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Vibrations of a Complex System with a Viscoelastic Inertial Interlayer

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Колебания сложных систем с вязкоупругим инерционным заполнителем

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Предложен аналитический метод решения задач о затухании свободных и вынужденных колебаний сложных систем, несущие слои которых выполнены из однородного упругого, а средний – из вязкоупругого инерционного материала. Малые поперечные колебания сложных систем обусловлены распределенной и подвижной нагрузкой. Выполнен динамический анализ слоистых конструкций в широком диапазоне изменения геометрических и механических характеристик слоя из вязкоупругого инерционного материала.

Ключевые слова: колебания, сложная система, вязкоупругий инерционный заполнитель, аналитическое решение.

Introduction. Compound systems coupled together by elastic constraints play an important role in various engineering and building structures. Vibration problems in engineering were considered by Timoshenko [1]. Vibration analysis for laminated plates is presented in [2–3] and in many other works. The stress-strain state of laminated orthotropic nonhomogeneous plates and shells was considered in [4–5]. Vibrations of an elastically connected rectangular double-plate compound system with moving loads are given in [6].

Vibration analysis of layered systems with vibration damping is a difficult problem. In the above complex cases, especially where viscosity and discrete elements occur, it is recommended to adopt a method of solving a dynamic problem for such a system in complex functions [7–8]. For the first time the property of orthogonality of complex modes of free vibrations for discrete systems with damping was presented in [7], for discrete-continuous systems with damping in [8], and for continuous systems with damping in [15, 16].

The goal of this paper is to present the solution and dynamic analysis of free and forced vibrations for a complex system with damping, which consists of a plate, viscoelastic inerlayer, and stiff foundation.

Statement of the Problem. Let us consider a problem of free and forced vibrations for a complex system with a viscoelastic inertial interlayer. The external layers of the complex system are made from an elastic plate and stiff

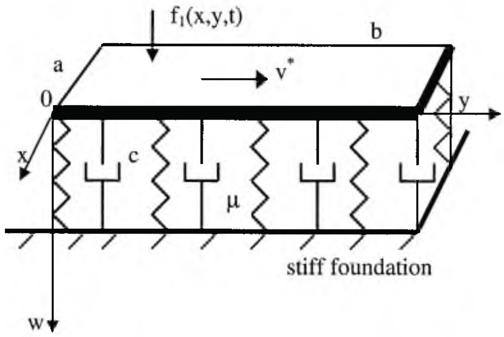

Fig. 1. Dynamic model for a complex system of an elastic plate with a viscoelastic inertial interlayer resting on stiff foundation.

foundation and are shown in Fig. 1. The elastic plate is described by the Kirchhoff–Love model and is simply supported at its ends. The viscoelastic inertial interlayer has the characteristics of a homogeneous continuous unidirectional Winkler's foundation and was described by the Voigt–Kelvin model [9–11].

In this paper, we consider two cases. In the first case, small-frequency transverse vibrations of a complex system are exited by a stationary dynamic load $f_1(x,y,t)$. In the second case, small-frequency transverse vibrations of a complex system are exited by a non-inertial moving load $f_1(x,\bar{y},t)$, $\bar{y}=v^*t$ with the speed v^*.

The phenomenon of small-frequency transverse vibrations for an elastic plate with a viscoelastic inertial interlayer resting on a stiff foundation is described by the following non-homogeneous system of conjugate partial differential equations:

$$D_1\Delta^2 w_1 + \mu_1 \frac{\partial^2 w_1}{\partial t^2} - \left(1 + c\frac{\partial}{\partial t}\right) E \frac{\partial w}{\partial z}\bigg|_{z=0} = f_1(x,y,t), \tag{1}$$

$$-\left(1 + c\frac{\partial}{\partial t}\right) Eh \frac{\partial^2 w}{\partial t^2} = 0, \tag{2}$$

where

$$\Delta^2 w_1 = \frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4}, \tag{3}$$

$$D_1 = \frac{E_1 h_1^3}{12(1-\nu_{1p}^2)}, \quad \mu_1 = \rho_1 h_1, \quad \mu = \rho h. \tag{4}$$

Here $w_1 = w_1(x,y,t)$, $w(x,y,z,t)$ are the transverse deflections of the plate and the viscoelatic inertial interlayer, respectively, E_1 and E are Young's moduli of the material of the plate and the interlayer, c is the damping coefficient of the interlayer (retardation time), ρ_1 and ρ are the mass densities of the material of the plate and the interlayer, respectively, h_1 and h are the thickness of the plate and the interlayer, a and b are the dimensions of the plate, ν_{1p} is Poisson's ratio, x and y are the coordinate axes, and $f_1(x,y,t)$ is the dynamic load acting on the complex system.

Separation of Variables. Presenting the solution of the problem under consideration in the form

$$\begin{bmatrix} w_1(x,y,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} W_1(x,y) \\ W(x,y,z) \end{bmatrix} \exp(i\nu t) \tag{5}$$

and substituting (5) in the system of differential equations (1), (2), by an assumption that $f_1(x,y,t)=0$, we obtain a homogeneous system of conjugate ordinary differential equations describing complex modes of free vibration of the plate and the viscoelastic inertial interlayer:

$$D_1\Delta^2 W_1 - \mu_1\nu^2 W_1 - (1+ic\nu)E\left.\frac{\partial W}{\partial z}\right|_{z=0} = 0, \tag{6}$$

$$\frac{d^2W}{dz^2} + \lambda^2 W = 0, \tag{7}$$

where

$$\Delta^2 W_1 = \frac{\partial^4 W_1}{\partial x^4} + 2\frac{\partial^4 W_1}{\partial x^2 \partial y^2} + \frac{\partial^4 W_1}{\partial y^4}, \tag{8}$$

$$\lambda = \sqrt{\frac{\mu\nu^2}{Eh(1+ic\nu)}}, \qquad i^2 = -1. \tag{9}$$

Here $W_1(x,y)$ and $W(x,y,z)$ are the complex modes of free vibration of the plate and the interlayer, and ν is the complex eigenfrequency of the complex system with damping.

Solution of a Boundary Value Problem. The solution for the inertial viscoelastic interlayer (7) is presented in the following form:

$$W(x,y,z) = C_1(x,y)\sin\lambda z + C_2(x,y)\cos\lambda z, \tag{10}$$

where $C_1(x,y)$ and $C_2(x,y)$ are constant coefficients.

We have assumed the following geometric conditions:

$$W|_{z=0} = W_1, \quad W|_{z=h} = 0. \tag{11}$$

On substitution of (10) and (11) in (7), Eq. (10) can be rewritten in the following form:

$$W(x,y,z)=W_1(x,y)\,(\cos\lambda z-\operatorname{ctg}\lambda z-\operatorname{ctg}\lambda h\sin\lambda z). \qquad (12)$$

Substituting (12) in Eq. (6), we obtain

$$D_1\Delta^2 W_1-[\mu_1\nu^2+(1+ic\nu)E\lambda\operatorname{ctg}\lambda h]W_1=0. \qquad (13)$$

The solution for W_1 is similar to that in [1]:

$$W_1(x,y)=X_1(x)Y_1(y). \qquad (14)$$

Substituting (9) and (14) in (13), we can rewrite Eq. (13) in the following form:

$$X_1^{IV}Y_1+2X_1^{II}Y_1^{II}+X_1Y_1^{IV}-\zeta^4X_1Y_1=0, \qquad (15)$$

where

$$\zeta^4=\frac{1}{D_1}[\mu_1\nu^2+(1+ic\nu)E\lambda\operatorname{ctg}\lambda h]. \qquad (16)$$

The quantities $X_1(x)$ and $Y_1(y)$ can be separated as follows [1]:

$$\frac{Y_1^{II}}{Y_1}=-d^2,\quad \frac{Y_1^{IV}}{Y_1}=g^4. \qquad (17)$$

Representing the solution of the differential equation (15) in the forms of

$$X_1=A\exp(r_1x)\quad\text{and}\quad Y_1=B\exp(r_2y), \qquad (18)$$

we obtain a characteristic equation in the form of an algebraic equation

$$[r_1^4-2d^2r_1^2+(d^4-\zeta^4)](r_2^2+d^2)=0,\quad d^4=g^4 \qquad (19)$$

with the following roots:

$$r_1=\pm i\alpha_v,\quad \alpha_v=\sqrt{-\beta^2\pm\zeta^2},\quad r_2=\pm i\beta,\quad \beta=d. \qquad (20)$$

The solution of the differential equation (6) consists of a system of solutions:

$$W_1(x,y)=\sum_{v=1}^{2}[A_v^*\sin\alpha_v x+A_v^{**}\cos\alpha_v x][B^*\sin\beta y+B^{**}\cos\beta y], \qquad (21)$$

where A_v^*, A_v^{**}, B^*, and B^{**} are constant coefficients.

In order to solve the boundary value problem, the following boundary conditions are used:

$$W_1\big|_{x=0}=0,\quad W_1\big|_{x=a}=0,\quad W_1\big|_{y=0}=0,\quad W_1\big|_{y=b}=0,$$
$$\left.\frac{d^2W_1}{dx^2}\right|_{x=0}=0,\quad \left.\frac{d^2W_1}{dx^2}\right|_{x=a}=0,\quad \left.\frac{d^2W_1}{dy^2}\right|_{y=0}=0,\quad \left.\frac{d^2W_1}{dy^2}\right|_{y=b}=0. \tag{22}$$

Substituting the sequences $\alpha_{1n}=\alpha_{2n}=\alpha_{n_1}$ and β_{n_2} in Eqs. (12) and (21), we obtain the following two complex sequences of free vibration modes for a plate and viscoelastic inertial interlayer:

$$W_{1n_1n_2}(x,y)=\sin\alpha_{n_1}x\sin\beta_{n_2}y, \tag{23}$$

$$W_{n_1n_2}(x,y,z)=\sin\alpha_{n_1}x\sin\beta_{n_2}y[\cos\lambda h\sin\lambda z], \tag{24}$$

where

$$\alpha_{n_1}=\frac{n_1\pi}{a},\quad \beta_{n_2}=\frac{n_2\pi}{b},\quad n_1=1,2,3,\ldots,\quad n_2=1,2,3,\ldots. \tag{25}$$

For $r_{1n_1}=\pm i\alpha_{n_1}$ and $r_{2n_2}=\pm i\beta_{n_2}$, on substituting Eqs. (16) and (25) in (20) and carrying out the transformations, we obtain the following equation of frequency:

$$\frac{n_1^2\pi}{a^2}+\frac{n_2^2\pi^2}{b^2}=\zeta^2_{n_1n_2}, \tag{26}$$

where

$$\zeta^2_{n_1n_2}=\pm\sqrt{\frac{1}{D_1}[\mu_1\nu^2_{n_1n_2}+(1+ic\nu_{n_1n_2})E\lambda_{n_1n_2}\operatorname{ctg}\lambda_{n_1n_2}h]} \tag{27}$$

and

$$\lambda_{n_1n_2}=\sqrt{\frac{\mu\nu^2_{n_1n_2}}{Eh(1+ic\nu_{n_1n_2})}}, \tag{28}$$

from which a sequence of complex eigenfrequencies is determined:

$$\nu_{n_1n_2}=i\eta_{n_1n_2}\pm\omega_{n_1n_2}. \tag{29}$$

Solution of the Initial Value Problem. Free vibration of a complex system with a viscoelastic inertial interlayer is represented in the form of a Fourier series based on the complex eigenfunctions, i.e.,

$$\begin{bmatrix} w_1(x,y,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} W_{1n_1n_2}(x,y) \\ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} W_{n_1n_2}(x,y,z) \end{bmatrix} \Phi_{n_1n_2} \exp(i\nu_{n_1n_2} t), \tag{30}$$

where $\Phi_{n_1n_2}$ is the Fourier coefficient.

From the system of equations (6) and (7) performing some algebraic transformations, adding the equations together and then integrating them on both sides within the limits from 0 to 1, we obtain the property of orthogonality of eigenfunctions for a complex system with an inertial viscoelastic interlayer:

$$\int_0^a \int_0^b \left[i(\nu_n + \nu_m) \left(\mu_1 W_{1n} W_{1m} + \int_0^h \mu W_n W_m dz \right) \right] dxdy +$$

$$+ c \int_0^l \int_0^a \int_0^b \frac{dW_n}{dz} \frac{dW_m}{dz} dxdydz = N_n \delta_{nm}, \tag{31}$$

where

$$N_{n_1n_2} = \int_0^a \int_0^b \left[2i\nu_{n_1n_2} \left(\mu_1 W_{1n_1n_2}^2 + \int_0^h \mu W_{n_1n_2}^2 dz \right) \right] dxdy +$$

$$+ c \int_0^a \int_0^b \int_0^h \left(\frac{dW_{n_1n_2}}{dz} \right)^2 dxdydz. \tag{32}$$

Here δ_{nm} is Kronecker's delta, and $n = (n_1, n_2)$, $m = (m_1, m_2)$.

The following initial conditions are the basis for solving the problem of free vibrations:

$$w_1(x,y,0) = w_{01}, \quad \overset{\circ}{w_1}(x,y,z,o) = \overset{\circ}{w}_{01}, \quad w_0(x,y,0) = w_0. \tag{33}$$

By applying conditions (33) in series (30) and taking into account the property of orthogonality (31), the formula for the complex Fourier coefficient is obtained:

$$\Phi_{n_1n_2} = \frac{1}{N_{n_1n_2}} \int_0^a \int_0^b \left[\mu_1 (i\nu_{n_1n_2} W_{1n_1n_2} w_{01} + W_{1n_1n_2} \overset{\circ}{w}_{01}) + \int_0^h \mu W_{n_1n_2} w_0 dz \right] dxdy +$$

$$+ c \int_0^a \int_0^b \int_0^h \frac{dW_{n_1n_2}}{dz} \frac{dw_0}{dz} dxdydz. \tag{34}$$

Free vibration of the complex system with a viscoelastic inertial interlayer is obtained in the following form:

$$w_1 = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} e^{-\eta_{n_1 n_2} t} \left| W_{1n_1 n_2} \right| \left| \Phi_{n_1 n_2} \right| \cos(\omega_{n_1 n_2} t + \varphi_{n_1 n_2} + \chi_{1n_1 n_2}), \tag{35}$$

$$w = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} e^{-\eta_{n_1 n_2} t} \left| W_{n_1 n_2} \right| \left| \Phi_{n_1 n_2} \right| \cos(\omega_{n_1 n_2} t + \varphi_{n_1 n_2} + \chi_{n_1 n_2}), \tag{36}$$

where

$$\begin{gathered} \left| W_{1n_1 n_2} \right| = \mathrm{mod}\, W_{1n_1 n_2}, \quad \left| W_{n_1 n_2} \right| = \mathrm{mod}\, W_{n_1 n_2}, \\ \chi_{1n_2 n_2} = \arg W_{1n_1 n_2}, \quad \chi_{n_1 n_2} = \arg \Phi_{n_1 n_2}. \end{gathered} \tag{37}$$

Solution of the Forced Vibration Problem. The first case concerns stationary forced vibrations. Small-frequency transverse vibrations of a complex system with damping are excited by dynamic loading $f_1(x,y,t)$ at the points x_0 and y_0 varying in time t (Fig. 1):

$$f_1(x,y,t) = P_1 \delta(x - x_0) \delta(y - y_0) \sin \omega_0 t, \tag{38}$$

where P_1 is the force, $\delta(x - x_0)$ and $\delta(y - y_0)$ are Dirac delta functions, and ω_0 is the real frequency of stationary forced vibrations.

The solution for complex modes of stationary forced vibrations for a complex system with damping are written in the form

$$W_1(x,y) = W_1^*(x,y) + W_1^{**}(x,y). \tag{39}$$

The general solution of the differential equation (1) consists of the system of solutions

$$W_1^*(x,y) = \sum_{v=1}^{2} [A_v^* \sin \alpha_v + A_v^{**} \cos \alpha_v x][B^* \sin \beta y + B^{**} \cos \beta y]. \tag{40}$$

In the case $\nu = \omega_0$ in Eq. (20)

$$\alpha_1 \neq \alpha_2 = \sqrt{-\beta^2 + \zeta_0^2}, \quad \beta = \frac{n_2 \pi}{b}, \tag{41}$$

where

$$\zeta_0^2 = \pm \sqrt{\frac{1}{D_1} [\mu_1 \omega_0^2 + (1 + ic\omega_0) E \lambda_0 \operatorname{ctg} \lambda_0 h]}. \tag{42}$$

Here

$$\lambda_0 = \sqrt{\frac{\mu\omega_0^2}{Eh(1+ic\omega_0)}}, \tag{43}$$

from which the frequencies of real stationary forced vibrations are determined: $\omega_0 = \{0...100000\}$.

The particular solution of the differential equation (1) consists of the system of solutions

$$W_1^{**}(x,y) =$$

$$= \frac{P_1}{D_1}\sum_{v=1}^{2}\frac{1}{\alpha_v\beta}\int_0^a\int_0^b \sin\alpha_v(x-\tau_1)\delta(\tau_1-x_0)\sin\beta(y-\tau_2)\delta(\tau_2-y_0)d\tau_1 d\tau_2. \tag{44}$$

On substituting (40) and (44) into (39), the complex modes of the stationary forced vibrations for a complex system with a viscoelastic inertial interlayer can be written in the following form:

$$W_1(x,y) = (A_1^* \sin\alpha_1 x + A_2^* \sin\alpha_2 x)B^* \sin\beta y +$$

$$+\frac{P_1}{D_1}\left[\frac{1}{\alpha_1}\sin\alpha_1(x-x_0)H(x-x_0) + \frac{1}{\alpha_2}\sin\alpha_2(x-x_0)\right]\times$$

$$\times\frac{1}{\beta}\sin\beta(y-y_0)H(y-y_0). \tag{45}$$

In order to solve the problem of forced vibrations, the boundary conditions (22) have been applied. The constants occurring in Eq. (45) are described in the following form:

$$A_1^{**} = A_2^{**} = 0, \quad A_1^* = -\frac{1}{\alpha_1}\frac{\sin\alpha_1(a-x_0)}{\sin\alpha_1 a}, \quad A_2^* = -\frac{1}{\alpha_2}\frac{\sin\alpha_2(a-x_0)}{\sin\alpha_2 a},$$
$$B^{**} = 0, \quad B^* = -\frac{P_1}{D_1}\frac{1}{\beta}\frac{\sin\beta(b-y_0)}{\sin\beta b}. \tag{46}$$

Forced vibrations of a complex system with damping are stationary and have the following form:

$$w_1(x,y,t) = W_1(x,y)\exp(i\omega_0 t). \tag{47}$$

Substituting (39) into (47) and performing trigonometric and algebraic transformations, we obtain forced vibrations of a complex system with a viscoelastic inertial interlayer:

$$w_1 = |W_1| \sin(\omega_0 t + \vartheta_1), \tag{48}$$

where $|W_1|$ is the amplitude of an elastic plate with a viscoelastic inertial interlayer and ϑ_1 is $\arg W_1(x,y)$.

The second case concerns non-stationary forced vibrations. Small-frequency transverse vibrations of a complex system with damping are excited by a non-inertial moving load $f_1(x,y,t)$ [12, 13] with the speed v^* (Fig. 1).

$$f_1(x,y,t) = P_1 \delta(y - y). \tag{49}$$

Here P_1 is the force, $\delta(y - \bar{y})$ is the Dirac delta function, $\bar{y} = v^* t$, $x = 0.5a$, and $v^* = \text{const}$.

In order to solve the differential equations (1), (2). the function of the load (49) is expanded by the operational method [14]:

$$f_1(x,y,t) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (\mu_1 W_{1n_1n_2} + \mu W_{n_1n_2}) f_{n_1n_2}, \tag{50}$$

where $W_{1n_1n_2}$ and $W_{n_1n_2}$ have been described by Eqs. (23) and (24).

The function of the displacement of a complex system with damping is presented in the form of a Fourier series as

$$\begin{bmatrix} w_1 \\ w \end{bmatrix} = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \begin{bmatrix} W_{1n_1n_2} \\ W_{n_1n_2} \end{bmatrix} T_{n_1n_2}. \tag{51}$$

Substituting (50) and (51) into the differential equations (1), (2), we obtain the following equation of motion:

$$\overset{\circ}{T}_{n_1n_2} - i v_{n_1n_2} T_{n_1n_2} = f_{n_1n_2}, \tag{52}$$

where $T_{n_1n_2}$ is the coefficient of the distribution of the dynamic loading function in the Fourier series.

Applying the property of orthogonality of eigenfunction (31), we derive formulas for the coefficients of load distribution, namely:

$$f_{n_1n_2} = \frac{1}{D_1 N_{n_1n_2}} i v_{n_1n_2} \int_0^a \int_0^b \left[W_{n_1n_2}(x,y) + \int_0^h W_{n_1n_2}(x,y,z) dz \right] P_1 \delta(y - \bar{y}) dx dy. \tag{53}$$

The solution of the differential equation (52) has the form [14]

$$T_{n_1 n_2} = \frac{1}{i\nu_{n_1 n_2}} \int_0^t [\exp(i\nu_{n_1 n_2})(t-\tau) - 1] f_{n_1 n_2}(\tau) d\tau. \tag{54}$$

On substituting (53) and (54) into (51), Eq. (51) can be rewritten in the following form:

$$w_1 = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left|W_{1n_1 n_2}\right| \left|T_{n_1 n_2}\right| \cos(\vartheta_{1n_1 n_2} + \zeta_{n_1 n_2}), \tag{55}$$

$$w = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \left|W_{n_1 n_2}\right| \left|T_{n_1 n_2}\right| \cos(\vartheta_{n_1 n_2} + \zeta_{n_1 n_2}), \tag{56}$$

where

$$\vartheta_{1n_1 n_2} = \arg W_{1n_1 n_2}, \quad \vartheta_{n_1 n_2} = \arg W_{n_1 n_2}, \quad \xi_{n_1 n_2} = \arg T_{n_1 n_2}. \tag{57}$$

Calculations. The calculations for a complex system with a viscoelastic inertial interlayer are presented. The external layers of the complex system are made of an elastic plate and stiff foundation. The thickness and mechanical characteristics of the plate, foundation, and the interlayer do not change. Numerical results are presented for the same parameters: $E_1 = 10^{10}$ N·m^{-2}, $\rho_1 = 2 \cdot 10^3$ N·s^2·m^{-4}, $\nu_{1p} = 0.3$, $h_1 = 0.2$ m, $h = 0.5$ m, $P = 2 \cdot 10^4$ N, $a = 10$ m, $b = 1000$ m, $c = 2.5$ s, and $v^* = 120$ m·s^{-2}.

The amplitude-frequency diagrams for a complex system with damping for real stationary frequencies in the range $0 < \omega_0 < 100000$ are presented in Figs. 2–3.

In the first case, small-frequency transverse vibrations of the complex system are excited by the force $f_1(x,y,t) = P_1 \delta(x - x_0)\delta(y - y_0)\sin(\omega_0 t)$ acting at the point $x_0 = 0.5a$, $y_0 = 0.5b$ and varying in time t. The amplitude-frequency diagrams of the complex system are presented in Fig. 2. Changes in the amplitude $|W_1|$ for the complex system at the point $x = 0.55a$, $y = 0.55b$ (variant “a”) and at the point $x = 0.7a$, $y = 0.8b$ (variant “b”) are also given there. In the case of variant “b” considered, the amplitude $|W_2|$ of the complex system is 50% smaller than the amplitude $|W_1|$ of the complex system for variant “a”.

The changes in the amplitude $|W_1|$ of the viscoelastic inertial interlayer at the point $x = 0.7a$, $y = 0.8b$ are shown in Fig. 3 for two thicknesses of the interlayer. In variant “a” for the thickness $z = 0.2h$, the amplitude $|W_1|$ of the interlayer is 36% smaller than the amplitude of the plate for $z = 0$. In variant “b” for the thickness $z = 0.5h$, the amplitude $|W_1|$ of the interlayer is 65% smaller than the amplitude of the plate for $z = 0$. After analyzing the results presented in Figs. 2–3, we state that a viscoelastic inertial interlayer can be a vibration damper for a plate loaded by the force $f_1(x,y,t)$ acting at the point x_0, y_0 and varying in time t.

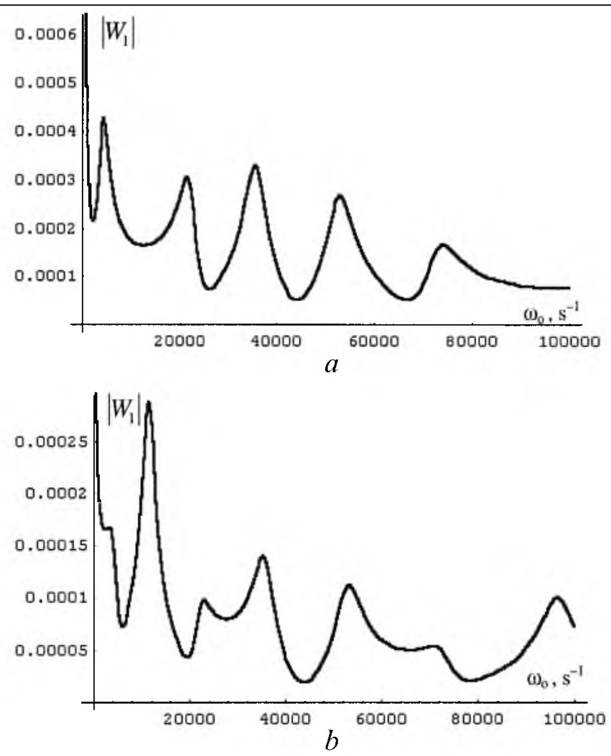

Fig. 2. The amplitude-frequency diagrams for an elastic plate with a viscoelastic inertial interlayer at the points: (a) $x = 0.55a$, $y = 0.55b$; (b) $x = 0.7a$, $y = 0.8b$.

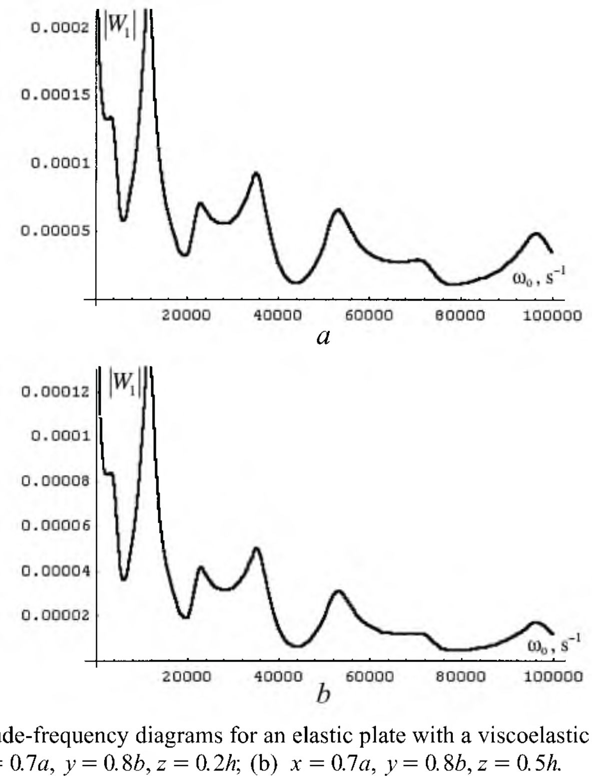

Fig. 3. The amplitude-frequency diagrams for an elastic plate with a viscoelastic inertial interlayer at the points: (a) $x = 0.7a$, $y = 0.8b, z = 0.2h$; (b) $x = 0.7a$, $y = 0.8b, z = 0.5h$.

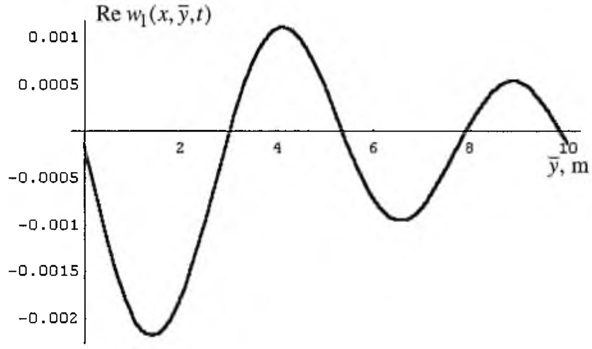

Fig. 4. The trajectory of dynamic displacement of a moving point of an elastic plate with a viscoelastic inertial interlayer for the concentrated non-inertial moving force for $x = 0.5a$ for time $t = 0.03$ s.

In the second case, small-frequency transverse vibrations of a complex system are exited by the moving concentrated force $f_1(x,y,t) = P_1\delta(y - y)$ with the speed v^*. The effect of the non-inertial moving force in the complex system with an inertial viscoelastic interlayer is presented in diagram 4. The diagram shows the real part of the trajectory of dynamic displacement of a moving point of the complex system $w_1(x,y,t)$ for $x = 0.5a$, $y = v^* t$ for time $t = 0.03$ s.

A model for a complex system consisting of an elastic plate with a viscoelastic inertial interlayer resting on a stiff foundation can play the role of a runway in an airport. In the first case, we might have been considering free and forced vibrations of an aircraft, which touches the runway of the airport. In the second case, we might have been considering the trajectory of dynamic displacement of an aircraft moving with the speed v^* after touching the runway of the airport. A complex system with a viscoelastic inertial interlayer of big thickness can be used for damping vibrations in practical problems.

Complex modes of free vibrations and the property of orthogonality of those modes presented in this paper are the basis for solving the problems of free and forced vibrations of a complex system with a viscoelastic inertial interlayer.

Резюме

Запропоновано аналітичний метод розв'язку задач щодо згасання вільних та вимушених коливань складних систем, несучі шари яких виконано з однорідного пружного, а середній – з в'язкопружного інерційного матеріалу. Малі поперечні коливання складних систем зумовлені розподіленим і рухомим навантаженням. Виконано динамічний аналіз шаруватих конструкцій у широкому діапазоні зміни геометричних і механічних характеристик шару з в'язкопружного інерційного матеріалу.

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