simplest nontrivial hafnian $\binom{\sigma_1 \sigma_2 \sigma_3 \sigma_4}{l_1\, l_2\, l_3\, l_4}$ of degree 2 is given below in two last lines of Eq. (4.6).

4.3 In view of the very important applied significance of the algorithm of constructing a family of homogeneous k-polynomials $\mathcal{P}_{sj}^G(m, \alpha, k)$ from the initial p-polynomials $\mathcal{P}_s^G(m, p)$, we give some examples:

$$1 \to j = 0 : 1;$$

$$p_l^\sigma \to j = 0 : Y_l^\sigma =: \genfrac{[}{]}{0pt}{}{\sigma}{l};$$

$$p_{l_1}^{\sigma_1} p_{l_2}^{\sigma_2} \to j = 0 : Y_{l_1}^{\sigma_1} Y_{l_2}^{\sigma_2} =: \genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_2}{l_1\, l_2},$$

$$j = 1 : (-\tfrac{1}{2})\{X_{l_1 l_2} g^{\sigma_1 \sigma_2} =: \binom{\sigma_1 \sigma_2}{l_1\, l_2}\};$$

$$p_{l_1}^{\sigma_1} p_{l_2}^{\sigma_2} p_{l_3}^{\sigma_3} \to j = 0 : Y_{l_1}^{\sigma_1} Y_{l_2}^{\sigma_2} Y_{l_3}^{\sigma_3} =: \genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_2 \sigma_3}{l_1\, l_2\, l_3},$$

$$j = 1 : (-\tfrac{1}{2})\big\{\binom{\sigma_1 \sigma_2}{l_1\, l_2}\genfrac{[}{]}{0pt}{}{\sigma_3}{l_3} + \binom{\sigma_1 \sigma_3}{l_1\, l_3}\genfrac{[}{]}{0pt}{}{\sigma_2}{l_2} + \binom{\sigma_2 \sigma_3}{l_2\, l_3}\genfrac{[}{]}{0pt}{}{\sigma_1}{l_1}\big\};$$

$$p_{l_1}^{\sigma_1} p_{l_2}^{\sigma_2} p_{l_3}^{\sigma_3} p_{l_4}^{\sigma_4} \to j = 0 : Y_{l_1}^{\sigma_1} Y_{l_2}^{\sigma_2} Y_{l_3}^{\sigma_3} Y_{l_4}^{\sigma_4} =: \genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_2 \sigma_3 \sigma_4}{l_1\, l_2\, l_3\, l_4},$$

$$j = 1 : (-\tfrac{1}{2})\big\{ \binom{\sigma_1 \sigma_2}{l_1\, l_2}\genfrac{[}{]}{0pt}{}{\sigma_3 \sigma_4}{l_3\, l_4} + \binom{\sigma_1 \sigma_3}{l_1\, l_3}\genfrac{[}{]}{0pt}{}{\sigma_2 \sigma_4}{l_2\, l_4} + \binom{\sigma_1 \sigma_4}{l_1\, l_4}\genfrac{[}{]}{0pt}{}{\sigma_2 \sigma_3}{l_2\, l_3} +$$

$$+ \binom{\sigma_2 \sigma_3}{l_2\, l_3}\genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_4}{l_1\, l_4} + \binom{\sigma_2 \sigma_4}{l_2\, l_4}\genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_3}{l_1\, l_3} + \binom{\sigma_3 \sigma_4}{l_3\, l_4}\genfrac{[}{]}{0pt}{}{\sigma_1 \sigma_2}{l_1\, l_2} \big\},$$

$$j = 2 : (-\tfrac{1}{2})^2\big\{ \binom{\sigma_1 \sigma_2}{l_1\, l_2}\binom{\sigma_3 \sigma_4}{l_3\, l_4} + \binom{\sigma_1 \sigma_3}{l_1\, l_3}\binom{\sigma_2 \sigma_4}{l_2\, l_4} +$$

$$+ \binom{\sigma_1 \sigma_4}{l_1\, l_4}\binom{\sigma_2 \sigma_3}{l_2\, l_3}\big\} =: (-\tfrac{1}{2})^2 \binom{\sigma_1 \sigma_2 \sigma_3 \sigma_4}{l_1\, l_2\, l_3\, l_4}. \tag{4.6}$$

5. Parametric Functions of FAs

5.1. We now formulate an algorithm of finding the parametric functions

$$\Delta(\alpha), \quad A(\alpha, k), \quad Y_l(\alpha, k), \quad X_{ll'}(\alpha),\ l, l' \in \mathcal{L},$$

of Feynman amplitudes. Of course, it is to be mentioned that we can use, in principle, any one of the available approaches. Contributions to this subject have been made by many authors. We give a very incomplete list of quotes here, namely, the papers by Chisholm [68], Nambu [69], Symanzik [70], Nakanishi [71], Schimamoto [72], Bjorken and Wu [73], Peres [74], Lam and Lebrun [75], Stepanov [76], Liu and Chow [77], Cvitanovic and Kinoshita [78], and the books by Todorov [79], Speer [80], Nakanishi [81], Zav'yalov [82], Smirnov [83], in which many other references can be found. Nevertheless, our algorithm seems to be very simple, but universal enough. It is named by the author [32, 84, 85] as *circuit-path* algorithm.

5.2. Suppose we have a connected graph $G(\mathcal{V}, \mathcal{L} \cup \mathcal{E})$ with sets of vertices, $\mathcal{V}$, internal lines, $\mathcal{L}$, and external lines, $\mathcal{E}$, and with a certain relation of incidence between $\mathcal{V}$ and $\Lambda \equiv \mathcal{L} \cup \mathcal{E}$ described by an oriented incidence matrix $e_{il} \equiv [e_{\mathcal{V}\Lambda}]_{il} = 0, \pm 1,\ v_i \in \mathcal{V},\ l \in \Lambda$.

In particular, $e_{il} = 0$, if line l is nonincident to the vertex v_i; $e_{il} = 1$, if line l is outgoing from the vertex v_i; and $e_{il} = -1$, if line l is incoming to the vertex v_i. The fact that the set of all lines Λ is separated from the very beginning into two mutually disjoint subsets $\mathcal{L}$ and $\mathcal{E}$ (their incident properties are different) is very important both from the algorithmic point of view and from potential possibilities. In so doing, we need not to replace here the set of external lines (incident to some vertex) by some effective line or to assign the same orientation to all external lines, as is usually done. Therefore, we can pose the task of constructing the parametric functions of the whole graph via the parametric functions of its subgraphs. As a result, the circuit-path approach is naturally arose, and the recursive structure of the parametric functions of FAs has been obtained [85, 86].

5.3. The set of external lines, $\mathcal{E}$, induces the single-valued decomposition of the set of all vertices, $\mathcal{V}$, into the subset of external vertices, $\mathcal{V}^{ext}$, and the subset of internal vertices, $\mathcal{V}^{int}$. The set of internal lines, $\mathcal{L}$, can be always decompose (as a rule, in more than one way) into two mutually disjoint subsets, $\mathcal{M}$ and $\mathcal{N}$, which determine some *skeleton tree* and the corresponding *co-tree* subgraphs of the graph G. So, we have the following decomposition of the set $\Lambda = \mathcal{E} \cup \mathcal{N} \cup \mathcal{M}$ of all lines of the graph G into mutually disjoint subsets, $\mathcal{E}$, $\mathcal{N}$, and $\mathcal{M}$. Then the circuit-path algorithm requires the following steps:

• Let us choose a subset $\mathcal{N} \subset \mathcal{L}$ such that the subgraph $G(\mathcal{V}, \mathcal{M} \cup \mathcal{E})$, where $\mathcal{M} := \mathcal{L}/\mathcal{N}$, is a *skeleton-tree*-type graph and the subgraph $G(\mathcal{V}, \mathcal{N} \cup \mathcal{E})$ is a *co-tree*-type graph. It is clear that this choice is ambiguous. It is shown in [84], however, that the parametric functions are independent of any choice of $\mathcal{N}$.

• Let us choose a vertex $v_j \in \mathcal{V}$ which will be referred as a *basis vertex*, (or *reference vertex*, or *zero point*). It is clear that this choice is also ambiguous. But it is shown in [84], that the parametric functions are again independent of any given choice of v_j. From the viewpoint of practical calculations, it seems reasonable to choose the basis vertex v_j as such a vertex, to which the largest number of external lines of the graph are incident.

• The choice of $\mathcal{N} \subset \mathcal{L}$ and the basis vertex v_j uniquely defines the notions of basis circuits $C(n)$, $n \in \mathcal{N}$ and basis paths $P(j|e)$, $e \in \mathcal{E}$.

The *basis circuit* $C(n)$ generated by the line $n \in \mathcal{N}$ is a union of the line n with the subset $\mathcal{M}(n) \subset \mathcal{M}$ which forms a chain in $\mathcal{M}$ between vertices incident to the line n, i.e. $C(n) := \{n\} \cup \mathcal{M}(n)$. The orientation in the circuit $C(n)$ is defined by the orientation of the line $n \in \mathcal{N}$.

The *basis path* $P(j|e)$ generated by the line $e \in \mathcal{E}$ and the basis vertex v_j is a union of the line e with the subset $\mathcal{M}(j|e) \subset \mathcal{M}$ which forms a chain in $\mathcal{M}$ between a vertex incident to the line $e \in \mathcal{E}$ and the basis vertex v_j, i.e. $P(j|e) := \{e\} \cup \mathcal{M}(j|e)$. The orientation in the path $P(j|e)$ is defined by the orientation of the line $e \in \mathcal{E}$.

• By analogy with the incidence matrix $e_{\mathcal{V}\Lambda}$ which can be referred, more precisely, as the *vertex-line* incidence matrix, one introduces topologically the *line-circuit* $e_{\Lambda\mathcal{N}}$ [77, 78, 81, 84, 85], and the *line-path* $e_{\Lambda\mathcal{E}}(j)$ [84, 85] incidence matrices, namely:

$$[e_{\Lambda\mathcal{N}}]_{ln} := \begin{cases} 0, & l \notin C(n), \\ \pm 1, & l \in C(n); \end{cases}$$

$$[e_{\Lambda\mathcal{E}}(j)]_{le} := \begin{cases} 0, & l \notin P(j|e), \\ \pm 1, & l \in P(j|e). \end{cases} \tag{5.1}$$

Here, the plus or minus sign depends on whether the orientation of the line $l \in \Lambda$ coincides or not with the orientation of the circuit $C(n)$ for $e_{\Lambda\mathcal{N}}$ or the path $P(j|e)$ for $e_{\Lambda\mathcal{E}}(j)$. As a result, the column-vector p_Λ of all momenta $p_l, l \in \Lambda$ and submatrices of $e_{\Lambda\mathcal{N}}$ and $e_{\Lambda\mathcal{E}}(j)$, whose rows are associated with the partition $\Lambda = \mathcal{E} \cup \mathcal{N} \cup \mathcal{M}$, can be represented as follows [84, 85]:

$$p_\Lambda = p_\Lambda^{\rm ext} + p_\Lambda^{\rm int}, \; p_\Lambda^{\rm ext} = e_{\Lambda\mathcal{E}}(j) k_\mathcal{E}, \; p_\Lambda^{\rm int} = e_{\Lambda\mathcal{N}} p_\mathcal{N};$$

$$e_{\mathcal{E}\mathcal{E}}(j) = 1_{\mathcal{E}\mathcal{E}}, \; e_{\mathcal{N}\mathcal{E}}(j|\mathcal{N}) = 0_{\mathcal{N}\mathcal{E}},$$

$$e_{\mathcal{M}\mathcal{E}}(j|\mathcal{N}) = -e^{-1}_{\{\mathcal{V}/j\}\mathcal{M}} e_{\{\mathcal{V}/j\}\mathcal{E}};$$

$$e_{\mathcal{E}\mathcal{N}} = 0_{\mathcal{E}\mathcal{N}}, \quad e_{\mathcal{N}\mathcal{N}} = 1_{\mathcal{N}\mathcal{N}},$$

$$e_{\mathcal{M}\mathcal{N}} = -e^{-1}_{\{\mathcal{V}/j\}\mathcal{M}} e_{\{\mathcal{V}/j\}\mathcal{N}}. \tag{5.2}$$

From now on, $k_\mathcal{E}$ and $p_\mathcal{N}$ are the column-vectors of the external momenta $k_e, e \in \mathcal{E}$, and the independent integration momenta $p_n, n \in \mathcal{N}$, respectively; $0_{\mathcal{A}\mathcal{B}}$ is the $|\mathcal{A}| \times |\mathcal{B}|$-rectangular matrix of zeros, and $1_{\mathcal{A}\mathcal{A}}$ is the $|\mathcal{A}|$-dimensional unit matrix. The matrices $e_{\{\mathcal{V}/j\}\mathcal{E}}$, $e_{\{\mathcal{V}/j\}\mathcal{N}}$, and $e_{\{\mathcal{V}/j\}\mathcal{M}}$ are submatrices of $e_{\mathcal{V}\Lambda}$. Their rows are defined by the set $(\mathcal{V}/v_j) \subset \mathcal{V}$, and their columns are defined by the subsets $\mathcal{E}, \mathcal{N}, \mathcal{M}$, respectively. The $(|\mathcal{V}|-1)$-dimensional square matrix $e_{\{\mathcal{V}/j\}\mathcal{M}}$ is nonsingular, and $\det[e_{\{\mathcal{V}/j\}\mathcal{M}}] = \pm 1$. In submatrices of the second and third lines of Eqs. (5.2), the subset $\mathcal{N}$ is pointed out explicitly, because of $e_{\mathcal{N}'\mathcal{E}}(j|\mathcal{N}) \neq 0_{\mathcal{N}'\mathcal{E}}$, and $e_{\mathcal{M}'\mathcal{E}}(j|\mathcal{N}) \neq e_{\mathcal{M}\mathcal{E}}(j|\mathcal{N})$ if $\mathcal{N}' \neq \mathcal{N}$, $\mathcal{L} = \mathcal{N} \cup \mathcal{M} = \mathcal{N}' \cup \mathcal{M}'$, but $e_{\mathcal{E}\mathcal{E}}(j|\mathcal{N}) = e_{\mathcal{E}\mathcal{E}}(j|\mathcal{N}') = 1_{\mathcal{E}\mathcal{E}}$.

• There exist the following very important "orthogonality" relations [84, 85, 87]:

$$e_{\mathcal{V}\Lambda}e_{\Lambda\mathcal{N}} = e_{\mathcal{V}\mathcal{L}}e_{\mathcal{L}\mathcal{N}} = 0_{\mathcal{V}\mathcal{N}},$$

$$e_{\{\mathcal{V}/j\}\Lambda}e_{\Lambda\mathcal{N}} = e_{\{\mathcal{V}/j\}\mathcal{L}}e_{\mathcal{L}\mathcal{N}} = 0_{\{\mathcal{V}/j\}\mathcal{N}},$$

$$[e_{\mathcal{V}\Lambda}e_{\Lambda\mathcal{E}}(j)]_{ie} = \delta_{ij}[e(\mathcal{V}^*)_{\mathcal{E}}]_e,$$

$$e_{\{\mathcal{V}/j\}\Lambda}e_{\Lambda\mathcal{E}}(j) = 0_{\{\mathcal{V}/j\}\mathcal{E}}, \tag{5.3}$$

where $e(\mathcal{V}^*)_{\mathcal{E}}$ is the vertex-line incidence matrix of the *"star"*-type graph $G^* := < \mathcal{V}^*, \mathcal{E} >$ with the one vertex $\mathcal{V}^*$ and the set of external lines $\mathcal{E}$ of the graph G. The graph $G^* := < \mathcal{V}^*, \mathcal{E} >$ is a result of the shrinking of all vertices $v_i \in \mathcal{V}$, and all internal lines $l \in \mathcal{L}$, of the graph G to the single *"black-hole"* vertex $\mathcal{V}^*$.

• By assigning the parameter $\alpha_l \geq 0$ to every internal line $l \in \mathcal{L}$, we define the *circuit* $C_{\mathcal{N}\mathcal{N}}(\alpha)$, *path* $E_{\mathcal{E}\mathcal{E}}(j|\alpha)$, and *path-circuit* $\Pi_{\mathcal{E}\mathcal{N}}(j|\alpha)$ matrices [84, 85], according to:

$$[C_{\mathcal{N}\mathcal{N}}(\alpha)]_{nn'} := [e^T_{\mathcal{L}\mathcal{N}}\alpha_{\mathcal{L}\mathcal{L}}e_{\mathcal{L}\mathcal{N}}]_{nn'} = \pm \sum_{l\in C(n)\cap C(n')} \alpha_l,$$

$$[E_{\mathcal{E}\mathcal{E}}(j|\alpha)]_{ee'} := [e^T_{\mathcal{L}\mathcal{E}}(j)\alpha_{\mathcal{L}\mathcal{L}}e_{\mathcal{L}\mathcal{E}}(j)]_{ee'} = \pm \sum_{l\in P(j|e)\cap P(j|e')} \alpha_l,$$

$$[\Pi_{\mathcal{E}\mathcal{N}}(j|\alpha)]_{en} := [e^T_{\mathcal{L}\mathcal{E}}(j)\alpha_{\mathcal{L}\mathcal{L}}e_{\mathcal{L}\mathcal{N}}]_{en} = \pm \sum_{l\in P(j|e)\cap C(n)} \alpha_l. \tag{5.4}$$

Here, the plus or minus sign depends on the mutual orientations of the sets, over which the summation is performed, on their intersection. The plus sign corresponds to the case of coinciding orientations. It is clear that the explicit form of these matrices in any given case can be easily obtained by inspecting the graph. From now on, $\alpha_{\mathcal{L}\mathcal{L}}$ is the diagonal $|\mathcal{L}|$-dimensional matrix, i.e., $[\alpha_{\mathcal{L}\mathcal{L}}]_{ll'} = \alpha_l\,\delta_{ll'}$.

• The parametric functions are derived by means of the use of the following matrices [84, 86]:

$$A_{\mathcal{E}\mathcal{E}}(j|\alpha) := E_{\mathcal{E}\mathcal{E}}(j|\alpha) - \Pi_{\mathcal{E}\mathcal{N}}(j|\alpha)C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha)\Pi^T_{\mathcal{E}\mathcal{N}}(j|\alpha),$$

$$Y_{\mathcal{L}\mathcal{E}}(j|\alpha) := e_{\mathcal{L}\mathcal{E}}(j) - e_{\mathcal{L}\mathcal{N}}C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha)\Pi^T_{\mathcal{E}\mathcal{N}}(j|\alpha),$$

$$X_{\mathcal{L}\mathcal{L}}(\alpha) := e_{\mathcal{L}\mathcal{N}}C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha)e^T_{\mathcal{L}\mathcal{N}},$$

$$\Delta(\alpha) := \det C_{\mathcal{N}\mathcal{N}}(\alpha). \tag{5.5}$$

So, the quadratic $A(\alpha, k)$ and linear $Y_l(\alpha, k)$, $l \in \mathcal{L}$, Kirchhoff forms in the external momenta k_e, $e \in \mathcal{E}$, and the line-correlator functions $X_{ll'}(\alpha)$, $l, l' \in \mathcal{L}$, are defined as [84, 86]

$$A(\alpha, k) := (k^T_{\mathcal{E}} \cdot A_{\mathcal{E}\mathcal{E}}(j|\alpha)k_{\mathcal{E}}) =$$

$$= \textstyle\sum_{e,e'\in\mathcal{E}}[A_{\mathcal{E}\mathcal{E}}(j|\alpha)]_{ee'}(k_e \cdot k_{e'}),$$

$$Y_l(\alpha, k) := Y_{l\mathcal{E}}(j|\alpha)k_{\mathcal{E}} = \textstyle\sum_{e\in\mathcal{E}}[Y_{l\mathcal{E}}(j|\alpha)]_e k_e,$$

$$Y_{l\mathcal{E}}(j|\alpha) = e_{l\mathcal{E}}(j) - e_{l\mathcal{N}}C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha)\Pi^T_{\mathcal{E}\mathcal{N}}(j|\alpha),$$

$$X_{ll'}(\alpha) = e_{l\mathcal{N}}C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha)e^T_{l'\mathcal{N}}. \tag{5.6}$$

Here, $e_{l\mathcal{N}}$ and $e_{l\mathcal{E}}(j)$ are the row-vectors of matrices (5.1)–(5.2) for the line $l \in \mathcal{L}$.

5.4. It should be mentioned that the functions $\Delta(\alpha)$ and $A(\alpha, k)$ do not depend on the orientation of internal lines. However, when the orientation of line l is changed, the parametric functions $Y_l(\alpha, k)$ and $X_{ll'}(\alpha)$ reverse their signs.

It is also useful to represent the quantities $A(\alpha, k)$ and $Y_{\mathcal{L}}(\alpha, k)$ in a form exhibiting a special role of the matrices $X_{\mathcal{L}\mathcal{L}}(\alpha)$ and $X_{\mathcal{N}\mathcal{N}}(\alpha)$ [78, 84]:

$$A(\alpha, k) = \left(p^{\text{ext}}_{\mathcal{L}}(k)^T \cdot [\,\alpha_{\mathcal{L}\mathcal{L}} - \alpha_{\mathcal{L}\mathcal{L}}X_{\mathcal{L}\mathcal{L}}(\alpha)\alpha_{\mathcal{L}\mathcal{L}}\,]\,p^{\text{ext}}_{\mathcal{L}}(k)\right),$$

$$p^{\text{ext}}_{\mathcal{L}}(k) = e_{\mathcal{L}\mathcal{E}}(j)k_{\mathcal{E}}, \quad p^{\text{ext}}_{\mathcal{E}}(k) = k_{\mathcal{E}}, \quad p^{\text{ext}}_{\mathcal{N}}(k) = 0_{\mathcal{N}},$$

$$Y_{\mathcal{L}}(\alpha, k) = [\,1_{\mathcal{L}\mathcal{L}} - X_{\mathcal{L}\mathcal{L}}(\alpha)\alpha_{\mathcal{L}\mathcal{L}}\,]\,p^{\text{ext}}_{\mathcal{L}}(k) =$$

$$= p^{\text{ext}}_{\mathcal{L}}(k) - Y^{\text{int}}_{\mathcal{L}}(\alpha, k),$$

$$Y^{\text{int}}_{\mathcal{L}}(\alpha, k) := X_{\mathcal{L}\mathcal{L}}(\alpha)\alpha_{\mathcal{L}\mathcal{L}}\,p^{\text{ext}}_{\mathcal{L}}(k),$$

$$X_{\mathcal{L}\mathcal{L}}(\alpha) := e_{\mathcal{L}\mathcal{N}}X_{\mathcal{N}\mathcal{N}}(\alpha)e^T_{\mathcal{L}\mathcal{N}},$$

$$X_{\mathcal{N}\mathcal{N}}(\alpha) := C^{-1}_{\mathcal{N}\mathcal{N}}(\alpha), \tag{5.7}$$

where $k_{\mathcal{E}}$ is the column-vector of the external momenta k_e, $e \in \mathcal{E}$. The following homogeneous properties hold:

$$\Delta(\rho\alpha) = \rho^{|\mathcal{C}|}\Delta(\alpha), \quad X_{ll'}(\rho\alpha) = \rho^{-1}X_{ll'}(\alpha),$$

$$A(\rho\alpha, \tau k) = \rho\tau^2 A(\alpha, k), \quad Y_l(\rho\alpha, \tau k) = \tau Y_l(\alpha, k),$$

$$\mathcal{P}^G_{sj}(m, \rho\alpha, \tau k) = \rho^{-j}\tau^{s-2j}\,\mathcal{P}^G_{sj}(m, \alpha, k). \qquad (5.8)$$

5.5. Now we exhibit some important properties of α-parametric functions [45]. Let us introduce the quantities

$$K^r_{\mathcal{LL}} := X_{\mathcal{LL}}\alpha_{\mathcal{LL}}, \quad K^l_{\mathcal{LL}} := \alpha_{\mathcal{LL}}X_{\mathcal{LL}},$$

$$L^i_{\mathcal{LL}} := 1_{\mathcal{LL}} - K^i_{\mathcal{LL}}, \quad i = r,\, l. \qquad (5.9)$$

Using Eqs. (5.3)–(5.5), we find that the matrices $K^i_{\mathcal{LL}}(\alpha)$ and $L^i_{\mathcal{LL}}(\alpha)$ are projectors with the properties

$$K^i_{\mathcal{LL}}K^i_{\mathcal{LL}} = K^i_{\mathcal{LL}}, \quad L^i_{\mathcal{LL}}L^i_{\mathcal{LL}} = L^i_{\mathcal{LL}}, \quad i = r, l,$$

$$K^i_{\mathcal{LL}}L^i_{\mathcal{LL}} = 0_{\mathcal{LL}}, \quad K^l_{\mathcal{LL}}\alpha_{\mathcal{LL}}L^r_{\mathcal{LL}} = 0_{\mathcal{LL}}, \qquad (5.10)$$

From Eqs. (5.10), we get some relations between products of $X_{\mathcal{LL}}$, $\alpha_{\mathcal{LL}}$, and $Y_{\mathcal{LE}}$:

$$\big(X_{\mathcal{LL}}\alpha_{\mathcal{LL}}\big)^m X_{\mathcal{LL}} = X_{\mathcal{LL}}\alpha_{\mathcal{LL}}X_{\mathcal{LL}} = X_{\mathcal{LL}},$$

$$\big(L^r_{\mathcal{LL}}\big)^m X_{\mathcal{LL}} = 0_{\mathcal{LL}},$$

$$\big(X_{\mathcal{LL}}\alpha_{\mathcal{LL}}\big)^m Y_{\mathcal{LE}} = X_{\mathcal{LL}}\alpha_{\mathcal{LL}}Y_{\mathcal{LE}} = 0_{\mathcal{LE}}, \qquad (5.11)$$

$$\big(L^r_{\mathcal{LL}}\big)^m Y_{\mathcal{LE}} = Y_{\mathcal{LE}},$$

$$\mathrm{Tr}\big[\big(K^i_{\mathcal{LL}}\big)^m\big] = \mathrm{Tr}\big[K^i_{\mathcal{LL}}\big] = \textstyle\sum_{l\in\mathcal{L}}\alpha_l X_{ll}(\alpha) = |\mathcal{N}|,$$

$$\mathrm{Tr}\big[\big(L^i_{\mathcal{LL}}\big)^m\big] = \mathrm{Tr}\big[L^i_{\mathcal{LL}}\big] = |\mathcal{M}|, \quad i = r, l,$$

and the relations between the quadratic $A(\alpha, k)$ and linear $Y_l(\alpha, k)$, $l \in \mathcal{L}$, Kirchhoff forms:

$$A(\alpha, k) = (Y^T_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}p^{\mathrm{ext}}_{\mathcal{L}}) = (p^{\mathrm{ext}\,T}_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}Y_{\mathcal{L}}) \equiv$$

$$\equiv \textstyle\sum_{l\in\mathcal{L}}\alpha_l(p^{\mathrm{ext}}_l(k) \cdot Y_l(\alpha, k)) =$$

$$= (Y^T_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}Y_{\mathcal{L}}) \equiv \textstyle\sum_{l\in\mathcal{L}}\alpha_l Y^2_l(\alpha, k). \qquad (5.12)$$

The following relations are also satisfied:

$$e_{\mathcal{VE}}k_{\mathcal{E}} + e_{\mathcal{VL}}Y_{\mathcal{L}}(\alpha, k) = 0_{\mathcal{V}}, \quad e^T_{\mathcal{LN}}\alpha_{\mathcal{LL}}Y_{\mathcal{L}}(\alpha, k) = 0_{\mathcal{N}},$$

$$e_{\mathcal{VL}}X_{\mathcal{LL}}(\alpha) = 0_{\mathcal{VL}},$$

$$K^r_{\mathcal{LL}}e_{\mathcal{LE}}(j) = -e_{\mathcal{LN}}Y_{\mathcal{NE}}(j|\alpha) = e_{\mathcal{LN}}K^r_{\mathcal{NL}}e_{\mathcal{LE}}(j),$$

$$K^r_{\mathcal{LL}}e_{\mathcal{LN}} = e_{\mathcal{LN}},$$

$$\big(Y^{\mathrm{int}\,T}_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}Y_{\mathcal{L}}\big) =$$

$$= \big(Y^{\mathrm{int}\,T}_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}\,p^{\mathrm{ext}}_{\mathcal{L}}\big) - \big(Y^{\mathrm{int}\,T}_{\mathcal{L}} \cdot \alpha_{\mathcal{LL}}Y^{\mathrm{int}}_{\mathcal{L}}\big) = 0. \qquad (5.13)$$

In our case of α-parametric functions, two relations in the first line of Eqs. (5.13) are analogs of the first and second Kirchhoff laws in electric networks. Similarly, in the third line of Eqs. (5.12), we find an analog of the well-known expression for a power dissipated in electric networks.

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САМОУЗГОДЖЕНА РЕНОРМАЛІЗАЦІЯ ЯК ЕФЕКТИВНА РЕАЛІЗАЦІЯ ГОЛОВНИХ ІДЕЙ *R*-ОПЕРАЦІЇ БОГОЛЮБОВА–ПАРАСЮКА

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Р е з ю м е

Книга природи написана мовою математики.
Галілео Галілей, [*Il Saggiatore, 1623*].

"... В зв'язку з тим, що в останній час знайдено тісний зв'язок між причиновістю та аналітичністю, не виключена ймовірність побудови віднімальної операції навіть в самому загальному випадку методами аналітичного продовження."
О.С. Парасюк, [[7], *с.566, останній абзац, 1956*].

Цю можливість реалізовано явно та ефективно засобами нашої самоузгодженої ренормалізації (СУР). Під самоузгодженістю розуміють, що всі формальні співвідношення між УФ-розбіжними фейнмановими амплітудами також автоматично зберігаються між їхніми регулярними значеннями, знайденими згідно з процедурою СУР. Самоузгоджена ренормалізація з однаковою ефективністю застосовна як до ренормовних, так і до неренормовних теорій. СУР має ефективні засоби для конструктивного розгляду нових задач: а) ренормалізаційних проблем, що пов'язані з симетріями, тотожностями Уорда та квантовими аномаліями; б) нових взаємозв'язків між скінченними зародковими та скінченними фізичними параметрами квантово-польових теорій. Наведено огляд головних ідей та властивостей СУР, а також чітко описано три взаємодоповнювальні алгоритми СУР, які подано у вигляді, максимально пристосованому для практичних застосувань.