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Transresonant Evolution of Spherical Waves Governed by the Perturbed Wave Equation

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Резонансные сферические волны, описываемые возмущенным волновым уравнением

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Выводится уравнение нелинейной акустики для радиальных сферических волн в твердом теле. Приближенное решение этого уравнения учитывает нелинейные, пространственные и диссипативные эффекты. Установлено, что в трансрезонансной частотной полосе могут возбуждаться нелинейные сферические волны, которые трудно классифицировать как хорошо известные солитон-, кноидал-, ударные или бриз-тип волны. Эти резонансные сферические волны также существенно отличаются от хорошо известных гладких сферических волн. Однако некоторые выражения для сферических волн напоминают известные решения для поверхностных волн.

1. **Introduction, a Governing Equation and Approximate Solutions**. A linear equation for spherically symmetric waves has a simple analytical solution. This solution was generalised for the case of resonant weakly nonlinear waves excited in gas [1, 2]. We use here this solution to study weakly nonlinear resonant spherical waves in solid resonators. With the help of this solution, the boundary problem reduces to the perturbed Korteweg–de Vries equation in time. A few solutions of this equation have been constructed. Contributions from nonlinear, spatial, transresonant, and dissipative effects can be seen from these solutions. According to these solutions, shock waves may be excited in an inviscid medium due to nonlinear effects. However, the formation of shock discontinuity is prevented due to viscosity and spatial dispersion. As a result of the competition among the nonlinear, dissipative, and spatial effects, periodic localized oscillating spherical excitations may be generated in resonators instead of the spherical shock waves. The shape and amplitude of these excitations depend on the excited frequency.

Let us consider spherically symmetric solid bodies and conical-type resonators having an approximately circular cross section. There, for the purely radial nonlinear waves, the following equations of continuity and motion are valid:

$$\rho_t + \rho_r u_t + \rho u_{rt} = -\frac{2}{r}\rho u_t, \tag{1}$$

$$\rho(u_{tt} + u_t u_{tr}) = \sigma_{r,r} + \frac{2}{r}(\sigma_r - \sigma_\varphi), \tag{2}$$

where u is the displacement, $\sigma_r = \lambda(u_r + 2r^{-1}u) + 2\mu u_r + \frac{4}{3}\eta(u_{tr} - r^{-1}u_t)$ and $\sigma_\varphi = \lambda(u_r + 2r^{-1}u) + \frac{2}{r}\mu u + \frac{2}{3}\eta(u_{tr} - r^{-1}u_t)$. Here the notations are standard and a viscoelastic model of a solid body is introduced. Equation (1) yields the following approximate expression: $\rho = \rho_0 - \rho_0(u_r + 2r^{-1}u)$, where ρ_0 is the undisturbed density. Using the expressions for ρ, σ_r, and σ_φ, and neglecting small terms, which are of the third order, we can rewrite Eq. (2) so that

$$\rho_0 u_{tt}(1 - u_r - 2r^{-1}u) + \rho_0 u_t u_{rt} = (\lambda + 2\mu)[u_{rr} + 2r^{-1}(u_r - r^{-1}u)] +$$

$$+\frac{2}{3r}\eta(u_r - r^{-1}u)_t + \frac{4}{3}\eta u_{trr}. \tag{3}$$

Then the displacement potential ϕ is introduced in (3) $(u = \phi_r)$. After integrating (3), we have

$$\rho_0\phi_{tt} - \rho_0\int \phi_{ttr}(\phi_{rr} + 2r^{-1}\phi_r)dr + 0.5\rho_0\phi_{tr}^2 =$$

$$= (\lambda + 2\mu)(\phi_{rr} + 2r^{-1}\phi_r) + \frac{4}{3}\eta(\phi_{rr} + 2r^{-1}\phi_r)_t. \tag{4}$$

Let the viscous term in (4) be of the second order. For this case, a linear wave equation follows from (4):

$$a_0^2(\phi_{rr} + 2r^{-1}\phi_r) = \phi_{tt},$$

where $a_0^2 = (\lambda + 2\mu)/\rho_0$. Using the wave equation, we can simplify (4). As a result, we have

$$\rho_0\phi_{tt}(1 - 0.5a_0^{-2}\phi_{tt}) + 0.5\rho_0(\phi_{rt})^2 = (\lambda + 2\mu)(\phi_{rr} + 2r^{-1}\phi_r) + \frac{4}{3}\eta a_0^{-2}\phi_{ttt}.$$

Let $\varphi = \phi_t$. For the latter case, we have the following equation of nonlinear acoustics for a homogeneous viscoelastic solid body:

$$a_0^2(\varphi_{rr} + 2r^{-1}\varphi_r) = \varphi_{tt} - a_0^{-2}\varphi_t\varphi_{tt} + \varphi_r\varphi_{rt} - \delta a_0^{-2}\varphi_{ttt}, \tag{5}$$

where $\delta = \frac{4}{3}\eta$. It is important that Eq. (5) differs from the equation of nonlinear acoustics for gas and liquid only by some coefficients [1, 2, 3]. Therefore, bellow we use the results obtained for resonant waves in spherical gas layers [1, 2, 4, 5].

We emphasize that Eq. (5) does not take into account the third order effects and the dissipative term is of the second order. The solution of (5) may be presented as

$$\varphi = \varphi_1 + \varphi_2, \tag{6}$$

where φ_1 and φ_2 are the first- and the second-order values, respectively. Substituting (6) into (5) and equating the values of the same order, we obtain a system of differential equations for φ_1 and φ_2:

$$\varphi_{1rr} + 2r^{-1}\varphi_{1r} = a_0^{-2}\varphi_{1tt}, \tag{7}$$

$$a_0^2(\varphi_{2rr} + 2r^{-1}\varphi_{2r}) = \varphi_{2tt} + \varphi_{1r}\varphi_{1rt} - a_0^{-2}\varphi_{1t}\varphi_{1tt} - \delta a_0^{-2}\varphi_{1ttt}. \tag{8}$$

The solution of the linear wave equation (7) is the sum of two travelling waves: $\varphi_1 = r^{-1}(f_1 + f_2)$. Here and below, $f_1 = f_1(\xi)$ and $f_2 = f_2(\eta)$, where $\xi = a_0 t - r$ and $\eta = a_0 t + r$. Now we can rewrite Eq. (8) in the form

$$a_0^2(\varphi_{2rr} + 2r^{-1}\varphi_{2r}) - \varphi_{2tt} = a_0\{r^{-2}(f_2' - f_1')(f_2'' - f_1'') -$$

$$-r^{-3}[(f_2 + f_1)(f_2'' - f_1'') - (f_1')^2 + (f_2')^2] + r^{-4}(f_2 + f_1)(f_2' + f_1') -$$

$$-r^{-2}(f_2' + f_1')(f_2'' + f_1'') - \delta r^{-1}(f_2''' + f_1''')\},$$

where the primes denote a derivative with respect to the argument. One can find φ_2 from this equation following [1, 2, 4, 5]. Finally, the approximate solution of (5) is

$$\varphi = r^{-1}(f_1 + f_2 + \psi_1 + \psi_2) + 0.25a_0^{-1}r^{-2}[(f_1 + f_2)^2]' -$$

$$-0.25a_0^{-1}r^{-1}\int\int r^{-1}(f_1' + f_2')(f_1'' + f_2'')d\xi d\eta + 0.25\delta a_0^{-1}r^{-1}(\eta f_1'' + \xi f_2''). \tag{9}$$

Here $\psi_1 = \psi_1(\xi)$ and $\psi_2 = \psi_2(\eta)$, and the functions ψ_1 and ψ_2 are of the second order. The functions f_1, f_2, ψ_1, and ψ_2 are unknown and must be found from the initial and boundary conditions. However, solution (9) is complicated by the integral. Let us simplify it to a form that is more convenient for satisfying the boundary conditions. Near any boundary surface $r = R$ and the multiplier $1/r$ under the integral is replaced by $1/R$. As a result, we have

$$\varphi = r^{-1}(f_1 + f_2 + \psi_1 + \psi_2) + 0.25a_0^{-1}r^{-2}[(f_1 + f_2)^2]' -$$

$$-0.25a_0^{-1}r^{-1}R^{-1}[0.5\eta(f_1')^2 + 0.5\xi(f_2')^2 + f_1'f_2 + f_2'f_1] +$$

$$+0.25\delta a_0^{-1}r^{-1}(\eta f_1'' + \xi f_2''). \tag{10}$$

Solution (10) satisfies Eq. (5) if the expression $\frac{1}{2}a_0 r^{-2}[(f_1' + f_2')^2]'(1 - rR^{-1})$ is of the third order. Thus, (10) is valid near the surface $r = R$, where $|1 - rR^{-1}| << 1$.

In this article, we examine only periodical oscillations. In this case, the velocity must not contain secular terms. The secular terms will be eliminated if we assume in (10)

$$\psi_i = \Psi_i + 0.125a_0^{-1}R^{-1}[\xi(f_i')^2 - 2f_i f_i'] - 0.25\delta a_0^{-1}\xi f_i'',$$

where $i = 1$ or 2; f_1, f_2, $\Psi_1 = \Psi_1(\xi)$, and $\Psi_2 = \Psi_2(\eta)$ are periodic functions. As a result, near the surface $r = R$, we have for steady-state oscillations

$$\varphi = r^{-1}(f_1 + f_2 + \Psi_1 + \Psi_2) + 0.25a_0^{-1}r^{-2}(1 - 0.5rR^{-1})[(f_1 + f_2)^2] -$$

$$-0.25a_0^{-1}R^{-1}[(f_1')^2 - (f_2')^2] + 0.5\delta a_0^{-1}(f_1'' - f_2''). \tag{11}$$

Both expression (9) and (11) are used below to solve a boundary problem.

2. **A Boundary Problem and Basic Equation**. Let us consider waves excited by an oscillating velocity at the surface $r = R$. Therefore, we have

$$\varphi_r = -\omega B \sin \omega t \quad (r = R), \tag{12}$$

$$4\pi r^2 \varphi_r = 0 \quad (r \to 0). \tag{13}$$

We have written (13) according to [6, p. 491]. When $r \to 0$, the stresses increase and the mechanical properties of the material can change strongly at the origin. As a result, Eq. (5) and solution (9) are not valid if $r = 0$. Therefore, it is possible that Eq. (13) is a rough approximation of the reality at $r = 0$. Let us assume that the influence of the origin is very local and does not change the wave pattern qualitatively. Using (9), we can rewrite condition (13) so that

$$r(f_2' - f_1' + \Psi_2' - \Psi_1') - f_1 - f_2 - \Psi_1 - \Psi_2 +$$

$$+0.25a_0^{-1}r^2\{r^{-2}[(f_1 + f_2)^2]'\}_r + 0.125\delta a_0^{-1}r^2[r^{-1}(\eta f_1'' + \xi f_2'')]_r +$$

$$+0.25a_0^{-1}\int\int r^{-1}(f_1' + f_2')(f_1'' + f_2'')d\xi d\eta -$$

$$-0.25a_0^{-1}r\Big[\int\int r^{-1}(f_1'+f_2')(f_1''+f_2'')d\xi d\eta\Big]_r=0, \tag{14}$$

where $r\to 0$. Equation (14) is satisfied if

$$\begin{aligned} &f_1(a_0t-r)=f(a_0t-r), \quad f_2(a_0t+r)=-f(a_0t+r),\\ &\Psi_1(a_0t-r)=\Psi(a_0t-r), \quad \Psi_2(a_0t+r)=\Psi(a_0t+r). \end{aligned} \tag{15}$$

In (14)

$$\int\int\frac{1}{r}(f_1'+f_2')(f_1''+f_2'')d\xi d\eta=2\int\int(f')_r[f''(\xi)-f''(\eta)]d\xi d\eta\to 0$$

because $\xi\to\eta$. Condition (12) is written now with the help of (11):

$$R(f_2'-f_1'+\Psi_2'-\Psi_1')-f_1-f_2-\Psi_1-\Psi_2+$$

$$+0.75a_0^{-1}R^{-1}(f_1+f_2)(f_1'+f_2')+$$

$$+0.25a_0^{-1}[(f_2')^2-(f_1')^2+(f_1+f_2)(-f_1'+f_2')]+$$

$$+0.5a_0^{-1}R(f_1'f_1''+f_2'f_2'')-0.5\delta a_0^{-1}R^2(f_1'''+f_2''')=-\omega BR^2\sin\omega t. \tag{16}$$

Here we must take into account (15). As a first approximation, it follows from (15) and (16) that

$$f(\xi)-f(\eta)+Rf'(\xi)+Rf'(\eta)=\omega BR^2\sin\omega t. \tag{17}$$

Following [1, 2, 4, 5], we assume $f(\xi)$ and $f(\eta)$ in the form

$$\begin{aligned} f(\xi)=&-R^{-1}\int F(\xi+R)d\xi-[a_0\omega^{-1}R^{-1}(\sin\omega Ra_0^{-1}-\\ &-\omega Ra_0^{-1}\cos\omega Ra_0^{-1})-1]F(\xi+R),\\ f(\eta)=&-R^{-1}\int F(\eta-R)d\eta-[a_0\omega^{-1}R^{-1}(\sin\omega Ra_0^{-1}-\\ &-\omega Ra_0^{-1}\cos\omega Ra_0^{-1})+1]F(\eta-R). \end{aligned} \tag{18}$$

Then Eq. (17) describes travelling waves

$$F[a_0t\pm(r-R)]=$$

$$=0.5\omega BR^2(\sin\omega a_0^{-1}R-\omega a_0^{-1}R\cos\omega a_0^{-1}R)^{-1}\cos\omega a_0^{-1}[a_0t\pm(r-R)]. \tag{19}$$

From (19), we obtain resonant frequencies as

$$\Omega_\gamma = \pi\gamma a_0 R^{-1}, \tag{20}$$

where $\gamma = 1.4303, 2.4590, 3.4709, \ldots$ [6, p. 506]. Linear solutions (19), (18), and (15) are not valid near the frequencies $\omega = \Omega_\gamma$. Therefore, Eq. (16) will be considered taking into account nonlinear terms. Considering the nonlinear terms, we assume that near a fast varying solution

$$\left| a_0 R^{-1} \int F(a_0 t) dt \right| << \left| F(a_0 t) \right|. \tag{21}$$

Taking into account (21), (18), and (15), we rewrite Eq. (16) in the following way:

$$\omega_1 F' + R^{-1}(\Psi_2' - \Psi_1') - R^{-2}(\Psi_1 + \Psi_2) - \delta a_0^{-1} F''' -$$

$$- 3a_0^{-1} R^{-3} F F' + a_0^{-1} R^{-1} F' F'' = -\omega B \sin\omega t, \tag{22}$$

where

$$\omega_1 = 2a_0 \omega^{-1} R^{-2} (\sin\omega R a_0^{-1} - \omega R a_0^{-1} \cos\omega R a_0^{-1}).$$

Equation (22) is complex for the integration. This equation can be simplified if we assume the following expressions in (11):

$$\begin{aligned} \Psi_1 &= \Psi_1(\xi) = \Psi(\xi) = 0.25 a_0^{-1} [F'(\xi + R)]^2, \\ \Psi_2 &= \Psi_2(\eta) = -\Psi(\eta) = -0.25 a_0^{-1} [F'(\eta - R)]^2. \end{aligned} \tag{23}$$

As a result, we obtain the following basic equation from (22), which is valid if (21) takes place:

$$\omega_1 F' - \delta a_0^{-1} F''' - 3a_0^{-1} R^{-3} F F' = -\omega B \sin\omega t. \tag{24}$$

Equation (24) resembles Eq. (3.17) from [7]. After integrating, we take a constant of integration in the form $c = a_0 B(1 - 8\pi^{-2} R^2)$. Then Eq. (24) may be rewritten as

$$(F - 2G\sqrt{\varepsilon}\pi^{-1})^2 + q_0 \varepsilon^{0.5} F''_{\tau\tau} = \varepsilon \cos^2\tau. \tag{25}$$

Here

$$\tau = \omega t/2, \quad G = \pi\omega_1 a_0 \varepsilon^{-0.5} R^3/6, \quad q_0 = -\delta\omega^2 \varepsilon^{-0.5} a_0^{-2} R^3/6, \quad \varepsilon = 4a_0^2 B R^3/3.$$

This equation has a nonlinear term that tends to produce a ‘discontinuity’ solution. The second term, which is generated due to viscosity of the medium, is responsible for the dispersion of the waves.

3. **Resonant Waves**. *Inviscid Medium*. For this case, Eq. (25) does not have a smooth periodic solution with the same period as forced oscillations. Therefore, following [7], we construct a discontinuous $2\pi/\omega$-periodic solution of (25):

$$F = \sqrt{\varepsilon}\cos\tau, \tag{26}$$

where $0 < \tau < \pi$. This solution is valid for exact resonance ($G = 0$). Now with the help of (18), we can calculate the functions $f(a_0 t \pm r)$. Then, taking into account (15) and (11), the expression for φ may be written. We recall again that (11) does not take into account correctly the second-order values far from the boundaries. Therefore, we must consider only the first-order terms in this expression so that

$$\varphi = \sqrt{\varepsilon}r^{-1}\Big\{\cos\frac{1}{2}[\omega t - \omega a_0^{-1}(r-R)] +$$

$$+\cos\frac{1}{2}[\omega t + \omega a_0^{-1}(r-R)] - 2a_0\omega^{-1}R^{-1}\sin\frac{1}{2}[\omega t - \omega a_0^{-1}(r-R)] +$$

$$+2a_0\omega^{-1}R^{-1}\sin\frac{1}{2}[\omega t + \omega a_0^{-1}(r-R)]\Big\}, \tag{27}$$

where $0 < \omega t \pm \omega a_0^{-1}(r-R) < 2\pi$. Within subsequent intervals of length π, we can find φ by the periodic continuation of $\cos\frac{1}{2}[\omega t \pm \omega a_0^{-1}(r-R)]$ and $\sin\frac{1}{2}[\omega t \pm \omega a_0^{-1}(r-R)]$ in (27). As a result, the solution is obtained with the same period as the forced oscillations. The function φ is discontinuous along straight lines: $\omega t \pm \omega a_0^{-1}(r-R) = 2n\pi$ ($n = 0, 1, 2, 3, ...$). Generally speaking, solutions (26) and (27) take place near the shock jumps, where condition (21) is valid.

Thus, according to the inviscid model of the material, resonant shock waves may be excited in a sphere. These waves are the sum of the spherical saw-tooth-like travelling waves. However, this result changes dramatically if we take into account the spatial dispersion [the second term in Eq. (25)].

Effect of Spatial Dispersion. Transresonant Process. If the combined effects of the nonlinearity and the spatial dispersion compensate each other, then soliton-like waves may be excited in the system. Following [8], we seek solutions of (25) for this case in the form $F = \sqrt{\varepsilon}[2G\pi^{-1} + \Phi(\tau)\cos\tau]$, where $\Phi(\tau)$ is an unknown function. As a result, we obtain the following equation:

$$\Phi'' - 2\Phi'\tan\tau - \Phi = q_0^{-1}(1-\Phi^2)\cos\tau. \tag{28}$$

We assume that $q_0 << 1$. We seek a localized fast varying solution of (28). Let

$$\Phi = \{A\,\mathrm{sech}^2[\gamma(\sin M^{-1}\tau - G)] + C\}\cos\tau, \tag{29}$$

where A, γ, and C are constant values, and $M = 1, 2, 3, \ldots$. Solution (29) is localized near the points, where $\sin M^{-1}\tau \approx G$. Solution (29) approximately satisfies Eq. (28) if $A = 6q_0\gamma^2 M^{-2}$, $\gamma^2 = 0.5M^2(1 - q_0^{-1}C)$, and $C_\pm = 4(q_0 \pm \pm\sqrt{q_0^2 + 3/4})/3$. If $q_0 << 1$, then $C_- \approx -1$, $\gamma^2 \approx 0.5q_0^{-1}M^2$, and $A \approx 3$. For the latter case, the expression for F written for the travelling waves is as follows:

$$F(2a_0\omega^{-1}p_\pm) = 2\sqrt{\varepsilon}\,G\pi^{-1} +$$

$$+\sqrt{\varepsilon}\{3\,\mathrm{sech}^2[M(\sin M^{-1}p_\pm - G)/\sqrt{2q_0}] - 1\}\cos^2 p_\pm, \tag{30}$$

where $p_\pm = \frac{1}{2}\omega t \pm \frac{1}{2}\omega a^{-1}(r - R)$. Strictly speaking, solution (30) is valid if $G \approx 0$. Then condition (21) takes place. According to (30), when $\sin M^1 p_\pm \approx G$, a peak of the function $F(2a_0\omega^{-1}p_\pm)$ is generated and then a crater occurs. This excitation resembles the so-called '*oscillon*' [9]. By contrast to oscillons, expressions (30), (18), (15), and (11) describe *travelling spherical oscillons.* Generally speaking, solution (30) defines a spectrum of subharmonic localized waves if $M = 2, 3, 4, \ldots$. If $M = 1$, oscillations are possible with a forced frequency ωt. Thus, near and at resonance, periodic resonant localized waves are predicted by (30).

Now, using (18) and (15), it is possible to find $f_1(\xi)$ and $f_2(\eta)$. Then we can write expressions for φ, stresses, and velocity. However, we emphasise that (11) does not take into account the second-order values far from the boundaries. Therefore, we must only consider the first-order terms in the expressions. For example, for the velocity we have

$$\varphi_r = -r^{-2}[rf'(\xi) + f(\xi)] - r^{-2}[rf'(\eta) - f(\eta)], \tag{31}$$

where the functions $f(a_0 t \pm r)$ are found approximately according to (30) and (18):

$$f(a_0 t \pm r) = -(\pm)\sqrt{3}\{2G\pi^{-1} + 3\,\mathrm{sech}^2[(\sin p_\pm - G)/\sqrt{2q_0}]\cos^2 p_\pm -$$

$$-\cos^2 p_\pm\} + a_0\omega^{-1}R^{-1}\sqrt{\varepsilon}\{p_\pm - 4G\pi^{-1}p_\pm + 0.5\sin 2p_\pm -$$

$$-2\sqrt{2q_0}\tanh[(\sin p_\pm - G)/\sqrt{2q_0}]\}. \tag{32}$$

We assume here $M = 1$. At the same time, according to (18), we have

$$f'(a_0 t \pm r) = -R^{-1}F(2a_0\omega^{-1}p_\pm), \tag{33}$$

where expression (30) is valid. One can see that the shape of the wave (31) may be complex. At the same time, the most important contributions in (31) are defined by expressions $3\mathrm{sech}^2[(\sin p_\pm - G)/\sqrt{2q_0}]\cos^2 p_\pm$. Thus, near and at resonance, the localized oscillating spherical waves are excited, which are quite different from the well-known saw-tooth spherical waves.

Free Oscillations or Trapped Waves. Oscillations may be generated due to a change in the velocity at $r=R$ at the moment $t=0$. We assume that, after a sufficiently long time, transient oscillations are damped, and we are interested in free nonlinear steady-state oscillations. Oscillations with a frequency of Ω_γ are considered. For this case, it follows from (24) that

$$\delta a_0^{-1}F'' + 1.5a_0^{-1}R^{-3}F^2 = c, \qquad (34)$$

where c is some constant of integration. We assume that c is defined by the initial condition. Let $F = E_1 + A\mathrm{sech}^2\left(E\sin\frac{1}{2}\Omega_\gamma t\right)\cos^2\frac{1}{2}\Omega_\gamma t$, where $|E| >> 1$. In the latter expression, the second term localizes near the points $\frac{1}{2}\Omega_\gamma t = (K-1)\pi$ $(K=1, 2, 3, ...)$. Then one can approximately find from (34): $E_1 = -(-0.66ca_0R^3)^{0.5}$, $E^2 = 0.1875\delta^{-1}a_0^2\Omega_\gamma^{-2}(-0.66ca_0R^{-3})^{0.5}$, and $A = -3E_1$. For the travelling waves,

$$F(2a_0\omega^{-1}p_\pm) = E_1[1 - 3\mathrm{sech}^2(E\sin\Omega_\gamma\omega^{-1}p_\pm)\cos^2\Omega_\gamma\omega^{-1}p_\pm]. \qquad (35)$$

Solution (35) is valid near the lines, where $\sin\Omega_\gamma\omega^{-1}p_\pm \approx 0$. Thus, if the coefficient δ ensures condition $|E| >> 1$, then nonlinear localized free spherical waves may be generated in resonators. These waves are defined by expressions (35), (18), (15), and (11).

4. **Discussion**. Thus, strongly localized spherical waves can travel within resonators according to the above analysis. Now we can calculate the stresses and velocity in the medium. We recall again that we must only consider the first-order terms for them. For example, we have expression (31) for the velocity. Pictures of the variation of the velocity $\varphi_r\varepsilon^{-0.5}$ are presented in Figs. 1–3. The dimensionless time τ and radius r/R are used, and $(2q_0)^{-0.5} = 3$ and $a_0 = 340$ m/s. There is strong amplification of the waves near $r=0$. Figures 1 and 2 display particularities of the transresonant process for the case of spherical waves. It is known that passage through resonance is a classic problem. However, usually one- or several-degree-of-freedom models are used. From Figures one can see that sometimes the properties of nonlinear waves may be very important. The waves depend on the excited frequency. They are localized and strongly amplified at resonance ($G=0$). The fast varying waves transform into harmonic waves when $|G|$ increases. If $|G| \approx 1$, two-peak localized waves with small amplitude are excited (Figs. 1 and 2). The process, which resembles a transresonant process, takes place at resonance if the dissipative effect changes (Fig. 3).

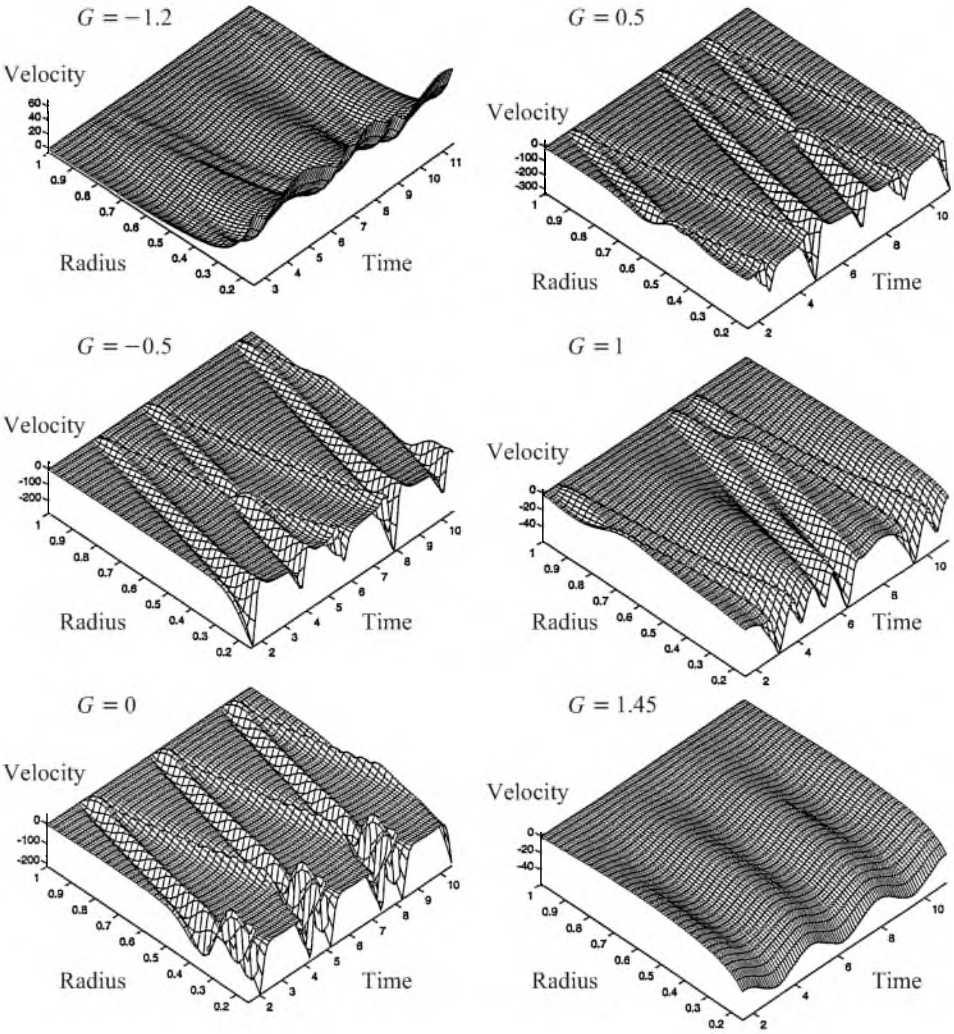

Fig. 1. Transresonant evolution of spherical waves (γ = 1.4303, q_0 = 0.02).

We have considered resonant localization of waves in spherical resonators. On the one hand, these waves are strictly different from the waves in elongated natural resonators [8, 10–12] and tubes [7, 12]. On the other hand, solutions (30) and (35) resemble the expressions for travelling localized plane surface waves (see solutions (35) and (39) from [8]). The localization of surface waves has been considered recently [9, 13, 14]. These localized waves are usually observed in parametrically excited dispersive systems [9–11, 13, 14].

Thus, solutions (30) and (35) of the perturbed wave equation describe a variety of wave processes in dispersive systems. One can see from (25) that in spherical systems dispersive effects are defined by viscous properties of the material. Periodic localized oscillating spherical waves are generated because spatial dispersive and nonlinear effects balance each other within the sphere. Thus, smooth localized waves rather than shock waves are formed in the system. This result agrees qualitatively with the data of numerical calculation [15].

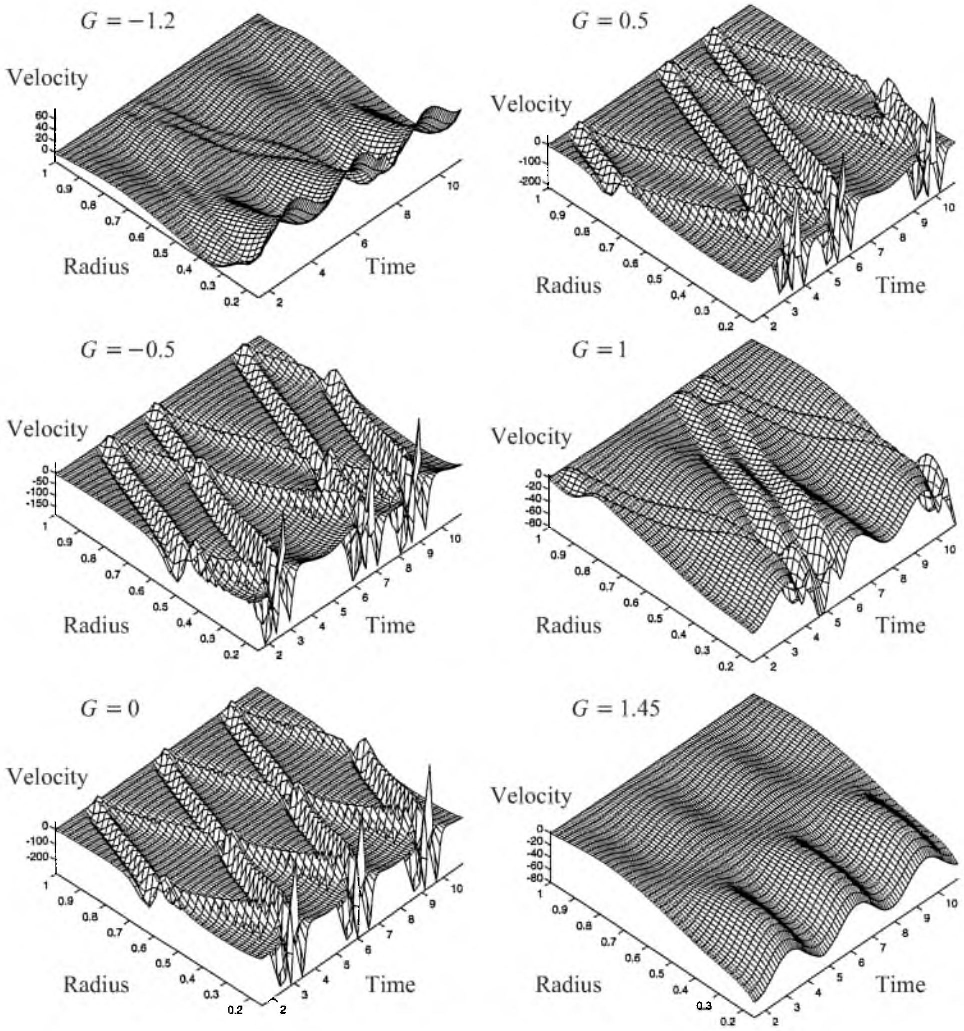

Fig. 2. Transresonant evolution of spherical waves ($\gamma = 2.4590$, $q_0 = 0.02$).

Localization also takes place because of the wave focusing [1, 14]. The order of the amplitude $O(\varepsilon^{0.5})$ of resonant spherical waves is the same as for plane waves in elongated resonators with fixed boundaries [7, 8, 10–12].

Resonant spherical nonlinear waves, in contrast to plane resonant waves, practically were not studied. At the same time, a spherical model for the simulation of different physical objects is very popular. Indeed, on the one hand, the model of a pulsating sphere is widely used in astrophysics [16]. On the other hand, this model is used to study sonoluminescence in liquids where the period of oscillation and space distances are very small [15]. However, the competition between nonlinear and spatial dispersive effects in resonant spherical systems has not been studied. Due to this competition, the distortion of harmonic waves into oscillating localized resonant waves can take place. Our study has been strictly limited by the aspect of nonlinear acoustics. However, the results presented may be interesting for various media and circumstances [10, 11, 17].

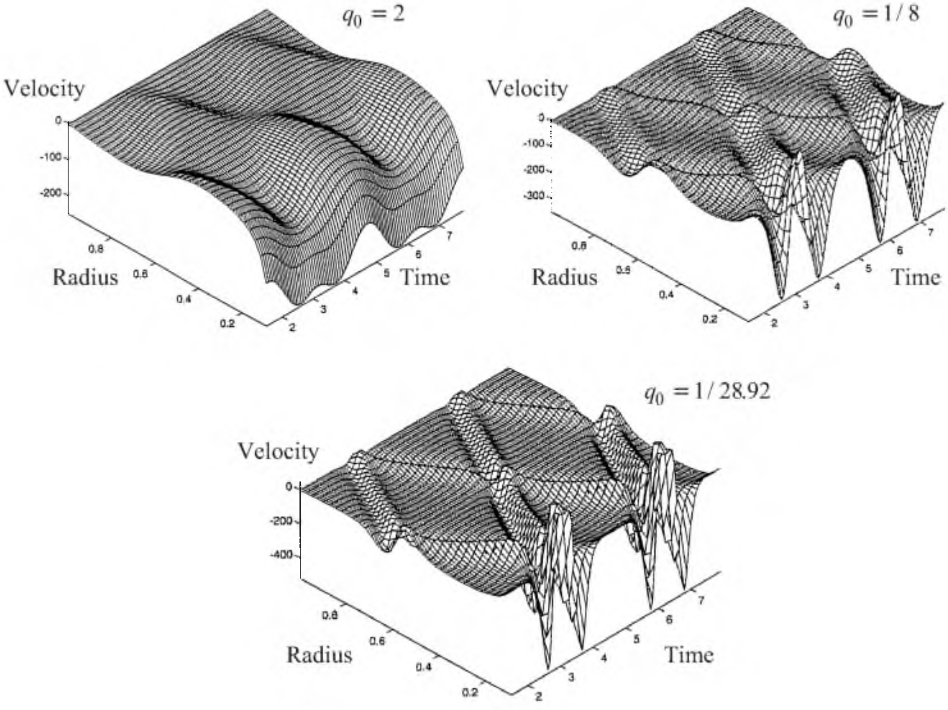

Fig. 3. Localization of waves due to dissipative and spatial dispersive effects.

Резюме

Виводиться рівняння нелінійної акустики для сферичних хвиль у твердому тілі. Наближений розв'язок цього рівняння ураховує нелінійні, просторові і дисипативні ефекти. Установлено, що у трансрезонансній смузі частот можуть збуджуватися нелінійні сферичні хвилі, які важко класифікувати як добре відомі солитон-, кноїдал-, ударні або бриз-тип хвилі. Ці резонансні сферичні хвилі також суттєво відрізняються від добре відомих гладких сферичних хвиль. Однак деякі вирази для сферичних хвиль нагадують відомі розв'язки для поверхневих хвиль.

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