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Vibrations of a Complex Systems with Damping under Dynamic Loading

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Колебания сложных систем с затуханием при динамическом нагружении

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На основе разработанного аналитико-численного метода решения задачи о свободных и вынужденных затухающих колебаниях на примере сложной системы, которая состоит из двух пластин, соединенных вязкоупругим слоем, выполнен численный анализ и изучены новые механические эффекты, обусловленные действием на рассматриваемую систему различного типа динамического нагружения. Расчеты проводились на основе моделей Тимошенко и Кирхгофа–Лява.

Introduction. Compound systems coupled together by viscoelastic constraints play an important role in various engineering and building structures. Since 1923, Timoshenko's model [1] for various compound constructions has been applied. Vibration analysis of laminated plates was presented in [2, 3] and in many other works.

The problem of nonaxisymmetric deformation of flexible rotational shells was solved in [4] with the use of the classical Kirchhoff–Love model and improved Timoshenko's model. The dynamic problem of elastic homogeneous bodies was presented in [5]. Vibration analysis of systems of solid and deformable bodies for complex motion was considered in [6].

Vibrations of elastically connected rectangular double-plate compound systems under moving loading are presented in [7]. Vibration analysis of compound systems with vibration damping is a difficult problem. In the above complex cases, especially where viscosity and discrete elements occur, it is recommended to adopt the method of solving a dynamic problem for a system in the domain changing of a complex function [8, 9]. The property of orthogonality of free vibrations of complex types was first described in [8] for discrete systems with damping, for discrete-continuous systems with damping in [9] and for continuous systems with damping in [10].

The goal of this paper is to present a method for solving the problem and dynamic analysis of free and forced vibrations for a complex system with damping, which consists of two elastic plates coupled by a viscoelastic interlayer, for various types of dynamic loading.

Statement of the Problem. Let us consider a problem of free and forced vibrations for a complex system with damping. External layers of the complex system are made from elastic materials as plates coupled by a viscoelastic interlayer (Fig. 1). The elastic plates are simply supported at their ends and described by Timoshenko's model [1]. The viscoelastic interlayer has the characteristics of a homogenous continuous unidirectional Winkler's foundation and is described by the Voigt–Kelvin model [11–13].

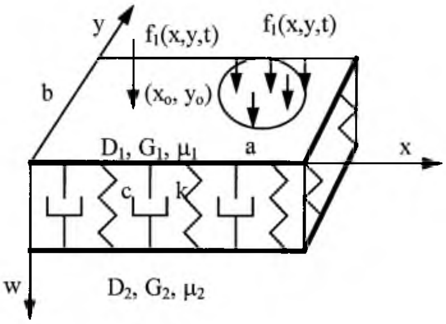

Fig. 1. Dynamic model of a complex system with damping.

In this paper, we consider two cases. In the first case, small-frequency transverse vibrations of a complex system with damping are excited by a stationary dynamic load $f_1(x,y,t)$ at the point x_0, y_0 with the load varying in time t. In the second case, small-frequency transverse vibrations of a complex system with damping are exited by a nonstationary dynamic load $f_1(x,y,t)$ varying in time t.

The phenomenon of small-frequency transverse vibrations for a complex system with damping is described by Timoshenko's model with the following non-homogeneous system of conjugate partial differential equations:

$$\begin{cases} D_1\left[\dfrac{\partial^2\psi_{11}}{\partial x^2}+\dfrac{\partial^2\psi_{11}}{\partial y^2}\right]+H_1\left(\dfrac{\partial w_1}{\partial x}+\psi_{11}\right)-\Xi_1\dfrac{\partial^2\psi_{11}}{\partial t^2}=0, \\ D_1\left[\dfrac{\partial^2\psi_{12}}{\partial x^2}+\dfrac{\partial^2\psi_{12}}{\partial y^2}\right]+H_1\left(\dfrac{\partial w_1}{\partial x}+\psi_{12}\right)-\Xi_1\dfrac{\partial^2\psi_{12}}{\partial t^2}=0, \\ \mu_1\dfrac{\partial^2 w_1}{\partial t^2}-H_1\left(\nabla^2 w_1+\dfrac{\partial\psi_{11}}{\partial x}+\dfrac{\partial\psi_{12}}{\partial y}\right)+(w_1-w_2)\left(k+c\dfrac{\partial}{\partial t}\right)=f_1(x,y,t), \\ D_2\left[\dfrac{\partial^2\psi_{21}}{\partial x^2}+\dfrac{\partial^2\psi_{21}}{\partial y^2}\right]+H_2\left(\dfrac{\partial w_2}{\partial x}+\psi_{21}\right)-\Xi_2\dfrac{\partial^2\psi_{21}}{\partial t^2}=0, \\ D_2\left[\dfrac{\partial^2\psi_{22}}{\partial x^2}+\dfrac{\partial^2\psi_{22}}{\partial y^2}\right]+H_2\left(\dfrac{\partial w_2}{\partial x}+\psi_{22}\right)-\Xi_2\dfrac{\partial^2\psi_{22}}{\partial t^2}=0, \\ \mu_2\dfrac{\partial^2 w_2}{\partial t^2}-H_2\left(\nabla^2 w_2+\dfrac{\partial\psi_{21}}{\partial x}+\dfrac{\partial\psi_{22}}{\partial y}\right)-(w_1-w_2)\left(k+c\dfrac{\partial}{\partial t}\right)=0, \end{cases} \tag{1}$$

where

$$D_1 = \frac{E_1 h_1^3}{12(1-\nu_{1p}^2)}, \; D_2 = \frac{E_2 h_2^3}{12(1-\nu_{2p}^2)}, \; H_1 = K_1' G_1 h_1, \; H_2 = K_2' G_2 h_2,$$

$$\mu_1 = \rho_1 h_1, \; \mu_2 = \rho_2 h_2, \; \Xi_1 = \frac{\rho_1 h_1^3}{12}, \; \Xi_2 = \frac{\rho_2 h_2^3}{12}, \; \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}$$

$$\psi_{1i} = \gamma_{1i} + \vartheta_{1i}, \; \psi_{2i} = \gamma_{2i} + \vartheta_{2i}, \; i = 1, 2.$$

Here $w_1 = w_1(x,y,t)$ and $w_2 = w_2(x,y,t)$ are the transverse deflections of plates I and II, respectively; $\psi_{11} = \psi_{11}(x,y,t)$, $\psi_{12} = \psi_{12}(x,y,t)$, $\psi_{21} = \psi_{21}(x,y,t)$, and $\psi_{22} = \psi_{22}(x,y,t)$ are the full angles of rectilinear elements of plates I and II turn; E_1 and E_2 are Young's moduli of the material of plates I and II, respectively; E is Young's modulus of the material of the interlayer; G_1 and G_2 are the shear moduli of the material of plates I and II, respectively; ρ_1 and ρ_2 are the mass densities of the material of plates I and II; K_1' and K_2' are the shear coefficients; k is the coefficient of elasticity of the interlayer; c is the coefficient of viscosity of the interlayer; h_1 and h_2 are the thickness of plates I and II; h is the thickness of the interlayer; a and b are the dimensions of the plates; γ_{1i} and γ_{2i} are the angles of the rectilinear element turn due to a cross shear; ϑ_{1i} and ϑ_{2i} are the angles of the middle surface turn in plane for $\alpha = \text{const}$ and $\beta = \text{const}$; ν_{1p} and ν_{2p} are Poisson's ratios; t is time; x, y, and z are the coordinate axes; $f_1(x,y,t)$ is the dynamic load acting on the complex system.

Separation of Variables. The analytical numerical method is based on the separation of variables. Presenting the solution of the problem considered in the form

$$\begin{bmatrix} w_1(x,y,t) \\ \psi_{11}(x,y,t) \\ \psi_{12}(x,y,t) \\ w_2(x,y,t) \\ \psi_{21}(x,y,t) \\ \psi_{22}(x,y,t) \end{bmatrix} = \begin{bmatrix} W_1(x,y) \\ \Psi_{11}(x,y) \\ \Psi_{12}(x,y) \\ W_2(x,y) \\ \Psi_{21}(x,y) \\ \Psi_{22}(x,y) \end{bmatrix} \exp(i\nu t) \tag{3}$$

and substituting (3) in the system of differential equations (1), by the assumption that $f_1(x,y,t) = 0$, we obtain a homogenous system of ordinary differential equations describing complex modes of free vibrations of a complex system with damping:

$$\begin{cases} D_1\left[\dfrac{d^2\Psi_{11}}{dx^2} + \dfrac{d^2\Psi_{11}}{dy^2}\right] + H_1\left(\dfrac{dW_1}{dx} + \Psi_{11}\right) + \Xi_1 \Psi_{11} \nu^2 = 0, \\ D_1\left[\dfrac{d^2\Psi_{12}}{dx^2} + \dfrac{d^2\Psi_{12}}{dy^2}\right] + H_1\left(\dfrac{dW_1}{dx} + \Psi_{12}\right) + \Xi_1 \Psi_{12} \nu^2 = 0, \end{cases} \tag{4a}$$

$$\begin{cases} H_1\left(\nabla^2 W_1 + \dfrac{d\Psi_{11}}{dx} + \dfrac{d\Psi_{12}}{dy}\right) - (W_1 - W_2)(k + icv) + \mu_1 W_1 v^2 = 0, \\ D_2\left[\dfrac{d^2\Psi_{21}}{dx^2} + \dfrac{d^2\Psi_{21}}{dy^2}\right] + H_2\left(\dfrac{dW_2}{dx} + \Psi_{21}\right) + \Xi_2 \Psi_{21} v^2 = 0, \\ D_2\left[\dfrac{d^2\Psi_{22}}{dx^2} + \dfrac{d^2\Psi_{22}}{dy^2}\right] + H_2\left(\dfrac{dW_2}{dx} + \Psi_{22}\right) + \Xi_2 \Psi_{22} v^2 = 0, \\ H_2\left(\nabla^2 W_2 + \dfrac{d\Psi_{21}}{dx} + \dfrac{d\Psi_{22}}{dy}\right) + (W_1 - W_2)(k + icv) + \mu_2 W_2 v^2 = 0, \end{cases} \tag{4b}$$

where $i^2 = -1$. Here $W_1 = W_1(x,y)$ and $W_2 = W_2(x,y)$ are the complex transverse vibration modes of plates I and II, v is the complex eigenfrequency of free vibrations of the complex system with damping.

Solution of the Boundary Value Problem. The solution of the differential equation (4) is sought in the form [1]

$$\begin{cases} W_1(x,y) = X_1(x) Y_1(y), \\ \Psi_{11}(x,y) = \Theta_{11}(x)\Gamma_{11}(y), \\ \Psi_{12}(x,y) = \Theta_{12}(x)\Gamma_{12}(y), \\ W_2(x,y) = X_2(x) Y_2(y), \\ \Psi_{21}(x,y) = \Theta_{21}(x)\Gamma_{21}(y), \\ \Psi_{22}(x,y) = \Theta_{22}(x)\Gamma_{22}(y). \end{cases} \tag{5}$$

Searching for a general solution of the system of differential equations (4) in the form

$$\begin{bmatrix} W_1(x,y) \\ \Psi_{11}(x,y) \\ \Psi_{12}(x,y) \\ W_2(x,y) \\ \Psi_{21}(x,y) \\ \Psi_{22}(x,y) \end{bmatrix} = \begin{bmatrix} A \\ C_{11} \\ C_{12} \\ B \\ D_{21} \\ D_{22} \end{bmatrix} \exp(r_1 x)\exp(r_2 y), \tag{6}$$

we obtain a homogeneous system of algebraic equations:

$$\begin{cases} A\dfrac{H_1}{D_1} r_1^2 + C_{11} s_1^{*} = 0, \\ A\dfrac{H_1}{D_1} r_2^2 + C_{12} s_1^{**} = 0, \end{cases} \tag{7a}$$

$$\begin{cases} A(r_1^2 + r_2^2 + p_1^{**}) + Bp_1^* - C_{11}r_1 - C_{12}r_2 = 0, \\ B\dfrac{H_2}{D_2}r_1^2 + D_{21}s_2^* = 0, \\ B\dfrac{H_2}{D_2}r_2^2 + D_{22}s_2^* = 0, \\ Ap_2^* + B(r_1^2 + r_2^2 + p_2^{**}) - D_{21}r_1 - D_{22}r_2 = 0, \end{cases} \qquad (7b)$$

where

$$p_1^* = \frac{1}{H_1}(k + icv), \quad p_1^{**} = \frac{1}{H_1}(\mu_1 v^2 - k - icv),$$

$$p_2^* = \frac{1}{H_2}(k + icv), \quad p_2^{**} = \frac{1}{H_2}(\mu_2 v^2 - k - icv),$$

$$s_1^* = r_1^2 + r_2^2 - \frac{H_1}{D_1}r_1 + \frac{\Xi_1}{D_1}v^2, \quad s_1^{**} = r_1^2 + r_2^2 - \frac{H_1}{D_1}r_2 + \frac{\Xi_1}{D_1}v^2, \qquad (8)$$

$$s_2^* = r_1^2 + r_2^2 - \frac{H_2}{D_2}r_1 + \frac{\Xi_2}{D_2}v^2, \quad s_2^{**} = r_1^2 + r_2^2 - \frac{H_2}{D_2}r_2 + \frac{\Xi_2}{D_2}v^2.$$

Constructing the determinant of the characteristic matrix of the system of equations (7) and equating it to zero

$$\begin{vmatrix} \dfrac{H_1}{D_1}r_1^2 & s_1^* & 0 & 0 & 0 & 0 \\ \dfrac{H_1}{D_1}r_2^2 & 0 & s_1^{**} & 0 & 0 & 0 \\ r_1^2 + r_2^2 + p_1^{**} & -r_1 & -r_2 & p_1^* & 0 & 0 \\ 0 & 0 & 0 & \dfrac{H_2}{D_2}r_1^2 & s_2^* & 0 \\ 0 & 0 & 0 & \dfrac{H_2}{D_2}r_2^2 & 0 & s_2^{**} \\ p_2^* & 0 & 0 & r_1^2 + r_2^2 + p_2^{**} & -r_1 & -r_2 \end{vmatrix} = 0, \qquad (9)$$

we obtain the characteristic equation in the form of the following algebraic equation:

$$r_1^8 + a_{17}r_1^7 + a_{16}r_1^6 + a_{15}r_1^5 + a_{14}r_1^4 + a_{13}r_1^3 + a_{12}r_1^2 + a_{11}r_1 + r_2^8 + a_{27}r_2^7 +$$

$$+ a_{26}r_2^6 + a_{25}r_2^5 + a_{24}r_2^4 + a_{23}r_2^3 + a_{22}r_2^2 + a_{21}r_2 + a_0 = 0 \qquad (10)$$

with the following roots: $r_{1j} = (-1)^{j-1} i\lambda_{1v}$, $r_{2j} = (-1)^{j-1} i\lambda_{2v}$, $j = (2v-1), 2v$, and $v = 1, 2, 3, 4$. Here $a_{17}, a_{16}, a_{15}, a_{14}, a_{13}, a_{12}, a_{11}, a_{27}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}, a_{21}$, and a_0 are constant coefficients.

Applying Euler's formulas, we construct the solution of the system of differential equations (4) in the form of the following system of solutions:

$$\begin{cases} W_1(x,y) = \sum_{v=1}^{4} (A_v^{*} \sin\lambda_{1v}x + A_v^{**}\cos\lambda_{1v}x)(A_v^{***}\sin\lambda_{2v}y + A_v^{****}\cos\lambda_{2v}y), \\ \Psi_{11}(x,y) = \sum_{v=1}^{4} (C_{11v}^{*}\cos\lambda_{1v}x + C_{11v}^{**}\sin\lambda_{1v}x)(C_{11v}^{***}\cos\lambda_{2v}y + C_{11v}^{****}\sin\lambda_{2v}y), \\ \Psi_{12}(x,y) = \sum_{v=1}^{4} (C_{12v}^{*}\cos\lambda_{1v}x + C_{12v}^{**}\sin\lambda_{1v}x)(C_{12v}^{***}\cos\lambda_{2v}y + C_{12v}^{****}\sin\lambda_{2v}y), \\ W_2(x,y) = \sum_{v=1}^{4} (B_v^{*}\sin\lambda_{1v}x + B_v^{**}\cos\lambda_{1v}x)(B_v^{***}\sin\lambda_{2v}y + B_v^{****}\cos\lambda_{2v}y), \\ \Psi_{21}(x,y) = \sum_{v=1}^{4} (D_{21v}^{*}\cos\lambda_{1v}x + D_{21v}^{**}\sin\lambda_{1v}x)(D_{21v}^{***}\cos\lambda_{2v}y + D_{21v}^{****}\sin\lambda_{2v}y), \\ \Psi_{22}(x,y) = \sum_{v=1}^{4} (D_{22v}^{*}\cos\lambda_{1v}x + D_{22v}^{**}\sin\lambda_{1v}x)(D_{22v}^{***}\cos\lambda_{2v}y + D_{22v}^{****}\sin\lambda_{2v}y). \end{cases} \tag{11}$$

Here $A_v^{*}, A_v^{**}, A_v^{***}, A_v^{****}, B_v^{*}, B_v^{**}, B_v^{***}, B_v^{****}, C_{11v}^{*}, C_{11v}^{**}, C_{11v}^{***}, C_{11v}^{****}$, $C_{12v}^{*}, C_{12v}^{**}, C_{12v}^{***}, C_{12v}^{****}, D_{21v}^{*}, D_{21v}^{**}, D_{21v}^{***}, D_{21v}^{****}, D_{22v}^{*}, D_{22v}^{**}, D_{22v}^{***}$, and D_v^{****} are constants and $\lambda_{1v} = \alpha_{1v} + i\beta_{1v}$ and $\lambda_{2v} = \alpha_{2v} + i\beta_{2v}$ are the parameters describing the roots of the characteristic equation (10).

In accordance with (7), the following relations exist between the constants of (9):

$$\begin{aligned} &a_v^{*} = \frac{B_v^{*}}{A_v^{*}}, \quad a_v^{**} = \frac{B_v^{**}}{A_v^{**}}, \quad a_v^{***} = \frac{B_v^{***}}{A_v^{***}}, \quad a_v^{****} = \frac{B_v^{****}}{A_v^{****}}, \\ &c_{11v}^{*} = \frac{C_{11v}^{*}}{A_v^{*}}, \quad c_{11v}^{**} = \frac{C_{11v}^{**}}{A_v^{**}}, \quad c_{11v}^{***} = \frac{C_{11v}^{***}}{A_v^{***}}, \quad c_{11v}^{****} = \frac{C_{11v}^{****}}{A_v^{****}}, \\ &c_{12v}^{*} = \frac{C_{12v}^{*}}{A_v^{*}}, \quad c_{12v}^{**} = \frac{C_{12v}^{**}}{A_v^{**}}, \quad c_{12v}^{***} = \frac{C_{12v}^{***}}{A_v^{***}}, \quad c_{12v}^{****} = \frac{C_{12v}^{****}}{A_v^{****}}, \\ &d_{21v}^{*} = \frac{D_{21v}^{*}}{B_v^{*}}, \quad d_{21v}^{**} = \frac{D_{21v}^{**}}{B_v^{**}}, \quad d_{21v}^{***} = \frac{D_{21v}^{***}}{B_v^{***}}, \quad d_{21v}^{****} = \frac{D_{21v}^{****}}{B_v^{****}}, \\ &d_{22v}^{*} = \frac{D_{22v}^{*}}{B_v^{*}}, \quad d_{22v}^{**} = \frac{D_{22v}^{**}}{B_v^{**}}, \quad d_{22v}^{***} = \frac{D_{22v}^{***}}{B_v^{***}}, \quad d_{22v}^{****} = \frac{D_{22v}^{****}}{B_v^{****}}, \end{aligned} \tag{12}$$

where

$$a_v^{*} = a_v^{**} = a_v^{***} = a_v^{****} = a_v = -\frac{-\lambda_{1v}^2 - \lambda_{2v}^2 + p_1^{**} + i(c_{11v}\lambda_{1v} + c_{12v}\lambda_{2v})}{p_1^{*}},$$

$$c_{11v}^{*} = c_{11v}^{***} = c_{11v} = \frac{-\lambda_{1v}^2 - i\dfrac{H_1}{D_1}\lambda_{1v} + \dfrac{\Xi_1}{D_1}\nu^2}{\dfrac{H_1}{D_1}\lambda_{1v}^2}, \quad c_{11v} = -c_{11v}^{**} = -c_{11v}^{****},$$

$$c_{12v}^{*} = c_{12v}^{***} = c_{12v} = \frac{-\lambda_{2v}^2 - i\dfrac{H_1}{D_1}\lambda_{2v} + \dfrac{\Xi_1}{D_1}\nu^2}{\dfrac{H_1}{D_1}\lambda_{2v}^2}, \quad c_{12v} = -c_{12v}^{**} = -c_{12v}^{****},$$

$$d_{21v}^{*} = d_{21v}^{***} = d_{21v} = \frac{-\lambda_{1v}^2 - i\dfrac{H_2}{D_2}\lambda_{1v} + \dfrac{\Xi_2}{D_2}\nu^2}{\dfrac{H_2}{D_2}\lambda_{1v}^2}, \quad d_{21v} = -d_{21v}^{**} = -d_{21v}^{****},$$

$$d_{22v}^{*} = d_{22v}^{***} = d_{22v} = \frac{-\lambda_{2v}^2 - i\dfrac{H_2}{D_2}\lambda_{2v} + \dfrac{\Xi_2}{D_2}\nu^2}{\dfrac{H_2}{D_2}\lambda_{2v}^2}, \quad d_{22v} = -d_{22v}^{**} = -d_{22v}^{****}.$$

(13)

On substitution of (12) in (11), the general solution of the system of differential equations (4) takes the following form:

$$\begin{cases} W_1(x, y) = \sum\limits_{v=1}^{4} (A_v^{*} \sin\lambda_{1v}x + A_v^{**} \cos\lambda_{1v}x)(A_v^{***} \sin\lambda_{2v}y + A_v^{****} \cos\lambda_{2v}y), \\ \Psi_{11}(x, y) = \sum\limits_{v=1}^{4} c_{11v}(A_v^{*} \cos\lambda_{1v}x - A_v^{**} \sin\lambda_{1v}x)(A_v^{***} \cos\lambda_{2v}y - A_v^{****} \sin\lambda_{2v}y), \\ \Psi_{12}(x, y) = \sum\limits_{v=1}^{4} c_{12v}(A_v^{*} \cos\lambda_{1v}x - A_v^{**} \sin\lambda_{1v}x)(A_v^{***} \cos\lambda_{2v}y - A_v^{****} \sin\lambda_{2v}y), \\ W_2(x, y) = \sum\limits_{v=1}^{4} a_v(A_v^{*} \sin\lambda_{1v}x + A_v^{**} \cos\lambda_{1v}x)(A_v^{***} \sin\lambda_{2v}y + A_v^{****} \cos\lambda_{2v}y), \\ \Psi_{21}(x, y) = \sum\limits_{v=1}^{4} a_v d_{21v}(A_v^{*} \cos\lambda_{1v}x - A_v^{**} \sin\lambda_{1v}x)(A_v^{***} \cos\lambda_{2v}y - A_v^{****} \sin\lambda_{2v}y), \\ \Psi_{22}(x, y) = \sum\limits_{v=1}^{4} a_v d_{22v}(A_v^{*} \cos\lambda_{1v}x - A_v^{**} \sin\lambda_{1v}x)(A_v^{***} \cos\lambda_{2v}y + A_v^{****} \sin\lambda_{2v}y). \end{cases}$$

(14)

In order to solve the boundary value problem, the following boundary conditions are used:

$$\begin{cases} W_1\big|_{x=0} = 0, & W_1\big|_{x=a} = 0, & W_2\big|_{x=0} = 0, & W_2\big|_{x=a} = 0, \\ W_1\big|_{y=0} = 0, & W_1\big|_{y=b} = 0, & W_2\big|_{y=0} = 0, & W_2\big|_{y=b} = 0, \\ \left.\dfrac{d\Psi_{11}}{dx}\right|_{x=0} = 0, & \left.\dfrac{d\Psi_{11}}{dx}\right|_{x=a} = 0, & \left.\dfrac{d\Psi_{21}}{dx}\right|_{x=0} = 0, & \left.\dfrac{d\Psi_{21}}{dx}\right|_{x=a} = 0, \\ \left.\dfrac{d\Psi_{12}}{dy}\right|_{y=0} = 0, & \left.\dfrac{d\Psi_{12}}{dy}\right|_{y=b} = 0, & \left.\dfrac{d\Psi_{22}}{dy}\right|_{y=0} = 0, & \left.\dfrac{d\Psi_{22}}{dy}\right|_{y=b} = 0. \end{cases} \qquad (15)$$

Substituting (14) in (15), we obtain a homogenous system of algebraic equations, which in the matrix notation has the following form:

$$\mathbf{YX} = 0. \qquad (16)$$

Here

$$\mathbf{X} = [A_1^*, A_2^*, A_3^*, A_4^*, A_1^{**}, A_2^{**}, A_3^{**}, A_4^{**}]^T$$

or

$$\mathbf{X} = [A_1^{***}, A_2^{***}, A_3^{***}, A_4^{***}, A_1^{****}, A_2^{****}, A_3^{****}, A_4^{****}]^T$$

are the vectors of the unknowns in the system of equations and

$$\mathbf{Y} = [Y_{i*j}]_{8*8} \qquad (17)$$

is the characteristic matrix of the system of equations (16).

The first four equations (16) are presented in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -\lambda_{11}c_{111} & -\lambda_{12}c_{112} & -\lambda_{13}c_{113} & -\lambda_{14}c_{114} \\ a_1 & a_2 & a_3 & a_4 \\ -\lambda_{11}a_1d_{211} & -\lambda_{12}a_2d_{212} & -\lambda_{13}a_3d_{213} & -\lambda_{14}a_4d_{214} \end{bmatrix} \begin{bmatrix} A_1^{**} \\ A_2^{**} \\ A_3^{**} \\ A_4^{**} \end{bmatrix} = 0, \qquad (18)$$

where $A_1^{**} = A_2^{**} = A_3^{**} = A_4^{**} = 0$.

The remaining four equations (16) give the following system of equations:

$$\begin{bmatrix} ss_{11} & ss_{12} & ss_{13} & ss_{14} \\ -\lambda_{11}c_{111}ss_{11} & -\lambda_{12}c_{112}ss_{12} & -\lambda_{13}c_{113}ss_{13} & -\lambda_{14}c_{114}ss_{14} \\ a_1ss_{11} & a_2ss_{12} & a_3ss_{13} & a_4ss_{14} \\ -\lambda_{11}a_1d_{211}ss_{11} & -\lambda_{12}a_2d_{212}ss_{12} & -\lambda_{13}a_3d_{213}ss_{13} & -\lambda_{14}a_4d_{214}ss_{14} \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \\ A_4^* \end{bmatrix} = 0, \qquad (19)$$

where $ss_{11} = \sin\lambda_{11}a$, $ss_{12} = \sin\lambda_{12}a$, $ss_{13} = \sin\lambda_{13}a$, and $ss_{14} = \sin\lambda_{14}a$.

The condition of solving the system of equations (19) is vanishing of the characteristic determinant, i.e.,

$$\begin{bmatrix} ss_{11} & ss_{12} & ss_{13} & ss_{14} \\ -\lambda_{11}c_{111}ss_{11} & -\lambda_{12}c_{112}ss_{12} & -\lambda_{13}c_{113}ss_{13} & -\lambda_{14}c_{114}ss_{14} \\ a_1 ss_{11} & a_2 ss_{12} & a_3 ss_{13} & a_4 ss_{14} \\ -\lambda_{11}a_1 d_{211}ss_{11} & -\lambda_{12}a_2 d_{212}ss_{12} & -\lambda_{13}a_3 d_{213}ss_{13} & -\lambda_{14}a_4 d_{214}ss_{14} \end{bmatrix} = 0. \qquad (20)$$

Expanding determinant (10), we obtained the following characteristic equation:

$$\sin\lambda_{11}a \sin\lambda_{12}a \sin\lambda_{13}a \sin\lambda_{14}a = 0, \qquad (21)$$

where $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_1$.

The characteristic equation (21) may be rewritten in the form

$$\sin\lambda_1 a = 0, \qquad (22)$$

where, in the general case,

$$\lambda_1 = \alpha_1 + i\beta_1 \qquad (23)$$

are complex numbers.

Substituting (23) in (22), we obtain the following equation:

$$\sin\alpha_1 a \cosh\beta_1 a + i\cos\alpha_1 a \sinh\beta_1 a = 0, \qquad (24)$$

which has the following roots:

$$\alpha_{n_1} = \frac{n_1\pi}{a}, \quad \beta_{n_1} = 0, \quad n_1 = 1, 2, 3, \ldots . \qquad (25)$$

Taking into account (25) in (23), we obtain the following identity:

$$\lambda_{n_1} = \alpha_{n_1} = \frac{n_1\pi}{a}. \qquad (26)$$

By analogy with equations (16)–(24), we obtain

$$\alpha_{n_2} = \frac{n_2\pi}{b}, \quad \beta_{n_2} = 0, \quad n_2 = 1, 2, 3, \ldots,$$
$$\lambda_{n_2} = \alpha_{n_2} = \frac{n_2\pi}{b}. \qquad (27)$$

Substituting $r_1 = i\lambda_{n_1}$ and $r_2 = i\lambda_{n_2}$ in equation (10) and carrying out all transformations, we obtain the following equation of frequency:

$$\nu^8 + a_{17}^{\#}\nu^7 + a_{16}^{\#}\nu^6 + a_{15}^{\#}\nu^5 + a_{14}^{\#}\nu^4 + a_{13}^{\#}\nu^3 + a_{12}^{\#}\nu^2 + a_{11}^{\#}\nu + a_0^{\#} = 0 \qquad (28)$$

from which a sequence of complex eigenfrequencies is determined:

$$\nu_{n_1 n_2} = i\eta_{n_1 n_2} \pm \omega_{n_1 n_2}, \tag{29}$$

where $a_{17}^{\#}$, $a_{16}^{\#}$, $a_{15}^{\#}$, $a_{14}^{\#}$, $a_{13}^{\#}$, $a_{12}^{\#}$, $a_{11}^{\#}$, and $a_{0}^{\#}$ are constant coefficients.

By substituting (29) into (13) and (14), the following formulas for the coefficients of amplitudes are obtained:

$$a_{n_1 n_2} = -\frac{-\lambda_{1n_1n_2}^2 - \lambda_{2n_1n_2}^2 + p_1^{**} + i(c_{11n_1n_2}\lambda_{1n_1n_2} + c_{12n_1n_2}\lambda_{2n_1n_2})}{p_1^*},$$

$$c_{11n_1n_2} = \frac{-\lambda_{1n_1n_2}^2 - i\dfrac{H_1}{D_1}\lambda_{1n_1n_2} + \dfrac{\Xi_1}{D_1}\nu_{n_1n_2}^2}{\dfrac{H_1}{D_1}\lambda_{1n_1n_2}^2},$$

$$c_{12n_1n_2} = \frac{-\lambda_{2n_1n_2}^2 - i\dfrac{H_1}{D_1}\lambda_{2n_1n_2} + \dfrac{\Xi_1}{D_1}\nu_{n_1n_2}^2}{\dfrac{H_1}{D_1}\lambda_{2n_1n_2}^2},$$

$$d_{21n_1n_2} = \frac{-\lambda_{1n_1n_2}^2 - i\dfrac{H_2}{D_2}\lambda_{1n_1n_2} + \dfrac{\Xi_2}{D_2}\nu_{n_1n_2}^2}{\dfrac{H_2}{D_2}\lambda_{n_1n_2}^2},$$

$$d_{22n_1n_2} = \frac{-\lambda_{n_1n_2}^2 - i\dfrac{H_2}{D_2}\lambda_{2n_1n_2} + \dfrac{\Xi_2}{D_2}\nu_{n_1n_2}^2}{\dfrac{H_2}{D_2}\lambda_{2n_1n_2}^2}, \tag{30}$$

where

$$\begin{aligned} p_1^* &= \frac{1}{H_1}(k + ic\nu_{n_1n_2}), \quad p_1^{**} = \frac{1}{H_1}(\mu_1\nu_{n_1n_2}^2 - k - ic\nu_{n_1n_2}), \\ p_2^* &= \frac{1}{H_2}(k + ic\nu_{n_1n_2}), \quad p_2^{**} = \frac{1}{H_2}(\mu_2\nu_{n_1n_2}^2 - k - ic\nu_{n_1n_2}). \end{aligned} \tag{31}$$

Substituting the sequences λ_{n_1}, λ_{n_2} and $a_{n_1n_2}$, $c_{11n_1n_2}$, $c_{12n_1n_2}$, $d_{21n_1n_2}$, $d_{22n_1n_2}$ in (14), we obtain the following six sequences of complex modes of free vibrations for a complex system with damping:

$$\begin{cases} W_{1n_1n_2}(x,y) = \sin\lambda_{n_1}x\sin\lambda_{n_2}y, \\ \Psi_{11n_1n_2}(x,y) = c_{11n_1n_2}\cos\lambda_{n_1}x\cos\lambda_{n_2}y, \\ \Psi_{12n_1n_2}(x,y) = c_{12n_1n_2}\cos\lambda_{n_1}x\cos\lambda_{n_2}y, \\ W_{2n_1n_2}(x,y) = a_{n_1n_2}\sin\lambda_{n_1}x\sin\lambda_{n_2}y, \\ \Psi_{21n_1n_2}(x,y) = a_{n_1n_2}d_{21n_1n_2}\cos\lambda_{n_1}x\cos\lambda_{n_2}y, \\ \Psi_{22n_1n_2}(x,y) = a_{n_1n_2}d_{22n_1n_2}\cos\lambda_{n_1}x\cos\lambda_{n_2}y. \end{cases} \tag{32}$$

Solution of the Initial Value Problem. In the case of $\nu = \nu_{n_1n_2}$, the complex equation of motion

$$T = \Phi\exp(i\nu t) \tag{33}$$

can be written in the following form:

$$T_{n_1n_2} = \Phi_{n_1n_2}\exp(i\nu_{n_1n_2}t), \tag{34}$$

where $\Phi_{n_1n_2}$ is the Fourier coefficient.

Free vibration of a complex system with damping is presented in the form of a Fourier series based on the complex eigenfunctions, i.e.:

$$\begin{bmatrix} w_1(x,y,t) \\ \psi_{11}(x,y,t) \\ \psi_{12}(x,y,t) \\ w_2(x,y,t) \\ \psi_{21}(x,y,t) \\ \psi_{22}(x,y,t) \end{bmatrix} = \begin{bmatrix} \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} W_{1n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \Psi_{11n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \Psi_{12n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty}\sum_{n_2 1}^{\infty} W_{2n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \Psi_{21n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \Psi_{22n_1n_2}(x,y) \end{bmatrix} \Phi_{n_1n_2}\exp(i\nu_{n_1n_2}t). \tag{35}$$

From the system of equations (4), performing some algebraic transformations, adding the equations together and then integrating them on both sides within the limits from 0 to a, and from 0 to b, we obtain the property of orthogonality of eigenfunctions for a complex system with damping using Timoshenko's model:

$$\int_0^a\int_0^b [\mu_1(W_{1m}V_{1n}+W_{1n}V_{1m})+\mu_2(W_{2m}V_{2n}+W_{2n}V_{2m})+$$

$$+\Xi_1(\Psi_{11m}Q_{11n}+\Psi_{11n}Q_{11m}+\Psi_{12m}Q_{12n}+\Psi_{12n}Q_{12m})+$$

$$+\Xi_2(\Psi_{21m}Q_{21n}+\Psi_{21n}Q_{21m}+\Psi_{22m}Q_{22n}+\Psi_{22n}Q_{22m})+$$

$$+c(W_{1n}-W_{2n})(W_{1m}-W_{2m})]dxdy = N_n\delta_{nm}, \tag{36}$$

where

$$N_n=\int_0^a\int_0^b [2(\mu_1 W_{1n}V_{1n}+\mu_2 W_{2n}V_{2n}+\Xi_1(\Psi_{11n}Q_{11n}+\Psi_{12n}Q_{12n})+$$

$$+\Xi_2(\Psi_{21n}Q_{21n}+\Psi_{22n}Q_{22n})+c(W_{1n}-W_{2n})^2]dxdy, \tag{37}$$

$$\begin{aligned}
&V_{1n}=iv_n W_{1n}(x,y), && V_{2n}=iv_n W_{2n}(x,y),\\
&V_{1m}=iv_m W_{1m}(x,y), && V_{2m}=iv_m W_{2m}(x,y),\\
&Q_{11n}=iv_n\Psi_{11n}(x,y), && Q_{21n}=iv_n\Psi_{21n}(x,y),\\
&Q_{11m}=iv_m\Psi_{11m}(x,y), && Q_{21m}=iv_m\Psi_{21m}(x,y),\\
&Q_{12n}=iv_n\Psi_{12n}(x,y), && Q_{22n}=iv_n\Psi_{22n}(x,y),\\
&Q_{12m}=iv_m\Psi_{12m}(x,y), && Q_{22m}=iv_m\Psi_{22m}(x,y),
\end{aligned} \tag{38}$$

Here δ_{nm} is Kronecker's delta, $n=(n_1, n_2)$, and $m=(m_1, m_2)$.

The following initial conditions are the basis for solving the problem of free vibrations:

$$\begin{aligned}
&w_1(x,y,0)=w_{01}, && w_2(x,y,0)=w_{02},\\
&\psi_{11}(x,y,0)=\psi_{011}, && \psi_{21}(x,y,0)=\psi_{021},\\
&\psi_{12}(x,y,0)=\psi_{012}, && \psi_{22}(x,y,0)=\psi_{022}.
\end{aligned} \tag{39}$$

By applying conditions (39) in series (35) and taking into account the property of orthogonality (36), the formula for a complex Fourier coefficient is obtained:

$$\Phi_{n_1n_2}=\frac{1}{N_{n_1n_2}}\int_0^a\int_0^b \{\mu_1(V_{1n_1n_2}w_{01}+W_{1n_1n_2}\overset{\circ}{w}_{01})+$$

$$+\mu_2(V_{2n_1n_2}w_{02}+W_{2n_1n_2}\overset{\circ}{w}_{02})+$$

$$+\Xi_1(Q_{11n_1n_2}\psi_{011}+\Psi_{11n_1n_2}\overset{\circ}{\psi}_{011}+Q_{12n_1n_2}\psi_{012}+\Psi_{12n_1n_2}\overset{\circ}{\psi}_{012})+$$

$$+\Xi_2(Q_{21n_1n_2}\psi_{021}+\Psi_{21n_1n_2}\overset{\circ}{\psi}_{021}+Q_{22n_1n_2}\psi_{022}+\Psi_{22n_1n_2}\overset{\circ}{\psi}_{022})+$$

$$+c[(W_{1n_1n_2}-W_{2n_1n_2})(w_{01}-w_{02})]\}dxdy. \tag{40}$$

By substituting (32), (34), and (40) into (35) and performing the trigonometric and algebraic transformations, the final form of free vibration of a complex system with damping is obtained:

$$\begin{cases} w_1=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|W_{1n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\chi_{1n_1n_2}),\\ \psi_{11}=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|\Psi_{11n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\theta_{11n_1n_2}),\\ \psi_{12}=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|\Psi_{12n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\theta_{12n_1n_2}),\\ w_2=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|W_{2n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\chi_{2n_1n_2}),\\ \psi_{21}=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|\Psi_{21n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\theta_{21n_1n_2}),\\ \psi_{22}=\sum\limits_{n_1=1}^{\infty}\sum\limits_{n_2=1}^{\infty}e^{-\eta_{n_1n_2}t}\left|\Phi_{n_1n_2}\right|\left|\Psi_{22n_1n_2}\right|\cos(\omega_{n_1n_2}t+\varphi_{n_1n_2}+\theta_{22n_1n_2}), \end{cases} \tag{41}$$

where

$$\left|W_{1n_1n_2}\right|=\sqrt{X_{1n_1n_2}^2+Y_{1n_1n_2}^2},\quad \left|W_{2n_1n_2}\right|=\sqrt{X_{2n_1n_2}^2+Y_{2n_1n_2}^2},$$

$$\left|\Psi_{11n_1n_2}\right|=\sqrt{\Lambda_{11n_1n_2}^2+\Omega_{11n_1n_2}^2},\quad \left|\Psi_{21n_1n_2}\right|=\sqrt{\Lambda_{21n_1n_2}^2+\Omega_{21n_1n_2}^2},$$

$$\left|\Psi_{12n_1n_2}\right|=\sqrt{\Lambda_{12n_1n_2}^2+\Omega_{12n_1n_2}^2},\quad \left|\Psi_{22n_1n_2}\right|=\sqrt{\Lambda_{22n_1n_2}^2+\Omega_{22n_1n_2}^2},$$

$$\chi_{1n_1n_2}=\arg W_{1n_1n_2},\quad \chi_{2n_1n_2}=\arg W_{2n_1n_2},$$

$$\theta_{11n_1n_2}=\arg\Psi_{11n_1n_2},\quad \theta_{21n_1n_2}=\arg\Psi_{21n_1n_2},$$

$$\theta_{12n_1n_2}=\arg\Psi_{12n_1n_2},\quad \theta_{22n_1n_2}=\arg\Psi_{22n_1n_2},$$

$$\left|\Phi_{n_1n_2}\right|=\sqrt{C_{n_1n_2}^2+D_{n_1n_2}^2},\quad \varphi_{n_1n_2}=\arg\Phi_{n_1n_2},$$

$$X_{1n_1n_2}=\mathrm{Re}W_{1n_1n_2},\ Y_{1n_1n_2}=\mathrm{Im}W_{1n_1n_2},\ X_{2n_1n_2}=\mathrm{Re}W_{2n_1n_2},\ Y_{2n_1n_2}=\mathrm{Im}W_{2n_1n_2},$$

$$\Lambda_{11n_1n_2}=\mathrm{Re}\Psi_{11n_1n_2},\ \Omega_{11n_1n_2}=\mathrm{Im}\Psi_{11n_1n_2},$$

$$\Lambda_{21n_1n_2} = \mathrm{Re}\,\Psi_{21n_1n_2}, \quad \Omega_{21n_1n_2} = \mathrm{Im}\,\Psi_{21n_1n_2},$$

$$\Lambda_{12n_1n_2} = \mathrm{Re}\,\Psi_{12n_1n_2}, \quad \Omega_{12n_1n_2} = \mathrm{Im}\,\Psi_{12n_1n_2},$$

$$\Lambda_{22n_1n_2} = \mathrm{Re}\,\Psi_{22n_1n_2}, \quad \Omega_{22n_1n_2} = \mathrm{Im}\,\Psi_{22n_1n_2},$$

$$C_{n_1n_2} = \mathrm{Re}\,\Phi_{n_1n_2}, \quad D_{n_1n_2} = \mathrm{Im}\,\Phi_{n_1n_2}. \tag{42}$$

Solution of the Forced Vibration Problem. In order to solve the differential equations (1), the function of loading is expanded from the operator method [15]

$$f_1(x,y,t) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} [\mu_1 W_{1n_1n_2} + \mu_2 W_{2n_1n_2} +$$

$$+ \Xi_1(\Psi_{11n_1n_2} + \Psi_{12n_1n_2}) + \Xi_2(\Psi_{21n_1n_2} + \Psi_{22n_1n_2})] f_{n_1n_2}, \tag{43}$$

where $W_{1n_1n_2}$, $\Psi_{11n_1n_2}$, $\Psi_{21n_1n_2}$, $W_{2n_1n_2}$, $\Psi_{12n_1n_2}$, and $\Psi_{22n_1n_2}$ have been described by equations (32).

The function of the displacement of a complex system with damping is presented in the form of a Fourier series:

$$\begin{bmatrix} w_1 \\ \psi_{11} \\ \psi_{12} \\ w_2 \\ \psi_{21} \\ \psi_{22} \end{bmatrix} = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \begin{bmatrix} W_{1n_1n_2} \\ \Psi_{11n_1n_2} \\ \Psi_{12n_1n_2} \\ W_{2n_1n_2} \\ \Psi_{21n_1n_2} \\ \Psi_{22n_1n_2} \end{bmatrix} T_{n_1n_2}. \tag{44}$$

Substituting (43) and (44) into the differential equations (1), we obtain the following equation of motion:

$$\mathring{T}_{n_1n_2} - i\nu_{n_1n_2} T_{n_1n_2} = f_{n_1n_2}, \tag{45}$$

where $T_{n_1n_2}$ are the coefficients of the distribution of the loading function in a Fourier series.

Applying the property of the eigenfunction orthogonality (36), we derive the coefficients for the load distribution, namely:

$$f_{n_1n_2} = \frac{1}{N_1 N_{n_1n_2}} i\nu_{n_1n_2} \int_0^a \int_0^b [W_{1n_1n_2} + W_{2n_1n_2} + \Psi_{11n_1n_2} + \Psi_{12n_1n_2} +$$

$$+\Psi_{21n_1n_2}+\Psi_{22n_1n_2}]f_1(x,y,t)dxdy. \qquad (46)$$

The solution of the differential equation (46) has the form [15]

$$T_{n_1n_2}=\frac{1}{i\nu_{n_1n_2}}\int_0^t[\exp(i\nu_{n_1n_2})(t-\tau)-1]f_{n_1n_2}(\tau)d\tau. \qquad (47)$$

On substituting (46) and (47) into (44), equation (44) can be rewritten in the following form:

$$\begin{cases} w_1=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|W_{1n_1n_2}||T_{n_1n_2}|\cos(\vartheta_{1n_1n_2}+\xi_{n_1n_2}),\\ \psi_{11}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|\Psi_{11n_1n_2}||T_{n_1n_2}|\sin(\theta_{11n_1n_2}+\xi_{n_1n_2}),\\ \psi_{12}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|\Psi_{12n_1n_2}||T_{n_1n_2}|\sin(\theta_{12n_1n_2}+\xi_{n_1n_2}),\\ w_2=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|W_{2n_1n_2}||T_{n_1n_2}|\cos(\vartheta_{2n_1n_2}+\xi_{n_1n_2}),\\ \psi_{21}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|\Psi_{21n_1n_2}||T_{n_1n_2}|\sin(\theta_{21n_1n_2}+\xi_{n_1n_2}),\\ \psi_{22}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|\Psi_{22n_1n_2}||T_{n_1n_2}|\sin(\theta_{22n_1n_2}+\xi_{n_1n_2}), \end{cases} \qquad (48)$$

where

$$\vartheta_{1n_1n_2}=\arg W_{1n_1n_2},\quad \vartheta_{2n_1n_2}=\arg W_{2n_1n_2},$$

$$\theta_{11n_1n_2}=\arg\Psi_{11n_1n_2},\quad \theta_{21n_1n_2}=\arg\Psi_{21n_1n_2},$$

$$\theta_{12n_1n_2}=\arg\Psi_{12n_1n_2},\quad \theta_{22n_1n_2}=\arg\Psi_{22n_1n_2},\quad \xi_{n_1n_2}=\arg T_{n_1n_2}. \qquad (49)$$

Solution of the Problem for Various Types of Dynamic Loading. Let us consider various types of dynamic loading for a complex system with damping. External layers of the complex system are made as plates of elastic material coupled by a viscoelastic interlayer and simply supported at their ends. The elastic plates are described by the Timoshenko and Kirchhoff–Love models. The viscoelastic interlayer has the characteristics of a homogenous continuous unidirectional Winkler's foundation and is described by the Voigt–Kelvin model [11–13].

We consider two cases. In the first case, small-frequency transverse vibrations of a complex system with damping are excited at the point $x_0 = 0.6a$, $y_0 = 0.6b$ by a stationary dynamic load varying in time t:

$$f_1(x,y,t) = P_1\delta(x - x_0)\delta(y - y_0)\sin(\omega_0 t). \tag{50}$$

In the second case, small-frequency transverse vibrations of the complex system with damping are excited by the dynamic nonstationary load $f_1(x,y,t)$ varying in time t [10]. We consider various types of nonstationary loading.

With the first type, small-frequency transverse vibrations of a complex system with damping are excited by a non-inertial moving concentrated load $f_1(x,y,t)$ varying in time t for $y = \text{const}$:

$$f_1(x^*,y,t) = P_1\delta(x - x^*) + M_1\delta'(x - x^*). \tag{51}$$

With the second type, small-frequency transverse vibrations of a complex system with damping are excited by an inertial moving concentrated load $f_1(x,y,t)$ varying in time t for $y = \text{const}$:

$$f_1(x^*,y,t) = -m_1\frac{d^2w(x^*,y,t)}{dt^2}\delta(x - x^*) - \Gamma_1\frac{d^2}{dt^2}\left[\frac{\partial w(x^*,y,t)}{\partial x}\right]\delta'(x - x^*). \tag{52}$$

With the third type, small-frequency transverse vibrations of a complex system with damping are excited by a nonuniform load $f_1(x,y,t)$ varying in time t:

$$f_1(x_1,y_1,t) = b(t) - m_1\frac{d^2w(x_1,y_1,t)}{dt^2}\delta(x - x_1)\delta(y - y_1). \tag{53}$$

Here P_1 is the force; M_1 is the moment; m_1 is the rubble mass; $\Gamma_1 = m_1 r^2$; r is the radius of the gyration of the mass; $\delta(...)$ and $\delta'(...)$ are the Dirac delta; $H(...)$ is the Heaviside function; $x^* = v^* t$ (v^* is a constant speed); $y = 0.5b$; $w(x,y,t)$ is the first iteration of the dynamic displacement of the complex system with damping from the force; $f_1(x_1,y_1,t)$ is the displacement of the complex system with damping under rubble; ν_n are complex eigenfrequencies of forced vibrations; x_0 and y_0 are coordinate rubble for time $t = 0$; and $b(t)$ is the displacement of the rubble in the direction of the axis z.

Accelerations occurring in equations (52) and (53) are expanded using the Renaudot formula [14]

$$\begin{aligned}
\frac{d^2}{dt^2}[w(x^*, y, t)] &= \frac{\partial^2 w(x^*, y, t)}{\partial t^2} + 2\frac{\partial^2 w(x^*, y, t)}{\partial x\partial t}v^* + \frac{\partial^2 w(x^*, y, t)}{\partial x^2}(v^*)^2,\\
\frac{d^2}{dt^2}\left[\frac{\partial w(x^*, y, t)}{\partial x}\right] &= \frac{\partial^3 w(x^*, y, t)}{\partial x\partial t^2} + 2\frac{\partial^3 w(x^*, y, t)}{\partial x^2\partial t}v^* + \frac{\partial^3 w(x^*, y, t)}{\partial x^3}(v^*)^2.
\end{aligned} \tag{54}$$

Calculations are made for the following data: $E_1 = E_2 = 10^8$ $\mathrm{N \cdot m^{-2}}$, $E = 10^5$ $\mathrm{N \cdot m^{-2}}$, $\rho_1 = \rho_2 = 2 \cdot 10^3$ $\mathrm{N \cdot s^2 \cdot m^{-4}}$, $h_1 = h_2 = 0.04$ m, $h = \{0.08, 0.12\}$ m, $\nu_{1p} = \nu_{2p} = 0.3$, $a = 4$ m, $b = 2$ m, $c = 2 \cdot 10^4$ $\mathrm{N \cdot s \cdot m^{-2}}$, $P_1 = 10^4$ N, $v^* = 20$ $\mathrm{m \cdot s^{-2}}$.

In order to solve the boundary value problem, the following boundary conditions are used:

for Timoshenko's model

$$\begin{cases} W_1\big|_{x=0} = 0, & W_1\big|_{x=a} = 0, & W_1\big|_{y=0} = 0, & W_1\big|_{y=b} = 0, \\ \left.\dfrac{d\Psi_{11}}{dx}\right|_{x=0} = 0, & \left.\dfrac{d\Psi_{11}}{dx}\right|_{x=a} = 0, & \left.\dfrac{d\Psi_{12}}{dy}\right|_{y=0} = 0, & \left.\dfrac{d\Psi_{12}}{dy}\right|_{y=b} = 0, \\ W_2\big|_{x=0} = 0, & W_2\big|_{x=a} = 0, & W_2\big|_{y=0} = 0, & W_2\big|_{y=b} = 0, \\ \left.\dfrac{d\Psi_{21}}{dx}\right|_{x=0} = 0, & \left.\dfrac{d\Psi_{21}}{dx}\right|_{x=a} = 0, & \left.\dfrac{d\Psi_{22}}{dy}\right|_{y=0} = 0, & \left.\dfrac{d\Psi_{22}}{dy}\right|_{y=b} = 0, \end{cases} \tag{55}$$

for the Kirchhoff–Love model

$$\begin{cases} W_1\big|_{x=0} = 0, & W_1\big|_{x=a} = 0, & W_1\big|_{y=0} = 0, & W_1\big|_{y=b} = 0, \\ \left.\dfrac{d^2W_1}{dx^2}\right|_{x=0} = 0, & \left.\dfrac{d^2W_1}{dx^2}\right|_{x=a} = 0, & \left.\dfrac{d^2W_1}{dy^2}\right|_{y=0} = 0, & \left.\dfrac{d^2W_1}{dy^2}\right|_{y=b} = 0, \\ W_2\big|_{x=0} = 0, & W_2\big|_{x=a} = 0, & W_2\big|_{y=0} = 0, & W_2\big|_{y=b} = 0, \\ \left.\dfrac{d^2W_2}{dx^2}\right|_{x=0} = 0, & \left.\dfrac{d^2W_2}{dx^2}\right|_{x=a} = 0, & \left.\dfrac{d^2W_2}{dy^2}\right|_{y=0} = 0, & \left.\dfrac{d^2W_2}{dy^2}\right|_{y=b} = 0. \end{cases} \tag{56}$$

In the first case, the effect of the stationary dynamic force (50) in a complex system, which consists of elastic plates coupled by a viscoelastic inertial interlayer resting on a stiff foundation $W_2 = 0$, is presented in Figs. 2 and 3. Figures 2 and 3 present the amplitude-frequency diagrams for complex systems with damping ($c = 10^{-6}$ s) with viscoelastic interlayers of small $h = 0.08$ m (Fig. 2) and large $h = 0.12$ m (Fig. 3) thickness for real stationary frequencies in the range $1700 < \omega_0 < 2000$ at the point $x = 0.7a$, $y = 0.7b$.

The amplitudes of forced vibrations of the plates of a complex system with damping loaded by a concentrated moving force calculated on the basis of Timoshenko's model are approximately 9–12% higher than those of the plates obtained with the use of the Kirchhoff–Love model (Figs. 2 and 3).

In the latter case, small-frequency transverse vibrations of a complex system with damping are excited by a nonstationary dynamic load $f_1(x, y, t)$ varying in time t.

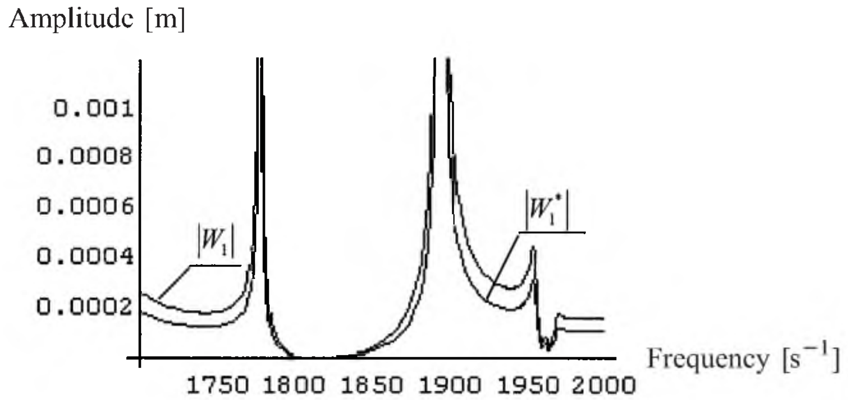

Fig. 2. The amplitude-frequency diagram for a complex system with damping with a stationary dynamic force $f_1(x,y,t)$ acting at the point x_0, y_0 for $h = 0.08$ m: (a) for the Kirchhoff–Love model $|W_1^*|$; (b) Timoshenko's model $|W_1|$.

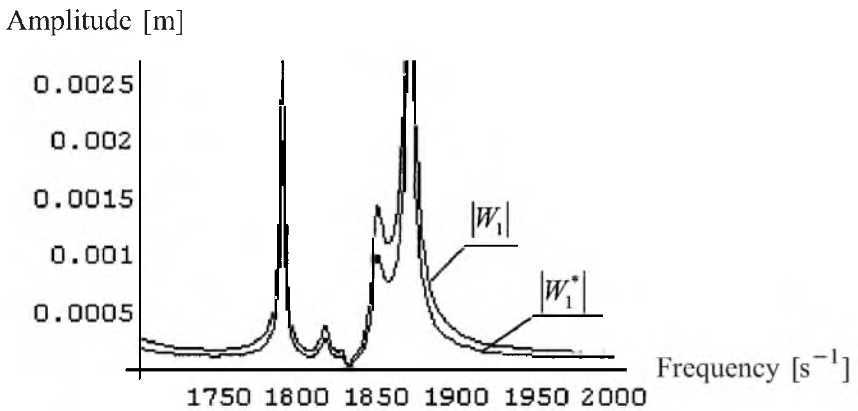

Fig. 3. The amplitude-frequency diagram for a complex system with damping with a stationary dynamic force $f_1(x,y,t)$ acting at the point x_0, y_0 for $h = 0.12$ m: (a) for the Kirchhoff–Love model $|W_1^*|$; (b) Timoshenko's model $|W_1|$.

Small-frequency transverse vibrations of a complex system with damping are excited by a non-inertial moving force $f_1(x,y,t) = P_1\delta(x - v^* t)$ with the speed v^* for the parameters: $y = 0.5b$ and $h = 0.08$ m.

The effect of non-inertial moving loading in a complex system with a viscoelastic interlayer is shown in Figs. 4 and 5 (the Kirchhoff–Love model (Fig. 4) and Timoshenko's model (Fig. 5)).

Calculations on the basis of Timoshenko's model for a complex system with damping loaded by a concentrated moving force gave the amplitudes of forced vibrations of the plates that are approximately 60% larger than those obtained with the use of the Kirchhoff–Love model.

Similar results were obtained for the problems of vibration of a complex system with damping excited by an inertial moving concentrated mass and nonuniform loading varying in time t.

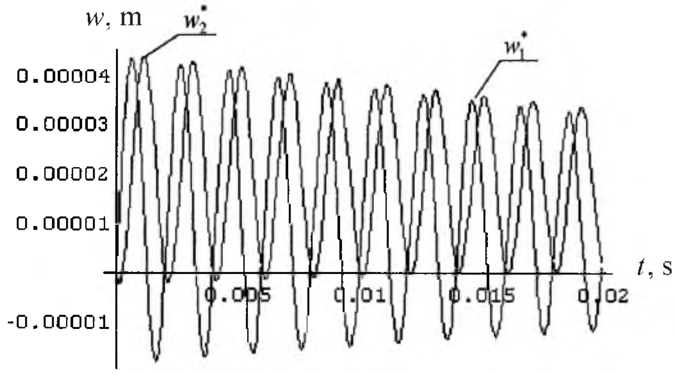

Fig. 4. Forced vibrations of a complex system with damping induced by a nonstationary force $f_1(x, y, t)$ moving with the speed $v^* = 20\ \mathrm{m \cdot s^{-2}}$, $h = 0.08$ m.

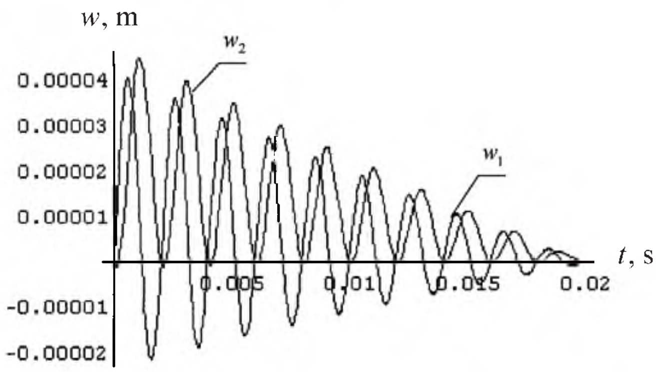

Fig. 5. Forced vibrations of a complex system with damping induced by a nonstationary force $f_1(x, y, t)$ moving with the speed $v^* = 20\ \mathrm{m \cdot s^{-2}}$, $h = 0.08$ m.

Conclusions. The analytical-numerical method presented in this paper can be applied to solutions of free and forced vibrations of various engineering structures consisting of plates and shells coupled by viscoelastic constraints for various types of dynamic loading.

Numerical investigations revealed that the problem of free and forced vibrations for various engineering structures consisting of plates and shells subjected to the action of a stationary dynamic force can be solved using the Kirchhoff–Love model. When we consider a problem of free and forced vibrations for various engineering structures under moving concentrated loading, it is necessary to use Timoshenko's model.

Резюме

На основі розробленого аналітико-числового методу розв'язку задачі про вільні і вимушені згасаючі коливання на прикладі системи, що складається з двох пластин, з'єднаних в'язкопружним шаром, виконано числовий аналіз і вивчено нові механічні ефекти, зумовлені дією на дану систему різного типу динамічного навантаження. Розрахунки проводилися на основі моделей Тимошенка і Кірхгофа–Лява.

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