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MINIMAX RECURSIVE STATE ESTIMATION FOR LINEAR DISCRETE-TIME DESCRIPTOR SYSTEMS

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This paper describes an approach to the online state estimation of systems described by a general class of linear noncausal time-varying difference descriptor equations subject to uncertainties. An approach is based on the notions of a linear minimax estimation and an index of causality introduced here for singular difference equations. The online minimax observer is derived by the application of the dynamical programming and Moore's pseudoinverse theory to the minimax estimation problem.

INTRODUCTION

There is a number of physical and engineering objects most naturally modelled as systems of differential and algebraic equations (DAEs) or descriptor systems: microwave circuits [1], flexible-link planar parallel platforms [2] and image recognition problems (noncasual image modeling) [3]. DAEs arise in economics [4]. Also nonlinear differential-algebraic systems are studied with help of linear DAEs by linearization: a batch chemical reactor model [5].

On the other hand there are many papers devoted to the mathematical processing of data, obtained from the measuring device during an experiment. In particular, the problem of observer design for continious-time DAEs was considered in [7] and discrete-time case was studied in [8]–[9]. The minimax state estimation for uncertain linear dynamical systems was investigated in [10]. Other approaches to state estimation with set-membership description of uncertainty were discussed in [12]–[14].

In [6] authors derive a so-called «3-block» form for the optimal filter and a corresponding 3-block Riccati equation using a maximum likelihood approach. A filter is obtained for a general class of time-varying descriptor models. Measurements are supposed to contain a noise with Gaussian distribution. The obtained recursion is stated in terms of the 3-block matrix pseudoinverse.

In [8] the filter recursion is represented in terms of a deterministic data fitting problem solution. The authors introduce an explicit form of the 3-block matrix pseudoinverse for a descriptor system with a special structure, so their filter coincides with obtained in [6].

In this paper we study an observer design problem for a general class of linear noncasual time-varying descriptor models with no restrictions on system structure. Suppose we are given an exact mathematical model of some real process and the vector x_k describes the system output at the moment k in the corresponding state space of the system. Also successive measurements $y_0 \dots y_k \dots$ of the system output x_k are supposed to be available with the noise $g_0 \dots g_k \dots$ of an uncertain nature. (For instance we do not have a-priory information about its distribution.) Further assume that the system input f_k, start point q and noise g_k are arbitrary elements of the given set G. The aim of this paper is to design a minimax observer $k \mapsto \hat{x}_k$ that gives an online guaranteed estimation of the output x_k on the basis of measurements y_k and the structure of G. In [9] minimax estimations were derived from the 2-point boundary value problem with conditions at $i = 0$ (start point) and $i = k$ (end point). Hence a recalculation of the whole history $\hat{x}_0 \dots \hat{x}_k$ is required if the moment k changes. Here we derive the observer $(k, y_k) \mapsto \hat{x}_k$ by applying dynamical programming methods to the minimax estimation problem similar to the posed one in [9]. We construct a map $\hat{x}$ that takes (k, y_k) to $\hat{x}_k$ making it possible to assign a unique sequence of estimations $\hat{x}_0 \dots \hat{x}_k \dots$ to given sequence of observations $y_0 \dots y_k \dots$ in the real time. A resulting filter recursion is stated in terms of pseudoinverse of positive semi-defined $n \times n$ - matrices.

Minimax estimation problem

Assume that $x_k \in \mathbf{R}^n$ is described by the equation

$$F_{k+1}x_{k+1} - C_k x_k = f_k, \quad k = 0,1,\dots, \tag{1}$$

with initial condition

$$F_0 x_0 = q, \tag{2}$$

and y_k is given by

$$y_k = H_k x_k + g_k, \quad k = 0,1,\dots, \tag{3}$$

where F_k, C_k are $m \times n$-matrices, H_k is $p \times n$-matrix. Since we deal with descriptor system we see that for any k there is a set of vectors $x_1^0 \dots x_k^0$ satisfying (1) while $f_i = 0,\ q = 0$. Thus the undefined inner influence caused, by $x_1^0 \dots x_k^0$, may appear in the system's output. Also we suppose the initial condition q, input $\{f_k\}$ and noise $\{g_k\}$ to be unknown elements of the given set. (Here and after $(\cdot,\cdot)$ denotes an inner product in an appropriate Euclidean space, $\|\mathbf{x}\| = (x,x)^{1/2}$.)

$$\Gamma = \Big\{ (q, \{f_k\}, \{g_k\}) : G(q, \{f_k\}, \{g_k\}) =$$

$$= (Sq, q) + \sum_{0}^{\infty} (S_k f_k, f_k) + (R_k g_k, g_k) \le 1 \Big\}, \tag{4}$$

where S, S_k, R_k are some symmetric positive-defined weight matrices with appropriate dimensions. The trick is to fix any N-partial sum of (4) so that $(q, \{f_k\}, \{g_k\})$ belongs to

$$\mathbf{G}^N := \{ (q, \{f_k\}, \{g_k\}) : (Sq, q) + \sum_{k=0}^{N-1} (S_k f_k, f_k) + \sum_{k=0}^{N} (R_k g_k, g_k) \le 1 \}. \tag{5}$$

Then we derive the estimation $\hat{x}_N = v(N, y_N, \hat{x}_{N-1})$ considering a minimax estimation problem for $\mathbf{G}^N$. Let us denote by $\mathbf{N}$ a set of all $(\{x_k\}, q, \{f_k\})$ such that (1) is held. The set $\mathbf{G}_y^N$ is said to be *a-posteriori set*, where

$$\mathbf{G}_y^N := \{\{x_k\} : (\{x_k\}, q, \{f_k\}) \in \mathbf{N},\ (q, \{f_k\}, \{y_k - H_k x_k\}) \in \mathbf{G}^N\}. \qquad (6)$$

It follows from the definition that $\mathbf{G}_y^N$ consists of all possible $\{x_k\}$, causing an output $\{y_k\}$, while $(q, \{f_k\}, \{g_k\})$ runs through $\mathbf{G}^N$. Thus, it's naturally to look for estimation x_N of **only** among the elements of $P_N(\mathbf{G}_y^N)$, where P_N denotes the projection that takes $\{x_0 \ldots x_N\}$ to x_N.

Definition 1. A linear function $(\ell, \hat{x}_N)$ is called a minimax a-posteriori estimation if the following condition holds:

$$\inf_{\{\tilde{x}_k\} \in \mathbf{G}_y^N} \sup_{\{x_k\} \in \mathbf{G}_y^N} |(\ell, x_N) - (\ell, \tilde{x}_N)| = \sup_{\{x_k\} \in \mathbf{G}_y^N} |(\ell, x_N) - (\ell, \hat{x}_N)|.$$

The non-negative number

$$\hat{\sigma}(\ell, N) = \sup_{\{x_k\} \in \mathbf{G}_y^N} |(\ell, x_N) - (\ell, \hat{x}_N)|$$

is called a minimax a-posteriori error in the direction ℓ. A map

$$N \mapsto I_N = \dim\{\ell \in \mathbf{R}^n : \hat{\sigma}(\ell, N) < +\infty\}$$

is called an index of causality for the pair of systems (1)–(3).

Now we say that a minimax estimation problem is to construct an a-posteriori linear minimax estimation $(\ell, \hat{x}_N)$ for the system (1) on the basis of the measurements (3) and a-posteriori set $\mathbf{G}_y^N$. A solution of the minimax estimation problem in the form of a recursive map $k \mapsto (\ell, \hat{x}_N)$ is presented in the next section.

Minimax online observer

Denote by $k \mapsto Q_k$ a recursive map that takes each natural number k to the matrix Q_k, where

$$Q_k = H_k' R_k H_k + F_k'[S_{k-1} - S_{k-1} C_{k-1} W_{k-1}^+ C_{k-1}' S_{k-1}] F_k,$$
$$Q_0 = F_0' S F_0 + H_0' R_0 H_0, W_k = Q_k + C_k' S_k C_k. \qquad (7)$$

Let $k \mapsto r_k$ be a recursive map that takes each natural number k to the vector $r_k \in \mathbf{R}^n$, where

$$r_k = F_k' S_{k-1} C_{k-1} W_{k-1}^+ r_{k-1} + H_k' R_k y_k, \quad r_0 = H_0' R_0 y_0 \qquad (8)$$

and to each natural number $i \in$ assign a number α_i, where

$$\alpha_i = \alpha_{i-1} + (R_i y_i, y_i) - (W_{i-1}^+ r_{i-1}, r_{i-1}), \ \alpha_0 = (Sg, g) + (R_0 y_0, y_0). \qquad (9)$$

The main result of this paper is formulated in the next theorem.

Theorema (minimax recursive estimation). Suppose we are given a natural number N and a vector $\ell \in \mathbf{R}^n$. Then a necessary and sufficient condition for a minimax a-posteriori error $\hat{\sigma}(\ell, N)$ to be finite is that

$$Q_N^+ Q_N \ell = \ell . \quad (10)$$

Under this condition we have

$$\hat{\sigma}(\ell, N) = [1 - \alpha_N + (Q_N^+ r_N, r_N)]^{\frac{1}{2}} (Q_N^+ \ell, \ell)^{\frac{1}{2}} \quad (11)$$

and

$$(\ell, \hat{x}_N) = (\ell, Q_N^+ r_N). \quad (12)$$

Corollary 1. The index of causality I_N for the pair of systems (1)–(3) can be represented as $I_N = \operatorname{rank}(Q_N)$.

Corollary 2 (minimax obsever). The online minimax observer is given by $k \mapsto \hat{x}_k = Q_k^+ r_k$ and (we assume here that $1/0 = +\infty$.)

$$\hat{\rho}(N) = \min_{\{x_k\}\in \mathbf{G}_y^N} \max_{\{\tilde{x}_k\}\in \mathbf{G}_y^N} \|x_N - \tilde{x}_N\|^2 =$$

$$= \max_{\{x_k\}\in \mathbf{G}_y^N} \|x_N - \hat{x}_N\|^2 = \frac{[1 - \alpha_N + (Q_N \hat{x}_N, \hat{x}_N)]}{\min_i \{\lambda_i(N)\}} \quad (13)$$

where $\lambda_i(N)$ are eigenvalues of Q_N. In this case all possible realisations of the state vector x_N of (1) fill the ellipsoid $P_N(\mathbf{G}_y^N) \subset \mathbf{R}^n$, where

$$P_N(\mathbf{G}_y^N) = \{x : (Q_N x, x) - 2(Q_N \hat{x}_N, x) + \alpha_N \le 1\}. \quad (14)$$

Remark 1. If $\lambda_{\min}(H_k' R_k H_k)$ grows for $k = i, i+1, \ldots$ then the minimax estimation error $\hat{\rho}(k)$ becomes smaller causing $\hat{x}_k$ to get closer to the real state vector x_k.

In [8] Kalman's filtering problem for descriptor systems was investigated from the deterministic point of view. Authors recover Kalman's recursion to the time-variant descriptor system by a deterministic least square fitting problem over the entire trajectory: find a sequence $\{\hat{x}_{0|k}, \ldots, \hat{x}_{k|k}\}$ that minimises the following fitting error cost

$$J_k(\{x_{i|k}\}_0^k) = \|F_0 x_{0|k} - g\|^2 + \|y_0 - H_0 x_{0|k}\|^2 +$$

$$+ \sum_{i=1}^{k} \|F_i x_{i|k} - C_{i-1} x_{i-1|k}\|^2 + \|y_i - H_i x_{i|k}\|^2$$

assuming that the $\operatorname{rank}\begin{bmatrix} F_k \\ H_k \end{bmatrix} \equiv n.$ According to [8] the successive optimal estimates $\{\hat{x}_{0|k}, \ldots, \hat{x}_{k|k}\}$ resulting from the minimisation of $\mathbf{J}_k$ can be found from the recursive algorithm

$$\hat{x}_{k|k} = P_{k|k}F_k'(E + C_{k-1}P_{k-1|k-1}C_{k-1}')^{-1}C_{k-1}\hat{x}_{k-1|k-1} +$$

$$+ P_{k|k}H_k'R_k y_k, \hat{x}_{0|0} = P_{0|0}(F_0'q + H_0'y_0),$$

$$P_{k|k} = (F_k'(E + C_{k-1}P_{k-1|k-1}C_{k-1}')^{-1}F_k + H_k'H_k)^{-1},$$

$$P_{0|0} = (F_0'F_0 + H_0'H_0)^{-1}. \tag{15}$$

Corollary 3 (Kalman's filter recursion). Suppose the rank$\begin{bmatrix} F_k \\ H_k \end{bmatrix} \equiv n$, and let $k \mapsto r_k$ be a recursive map that takes each natural number k to the vector $r_k \in \mathbf{R}^n$, where

$$r_k = H_k'y_k + F_k'C_{k-1}(C_{k-1}'C_{k-1} + Q_{k-1})_{k-1}^{+}r_{k-1},$$

$$r_0 = F_0'q + H_0'y_0. \tag{16}$$

Then $Q_k^{+}r_k = \hat{x}_{k|k}$ for each $k \in \mathbf{N}$, where $\hat{x}_{k|k}$ is given by (15) and $I_k = n$.

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Proof of Theorem. By definition, put

$$\mathbf{F} = \begin{pmatrix} F_0 & 0_{mn} & 0_{mn} & \dots & 0_{mn} \\ 0_{mn} & & & \dots & \\ -C_0 & F_1 & 0_{mn} & \dots & 0_{mn} \\ 0_{mn} & & & \dots & \\ 0_{mn} & -C_1 & F_2 & \dots & 0_{mn} \\ 0_{mn} & & & \dots & \\ \vdots & \vdots & \vdots & \dots & \\ 0_{mn} & 0_{mn} & 0_{mn} & \dots & -C_{N-1} \\ F_N & & & & \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} H_0 & 0_{pn} & \dots & 0_{pn} \\ 0_{pn} & H_1 & \dots & 0_{pn} \\ \vdots & \vdots & \dots & \vdots \\ 0_{pn} & 0_{pn} & \dots & H_N \end{pmatrix}, \ \mathbf{X} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} q \\ f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}.$$

By direct calculation we obtain $(\ell, x_N) = (\mathbf{L}, \mathbf{X})$,

$$\mathbf{G}_y^N = \{\mathbf{X} : \|\mathbf{F}\mathbf{X}\|_1^2 + \|\mathbf{Y} - \mathbf{H}\mathbf{X}\|_2^2) \le 1\},$$

where $\|\mathbf{F}\|_1^2 = (Sq, q) + \sum_0^{N-1}(S_k f_k, f_k)$, $\|\cdot\|_2$ is indused by R_k on the same way. This implies

$$\sup_{\{x_k\}\in \mathbf{G}_y^N} |(\ell, x_N - \widetilde{x}_N)| = \sup_{\mathrm{X}\in \mathbf{G}_y^N} |(\mathbf{L},\mathbf{X}) - (\mathbf{L},\widetilde{\mathbf{X}})|.$$

Denote by $\mathbf{M}$ the set $R[\mathbf{F}'\,\mathbf{H}']$. We obviously get

$$\mathbf{L}\in\mathbf{M} \Leftrightarrow \sup_{\mathbf{X}\in\mathbf{G}_y^N} |(\mathbf{L},\mathbf{X}) - (\mathbf{L},\widetilde{\mathbf{X}})| < +\infty .$$

The application of Corollary 4 yields (10). Consider a vector $\mathbf{L}\in\mathbf{M}$. Clearly

$$\inf_{\mathbf{X}\in\mathbf{G}_y^N} (\mathbf{L},\mathbf{X}) \le (\mathbf{L},\mathbf{X}) \le \sup_{\mathbf{X}\in\mathbf{G}_y^N} (\mathbf{L},\mathbf{X}),\ \ \mathbf{X}\in\mathbf{G}_y^N .$$

Let c denotes $\frac{1}{2}(\sup_{\mathbf{X}\in\mathbf{G}_y^N}(\mathbf{L},\mathbf{X}) + \inf_{\mathbf{X}\in\mathbf{G}_y^N}(\mathbf{L},\mathbf{X}))$. Therefore

$$\sup_{\mathbf{X}\in\mathbf{G}_y^N} |(\mathbf{L},\mathbf{X}) - (\mathbf{L},\widetilde{\mathbf{X}})| = \frac{1}{2}(\mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_y^N) + (\mathrm{s}(-\mathbf{L}\,|\,\mathbf{G}_y^N)) + |\,\mathrm{c} - (\mathbf{L},\widetilde{\mathbf{X}})|$$

hence

$$\hat{\sigma}(\ell, N) = \frac{1}{2}(\mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_y^N) + \mathrm{s}(-\mathbf{L}\,|\,\mathbf{G}_y^N)),\ (\ell,\hat{x}_N) = \frac{1}{2}(\mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_y^N) - \mathrm{s}(-\mathbf{L}\,|\,\mathbf{G}_y^N)), \quad (17)$$

where $\mathrm{s}(\cdot\,|\,\mathbf{G}_y^N)$ denotes the support function of $\mathbf{G}_y^N$. Clearly, $\mathbf{G}_y^N$ is a convex closed set. Hence the equality $(\mathbf{L},\widetilde{\mathbf{X}}) = (\ell,\hat{x}_N)$ is held for some $\widetilde{\mathbf{X}}\in\mathbf{G}_y^N$. Thus, to conclude the proof we have to calculate $\mathrm{s}(\mathbf{L},\mathbf{G}_y^N)$. Let

$$\mathbf{G}_0^N = \{\mathbf{X} : \|\mathbf{FX}\|^2 + \|\mathbf{HX}\|^2 \le \beta_N\}, \quad (18)$$

where $\beta_N = 1 - \alpha_N + (Q_N^+ r_N, r_N) \ge 0$.

Lemma 1.

$$\mathrm{s}(\mathbf{L},\mathbf{G}_y^N) = (\ell, Q_N^+ r_N) + \mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_0^N). \quad (19)$$

It follows from the definition of $\mathbf{G}_0^N$ that $\mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_0^N) = \mathrm{s}(-\mathbf{L}\,|\,\mathbf{G}_0^N)$ hence (17) implies

$$(\ell,\hat{x}_N) = (\ell, Q_N^+ r_N),\ \ \hat{\sigma}(\ell) = \mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_0^N).$$

The application of Lemma 2 completes the proof.

Lemma 2.

$$\mathrm{s}(\mathbf{L}\,|\,\mathbf{G}_0^N) = \begin{cases} \sqrt{\beta_N}\,(Q_N^+\ell,\ell)^{\frac{1}{2}}, & [E - Q_N^+ Q_N]\ell = 0, \\ +\infty, & [E - Q_N^+ Q_N]\ell \ne 0. \end{cases} \quad (20)$$

Let r_k denote $\mathbf{R}^n$ — valued recursive map

$$r_k = F_k'(S_{k-1} - S_{k-1}C_{k-1}P_{k-1}^+ C_{k-1}' S_{k-1}) f_{k-1} + F_k' S_{k-1} C_{k-1} W_{k-1}^+ r_{k-1} + H_k' R_k y_k ,$$

$$r_0 = F_0' S q + H_0' R_0 y_0,\ \ P_k = C_k' S_k C_k + Q_k , \quad (21)$$

and set

$$\mathbf{J}(\{x_k\}) = \|F_0 x_0 - g\|_S^2 + \|y_0 - H_0 x_0\|_0^2 +$$

$$+\sum_{k=1}^{N} \|F_k x_k - C_{k-1} x_{k-1} - f_{k-1}\|_{k-1}^2 + \|y_k - H_k x_k\|_k^2 ,$$

where $\|g\|_S^2 = (Sg, g)$, $\|f_k\|_k^2 = (S_k f_k, f_k)$, $\|y_i\|_i^2 = (R_i y_i, y_i)$.

Lemma 3. Let $x \mapsto \hat{x}_k$ be a recursive map that takes any $k \in$ natural to $\hat{x}_k \in \mathbf{R}^n$, where

$$\hat{x}_k = P_k^+ (C_k' S_k (F_{k+1} \hat{x}_{k+1} - f_k) + r_k), \quad \hat{x}_N = Q_N^+ r_N . \tag{22}$$

Then

$$\min_{\{x_k\}} \mathbf{J}(\{x_k\}) = \mathbf{J}(\{\hat{x}_k\}) .$$

Proof. By definition put $\Phi(x_0) := \|F_0 x_0 - g\|_S^2 + \|y_0 - H_0 x_0\|_0^2$, $\Phi_i(x_i, x_{i+1}) := \|F_{i+1} x_{i+1} - C_i x_i - f\|_i^2 + \|y_{i+1} - H_{i+1} x_{i+1}\|_{i+1}^2$.

Then we obviously get

$$\mathbf{J}(\{x_k\}) = \Phi(x_0) + \sum_{i=0}^{N-1} \Phi_i(x_i, x_{i+1}) . \tag{23}$$

Let us apply a modification of Bellman's method (so-called «Kyivsky vinyk» method) to the nonlinear programming task

$$\mathbf{J}(\{x_k\}) \to \min_{\{x_k\}} .$$

By definition put

$$\ell_1(x_1) := \min_{x_0} \{\Phi(x_0) + \Phi_0(x_0, x_1)\} .$$

Using (7) and (21) one can get

$$\Phi(x_0) = (Q_0 x_0, x_0) - 2(r_0, x_0) + \alpha_0 \geq 0, \alpha_0 := \|g\|_S^2 + \|y_0\|_0^2 .$$

On the other hand it's clear that

$$\ell_1(x_1) = \Phi(\hat{x}_0) + \Phi_0(\hat{x}_0, x_1) = (Q_1 x_1, x_1) - 2(r_1, x_1) + \alpha_1 \geq 0,$$

where $\hat{x}_0 = P_0^+ (r_0 + C_0' S_0 (F_1 x_1 - f_0))$

$$\alpha_1 := \alpha_0 + \|y_1\|_1^2 + \|f_0\|_0^2 - (P_0^+ (r_0 - C_0' S_0 f_0), r_0 - C_0' S_0 f_0) .$$

Considering $\ell_1(x_1)$ as an induction base and assuming that

$$\ell_{i-1}(x_{i-1}) = \min_{x_{i-2}} \{\Phi_{i-2}(x_{i-2}, x_{i-1}) + \ell_{i-2}(x_{i-2})\} =$$

$$= (Q_{i-1} x_{i-1}, x_{i-1}) - 2(r_{i-1}, x_{i-1}) + \alpha_{i-1}$$

now we are going to prove that

$$\ell_i(x_i) = \min_{x_{i-1}} \{\Phi_{i-1}(x_{i-1}, x_i) + \ell_{i-1}(x_{i-1})\} = (Q_i x_i, x_i) - 2(r_i, x_i) + \alpha_i . \quad (24)$$

Note that [11] for any convex function $(x, y) \mapsto f(x, y)$

$$y \mapsto \min\{f(x, y) \mid (x, y) : P(x, y) = y\},\ P(a, b) = b$$

is convex. Thus taking into account the definition of $\ell_1(x_1)$ one can prove by induction that ℓ_{i-1} is convex and

$$\Phi_{i-1}(x_{i-1}, x_i) + \ell_{i-1}(x_{i-1}) \geq 0 .$$

Hence (the function $x \mapsto (Ax, x) - 2(x, q) + c$ is convex if $A = A' \geq 0$) $Q_{i-1} \geq 0$, the set of global minimums Ψ_{i-1} of the quadratic function.

$$x_{i-1} \mapsto \Phi_{i-1}(x_{i-1}, x_i) + (Q_{i-1}x_{i-1}, x_{i-1}) - 2(r_{i-1}, x_{i-1}) + \alpha_{i-1}$$

is non-empty and $\hat{x}_{i-1} \in \Psi_i$, where (The vector $\hat{x}_{i-1}$ has the smallest norm among other points of the minimum.)

$$\hat{x}_{i-1} = (Q_{i-1} + C'_{i-1}S_{i-1}C_{i-1})^+ (C'_{i-1}S_{i-1}(F_i x_i - f_{i-1}) + r_{i-1}) .$$

This implies

$$\ell_i(x_i) = \Phi_{i-1}(\hat{x}_{i-1}, x_i) + \ell_{i-1}(\hat{x}_{i-1}) = (Q_i x_i, x_i) - 2(r_i, x_i) + \alpha_i ,$$

where

$$\alpha_i = \alpha_{i-1} + (R_i y_i, y_i) + (S_{i-1}f_{i-1}, f_{i-1}) -$$

$$-(P_{i-1}^+(r_{i-1} - C'_{i-1}S_{i-1}f_{i-1}), r_{i-1} - C'_{i-1}S_{i-1}f_{i-1}) .$$

Therefore, we obtain

$$\min_{x_N} \ell_N(x_N) = \ell_N(\hat{x}_N) = \alpha_N - (r_N, Q_N^+ r_N), \hat{x}_N = Q_N^+ r_N$$

so that $\min_{\{x_k\}} \mathbf{J}(\{x_k\}) = \mathbf{J}(\{\hat{x}_k\})$.

Corollary 4. Suppose $\mathbf{L} = [0 \ldots \ell]$; then

$$\mathbf{L} \in \mathbf{R}[\mathbf{F'H'}] \Leftrightarrow [E - Q_N^+ Q_N]\ell = 0$$

and

$$\left\| [\mathbf{F'H'}]^+ \mathbf{L} \right\|^2 = (Q_N^+ \ell, \ell) .$$

Proof. Suppose $S_k = E,\ R_k = E$ for a simplicity. If $\mathbf{L} \in \mathbf{R}[\mathbf{F'H'}]$ then

$$F'_N z_N + H'_N u_N = \ell, \quad F'_k z_k + H'_k u_k - C'_k z_{k+1} = 0 (*),$$

for some $z_k \in \mathbf{R}^m$, $u_k \in \mathbf{R}^p$. Let's find the projection $\{(\hat{z}_k, \hat{u}_k)\}_{k=0}^N$ of the vector $\{(z_k, u_k)\}_{k=0}^N$ onto the range of the matrix $[\begin{smallmatrix}\mathbf{F}\\ \mathbf{H}\end{smallmatrix}]$. Lemma 3 implies

$$\hat{z}_0 = F_0 \hat{x}_0, \hat{z}_k = F_k \hat{x}_k - C_{k-1}\hat{x}_{k-1},\ \hat{u}_k = H_k \hat{x}_k, (**)$$

where

$$\hat{x}_k = P_k^+(C'_k F_{k+1}\hat{x}_{k+1} + r_k - C'_k z_{k+1}),\ \hat{x}_N = Q_N^+ r_N,$$

$$r_k = F'_k C_{k-1} P_{k-1}^+ r_{k-1} + F'_k (E - C_{k-1} P_{k-1}^+ C'_{k-1}) z_k +$$

$$+ H'_k u_k, r_0 = F'_0 z_0 + H'_0 u_0, P_k = C'_k C_k + Q_k$$

(*) implies $r_k = C'_k z_{k+1}$, $k = 0,\dots,N-1$, $r_N = \ell$ thus $\hat{x}_N = Q_N^+ \ell$, $\hat{x}_k = P_k^+ C'_k F_{k+1}\hat{x}_{k+1}$ or $\hat{x}_k = \Phi(k,N) Q_N^+ \ell$,

$$\Phi(k,N) = P_k^+ C'_k F_{k+1} \Phi(k+1,N), \Phi(s,s) = E.$$

Combining this with (**) we obtain

$$\hat{z}_k = (F_k \Phi(k,N) - C_{k-1}\Phi(k-1,N)) Q_N^+ \ell,$$

$$\hat{u}_k = H_k \Phi(k,N) Q_N^+ \ell,\ \hat{z}_0 = F_0 \Phi(0,N) Q_N^+ \ell. \tag{25}$$

By definition, put $U(0) = Q_0$,

$$U(k) = \Phi'(k-1,k) U(k-1) \Phi(k-1,k) + H'_k H_k + F_k (E - C_{k-1} P_{k-1}^+ C'_{k-1})^2 F_k.$$

It now follows that

$$\left\| [\mathbf{F'H'}]^+ \mathbf{L} \right\|^2 = \sum_0^N \|\hat{z}_N\|^2 + \|\hat{u}_N\|^2 = (U(N) Q_N^+ \ell, Q_N^+ \ell).$$

It's easy to prove by induction that $Q_k = U(k)$.

Since

$$\mathbf{L} \in R[\mathbf{F'H'}]$$

we obtain by substituting $\hat{z}_k, \hat{u}_k$ into (*)

$$F'_N \hat{z}_N + H'_N \hat{u}_N = \ell.$$

On the other hand (7) and (25) imply

$$F'_N \hat{z}_N + H'_N \hat{u}_N = \ell \Rightarrow [E - Q_N^+ Q_N]\ell = 0.$$

Suppose that $[E - Q_N^+ Q_N]\ell = 0$. To conclude the proof we have to show that

$$(\ell, x_N) = (Q_N^+ \ell, Q_N x_N) = 0,\ \forall [x_0 \dots x_N] \in \mathbf{N}[\mathbf{F'H'}].$$

By induction, fix $N = 0$. If $F_0 x_0 = 0,\ H_0 x_0 = 0$, then $Q_0 x_0 = 0$. We say that $[x_0 \dots x_k] \in \mathbf{N}[\mathbf{F'H'}]$ if

$$F_0 x_0 = 0,\ H_0 x_0 = 0,\ F_s x_s = C_{s-1} x_{s-1},\ H_s x_s = 0.$$

Suppose $Q_{k-1} x_{k-1} = 0,\ \forall [x_0 \dots x_{k-1}] \in \mathbf{N}[\mathbf{F'H'}]$ and fix any $[x_0 \dots x_k] \in \mathbf{N}[\mathbf{F'H'}]$. Then $F_k x_k = C_{k-1} x_{k_1}$, $H_k x_k = 0$. Combining this with (7) we obtain

$$Q_k x_k = F'_k (E - C_{k-1} P_{k-1}^+ C'_{k-1}) C_{k-1} x_{k-1} (*).$$

We show that $Q_k \geq 0$ in the proof of Theorem 1. One can see that

$$\begin{bmatrix} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{bmatrix}^{+} = [(C'_{k-1}C_{k-1} + Q_{k-1})^{+} C'_{k-1}, (C'_{k-1}C_{k-1} + Q_{k-1})^{+} Q_{k-1}^{\frac{1}{2}}].$$

Since

$$\begin{bmatrix} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{bmatrix}^{+}\begin{bmatrix} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{bmatrix} x_{k-1} = \begin{bmatrix} C_{k-1} \\ Q_{k-1}^{\frac{1}{2}} \end{bmatrix} x_{k-1}$$

we obviously get

$$C_{k-1}(C'_{k-1}C_{k-1} + Q_{k-1})^{+} C'_{k-1}C_{k-1}x_{k-1} = C_{k-1}x_{k-1} \Rightarrow Q_k x_k = 0$$

as it follows from $(*)$. This completes the proof.

Proof of Lemma 1. Taking into account the definitions of the matrices $\mathbf{F}, \mathbf{H}$ and (6) we clearly have

$$\mathbf{G}_y^N = \{\mathbf{X} : \|\mathbf{FX}\|^2 + \|\mathbf{Y} - \mathbf{HX}\|^2 \le 1\}.$$

Let $\hat{\mathbf{X}}$ be a minimum of the quadratic function $\mathbf{X} \mapsto \|\mathbf{FX}\|^2 + \|\mathbf{Y} - \mathbf{HX}\|^2$. It now follows that

$$\mathbf{G}_y^N = \hat{\mathbf{X}} + \mathbf{G}_0^N \Rightarrow \mathrm{s}(\mathbf{L} \mid \mathbf{G}_0^N) = (\mathbf{L}, \hat{\mathbf{X}}) + \mathrm{s}(\mathbf{L} \mid \mathbf{G}_0^N).$$

The application of Lemma 3 yields

$$(\mathbf{L}, \hat{\mathbf{X}}) = (\ell, Q_N^{+} r_N).$$

This completes the proof.

Proof of Lemma 2. Suppose the function $f : \mathbf{R}^n \to R^1$ is convex and closed. Then [11] the support function $s(\cdot \mid \{x : f(x) \le 0\})$ of the set $\{x : f(x) \le 0\}$ is given by

$$s(z \mid \{x : f(x) \le 0\}) = \mathrm{cl} \inf_{\lambda \ge 0} \left\{ \lambda f^{*}\left(\frac{z}{\lambda}\right) \right\}.$$

To conclude the proof it remains to compute the support function of $\mathbf{G}_0^N$ according to this rule and then apply Corollary 4.

Proof of Corollary 3. The proof is by induction on k. For $k = 0$, there is nothing to prove. The induction hypothesis is $P_{k-1|k-1} = Q_{k-1}^{-1}$. Suppose S is $n \times n$-matrix such that $S = S' > 0$, A is $m \times n$-matrix; then

$$A(S^{-1} + A'A)^{-1} = (E + ASA')^{-1} AS. \quad (26)$$

Using (26) we get

$$ASA' = [E + ASA']A[A'A + S^{-1}]^{-1} A'. \quad (27)$$

Combining (27) with the induction assumption we get the following

$$E + C_{k-1}P_{k-1|k-1}C'_{k-1} = E + [E + C_{k-1}P_{k-1|k-1}C'_{k-1}] \times$$

$$\times C_{k-1}[Q_{k-1}+C'_{k-1}C_{k-1}]^{-1}C'_{k-1}.$$

By simple calculation from the previous equality follows

$$E-C_{k-1}(Q_{k-1}+C'_{k-1}C_{k-1})^{-1}C'_{k-1}=(E+C_{k-1}P_{k-1|k-1}C'_{k-1})^{-1}.$$

Using this and (7), 15) we obviously get $Q_k^{-1}=P_{k|k}$.

It follows from the definitions that $Q_0^{-1}r_0=\hat{x}_{0|0}$. Suppose that $Q_{k-1}^{-1}r_{k-1}=$ $=\hat{x}_{k-1|k-1}$. The induction hypothesis and (26) imply

$$(E+C_{k-1}P_{k-1|k-1}C'_{k-1})^{-1}C_{k-1}\hat{x}_{k-1|k-1}=C_{k-1}(C'_{k-1}C_{k-1}+Q_{k-1})^{-1}_{k-1}r_{k-1}.$$

Combining this with (15), (16) and using $Q_k^{-1}=P_{k|k}$ we obtain

$$\hat{x}_{k|k}=Q_k^{-1}(F'_kC_{k-1}(C'_{k-1}C_{k-1}+Q_{k-1})^{+}_{k-1}r_{k-1}+H'_ky_k).$$

This concludes the proof.

Proof of Corollary 2. If $I_k<n$ then $\operatorname{rank}(Q)<n$ hence $\lambda_{min}(Q_k)=0$. In this case there is a direction $\ell\in\mathbf{R}^n$ such that $\hat{\sigma}(\ell,k)=+\infty$. So $\hat{\rho}(k)=+\infty$.

If $I_k=n$ then it follows from formula (11) that

$$\min_{\{x_k\}\in\mathbf{G}_y^N}\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}\|x_N-\tilde{x}_N\|^2=\min_{\{x_k\}\in\mathbf{G}_y^N}\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}\{\max_{\|l\|=1}|(\ell,x_N-\tilde{x}_N)|\}^2=$$

$$=\{\min_{\mathbf{G}_y^N}\max_{\|l\|=1}\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}|(\ell,x_N-\tilde{x}_N)|\}^2\geq$$

$$\geq\{\max_{\|l\|=1}\min_{\{x_k\}\in\mathbf{G}_y^N}\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}|(\ell,x_N-\tilde{x}_N)|\}^2=$$

$$=[1-\alpha_N+(Q_N^+r_N,r_N)]\max_{\|l\|=1}(Q_N^+\ell,\ell)=\frac{[1-\alpha_N+(Q_N^+r_N,r_N)]}{\min_i\{\lambda_i(N)\}}. \quad (28)$$

On the other hand formula (11) implies

$$\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}\|\hat{x}_N-\tilde{x}_N\|^2=\{\max_{\|l\|=1}\max_{\{\tilde{x}_k\}\in\mathbf{G}_y^N}|(\ell,x_N-\tilde{x}_N)|\}^2=$$

$$=\{\max_{\|l\|=1}[1-\alpha_N+(Q_N^+r_N,r_N)]^{\frac{1}{2}}(Q_N^+\ell,\ell)^{\frac{1}{2}}\}^2. \quad (29)$$

Using (28)–(29), we get (13).

Since (29) we see that the condition $I_N=n$ implies $\mathbf{G}_y^N$ is a bounded set. On the other hand $I_N=n$ implies $[E-Q_N^+Q_N]=0$ for the given N. It follows from Lemmas 1, 2 that

$$s(\ell \mid P_N(\mathbf{G}_y^N)) = s(P_N'\ell \mid \mathbf{G}_y^N) = s(\mathbf{L} \mid \mathbf{G}_y^N) = (\ell, Q_N^+ r_N) + \sqrt{\beta_N}\,(Q_N^+\ell, \ell)^{\frac{1}{2}}, \quad (30)$$

for any $\ell \in \mathbf{R}^n$. By Young's theorem [11], (30), so that

$$P_N(\mathbf{G}_y^N) = \{x \in \mathbf{R}^n : (x, \ell) \le s(\ell \mid P_N(\mathbf{G}_y^N)),\ \forall\, \ell \in \mathbf{R}^n\} =$$

$$= \{x \in \mathbf{R}^n : \sup_{\ell}\{(x,\ell) - (\ell, \hat{x}_N) - \sqrt{\beta_N}\,(Q_N^+\ell, \ell)^{\frac{1}{2}}\} \le 0\} =$$

$$= \{x \in \mathbf{R}^n : (Q_N x, x) - 2(Q_N \hat{x}_N, x) + \alpha_N \le 1\}.$$

This completes the proof.

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