UDC 539.4

Dynamic Analysis of Sandwich Cylindrical Shell

K. Cabańska-Płaczkiewicz

Pedagogical University, Department of Mathematics Technology and Natural Science Institute of Technics, Bydgoszcz, Poland

УДК 539.4

Динамический анализ многослойных цилиндрических оболочек

К. Цабанска-Плашкевич

Педагогический университет, отделение математических технологий и естественных наук Технического института, Быдгощ, Польша

Выполнен динамический анализ цилиндрических оболочек и изучены новые механические эффекты в распределении напряжений, деформаций и перемещений. Цилиндрические оболочки рассматриваются на основе гипотезы Кирхгофа–Лява и уточненной модели Тимошенко.

Cylindrical shells with wide range of geometrical and physicomechanical parameters are component parts of many modern structures subject to the action of different loads. The development of various vibration models becomes an urgent problem due to the structural features of layered shells operating under different conditions of mechanical loading. The classical theory of cylindrical shells, which is based on the Kirchhoff–Love hypothesis, is widely used for the evaluation of the stress-strain state or vibration of isotropic thin elastic shells. Among numerous precise models applied to the investigation of shells made of modern materials due to their practical validity, visualization, and completeness, the Timoshenko shear model is used. The behavior of thick-walled shells under such conditions is fairly peculiar and complex and requires a more detailed investigation taking into account the changes in geometry, material behavior, mode of loading, conditions of the boundary, and many other factors.

The dynamic problem of elastic homogeneous bodies was presented in [1, 2]. The problem of simulation of the acoustic properties of the larger human blood vessel was considered in [3]. Simple and complex vibration systems were considered in [4]. The coupled problems of the thermomechanical behavior of viscoelastic bodies under harmonic loading were presented in [5]. The problem of nonaxisymmetric deformation of flexible rotational shells was solved in [6] with the use of the classical Kirchhoff–Love model and improved Timoshenko model. Free vibrations of the elements of shell constructions were described in [7].

The goal of this paper is to perform the dynamic analysis of cylindrical shells and discover new mechanical effects in the distribution of stresses, deformations, and displacements. We present the dynamic analysis of elastic layered cylindrical shells of finite length l for different values of thickness h. We consider two models of deformation of a straight line shell element that is normal to the undeformed coordinate surface. The first one is based on the Kirchhoff–Love hypothesis according to which this element remains straight and normal and its length does not change in the process of deformation. The second one is the improved Timoshenko model, which is also based on the hypothesis of straight line, but, in this case, the shell element, initially normal to the surface, does not remain normal to the deformed surface. These kinematic assumptions are supplemented with static assumptions according to which the normal stresses on the squares that are parallel to the coordinate surface, as compared with the stresses on the other squares, can be neglected. The inertial forces associated with the displacement of a surface element along the coordinate axes are also taken into account. The layers of the shell are deformed without mutual separation on the entire surface of contact.

Due to the assumptions made, the displacements and deformations of arbitrary points of the shell are determined by the displacements and deformations of the coordinate surface. Thus, the displacements of the shell points located at a distance z from the coordinate surface are determined as a near function of the displacements of this surface:

$$U(s,\theta,r,t)=u(s,\theta,t)+r\psi_1(s,\theta,t),$$
$$V(s,\theta,r,t)=v(s,\theta,t)+r\psi_2(s,\theta,t),$$
$$W(s,\theta,r,t)=w(s,\theta,t),$$

where φ_1, φ_2 are the angles of rotation of the normal in the planes $\theta=\text{const}$ and $s=\text{const}$, respectively, for the classical version of the theory, or the full angles of rotation for the improved version of the theory with shear deformations taken into account.

Investigations were carried out for both models for radial displacements w in the cylindrical coordinates r, z, θ $(-h/2\le r\le h/2,\ 0\le z\le 1,\ 0\le\theta\le 2\pi)$ under axisymmetric loading.

One-Layered Cylindrical Shell. According to the assumptions of the classical Kirchhoff–Love theory, we have $\varepsilon_r=\varepsilon_{zr}=\varepsilon_{\theta r}=0$, the normal stresses $\sigma_r=0$, and $w(z,r,\theta,t)=f(z,r,t)$. In the mathematical model of the problem on the basis of the classical Kirchhoff–Love model, the system of conjugate partial differential equations describing the phenomenon of small transverse vibrations in the considered physical model has the form

$$\begin{aligned}&\frac{Eh}{1-\nu^2}\frac{d^2u}{dz^2}-\frac{\nu}{R}\frac{dw}{dz}-\rho h\frac{d^2u}{dt^2}=0,\\&D\frac{d^4w}{dz^4}+\frac{Eh}{(1-\nu^2)R}\left(\frac{w}{R}+\nu\frac{du}{dz}\right)+\rho h\frac{d^2w}{dt^2}=0,\end{aligned}\tag{1}$$

where

$$D=\frac{Eh^3}{12(1-\nu^2)}.$$

A solution of this problem was obtained in the following form:

$$W=\sum_{n=1}^{\infty} e^{-\eta_n t}\cos\left(\frac{n\pi z}{l}\right)\cos(\omega_n t+\varphi_n),$$
$$\varphi_z=\frac{\partial w}{\partial z}. \tag{2}$$

Function of stresses has been accepted as

$$\sigma_r=\sum_{t=0}^{T}\sum_{n=1}^{N}\sigma_r(r)\sin\frac{n\pi z}{l}\sin\omega t,$$
$$\sigma_z=\sum_{t=0}^{T}\sum_{n=1}^{N}\sigma_z(r)\cos\frac{n\pi z}{l}\sin\omega t, \tag{3}$$
$$\sigma_\theta=\sum_{t=0}^{T}\sum_{n=1}^{N}\sigma_\theta(r)\cos\frac{n\pi z}{l}\sin\omega t,$$

where $\sigma_r(r), \sigma_z(r), \sigma_\theta(r)$ are functions varying across the thickness h and length l of the shell at time t.

Under the assumptions of the improved Timoshenko model, in which the influences of forces of rotational inertia and shearing deformation are taken into account, one has deformations $\varepsilon_{zr}\neq 0, \varepsilon_{\theta r}\neq 0$ and normal stresses $\sigma_r=0$. The mathematical model of the problem on the basis of the improved Timoshenko model is represented by the following system of conjugate partial differential equations

$$D\frac{d^2\psi_z}{dz^2}+k'Gh\left[\frac{dw}{dz}-\psi_z\right]-\frac{\rho h^3}{12}\frac{d^2\psi_z}{dt^2}=0,$$
$$\rho h\frac{d^2w}{dt^2}-\frac{1}{k'Gh}\left[\frac{d^2w}{dz^2}-\frac{d\psi_z}{dz}\right]+\frac{Eh}{(1-\nu^2)R}\left(\frac{w}{R}+\nu\frac{du}{dz}\right)=0. \tag{4}$$

In this case, we found a solution of the problem in the form

$$w=\sum_{n=1}^{\infty} e^{-\eta_n t}\left[D\frac{d^2\psi_z}{dz^2}+\frac{\rho h^3}{12}\frac{d^2\psi_z}{dt^2}+k'Gh\psi_z\right]\cos(\omega_n t+\varphi_n),$$
$$\psi_z=\sum_{n=1}^{\infty} e^{\eta_n t}\frac{1}{k'Gh}\left[\frac{Eh}{(1-\nu^2)R^2}\left(w+\nu\frac{du}{dz}\right)+\rho h\frac{d^2w}{dt^2}-\frac{d^2w}{dz^2}\right]\cos(\omega_n t+\varphi_n). \tag{5}$$

Here, w, u are displacements of the shell, ψ_z is the angle of rotation of a cross section of the shell, ρ is the mass density of the material of the shell, h is the height of the shell, ν is Poisson's ratio, G is the Kirchhoff modulus of the material of the shell, E is Young's modulus of the material of the shell, k' is the shear coefficient, and t is time.

First, we consider a cylindrical shell under load $q_n = q_0 \sin(n\pi z / l)$ distributed symmetrically with respect to z-axis. Elastic cylindrical shell with finite length l, rigidity D, and external radius R is considered with free supports on its ends:

$$w|_{z=0} = 0, \quad w|_{z=l} = 0. \tag{6}$$

Numerical calculations are performed for the following data: $l = 120$ mm, $h = 2$; 6 mm, $E = 2.1 \cdot 10^{11}$ N/mm^2, $\nu = 0.3$, $n = 1, 2, \ldots, 20$.

Some results concerning the solution of the problem are shown in Figs. 1 and 2. The diagrams present free vibrations for displacements $w(z,t)$ in the middle section of the cylindrical shell for different thicknesses $h / R = 1/10$; 1/30 and the initial condition $w_0 = 0.02 \sin(n\pi z / l)$ [m]. The two-dimensional diagrams display the distribution with length $0 < z < 0.2\, l$ at time $t = 0.48$ s for various $n, \delta = l / nh$. The diagrams marked with "a" correspond to the cases where the classical Kirchhoff–Love model was applied, whereas the diagrams marked with "b" correspond to the cases where the improved Timoshenko model was used.

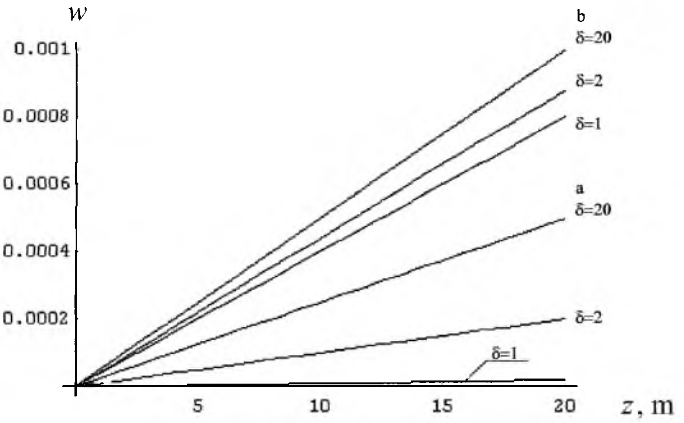

Fig. 1. Distribution of free vibrations of the cylindrical shell for $h / R = 1/10$.

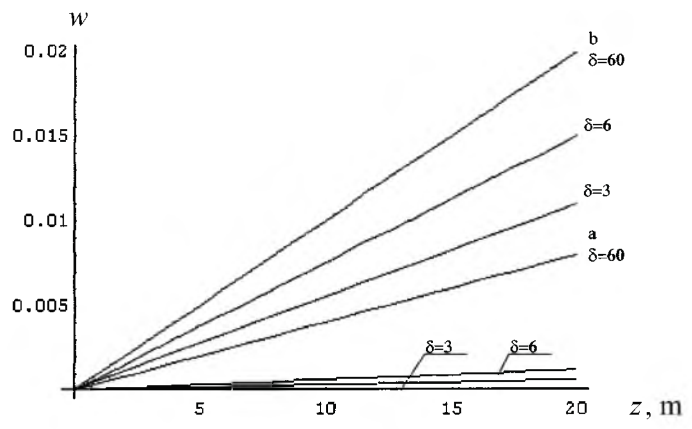

Fig. 2. Distribution of free vibrations of the cylindrical shell for $h / R = 1/30$.

Analyzing the results for $h / R = 1/10$, $\delta = 1$, 2 (Fig. 1) and $h / R = 1/30$, $\delta = 3$, 6 (Fig. 2), one can note that, as $n = 1$, 10, 20 increases, the free vibrations w decrease and the difference between the results obtained for the Timoshenko and Kirchhoff–Love models increases. As δ decreases, the difference between the values of displacement for the two models considered increases.

The distributions of stresses $\sigma_z, \sigma_r, \sigma_\theta, \tau_{rz}$ and deformations $\varepsilon_r, \varepsilon_z$ for cylindrical shells for $z = 0.5l$ across the thickness $-h/2 < r < h/2$ are shown in Fig. 3. The spatial curves of stresses and deformations are presented for the external load $q_n = q_0 \sin(n\pi z / l)$, which affects the cylindrical shell, with respect to time. The components of stresses and deformations vary nonlinearly across the thickness. The tangential stresses τ_{rz} vary across the thickness according to the parabolic law. The distribution of stresses τ_{rz} is symmetrical with respect to the $z = 0.5l$ axis. The normal stresses depend on r. Strain distribution $\varepsilon_z, \varepsilon_r$ (both axial and radial ones) also have the form of a centrally symmetrical parabola. Considering the section $z = 0.2\,l$, we see that the longitudinal σ_z, radial σ_r, and circular σ_θ stresses reach values that are smaller than those for $z = 0.5l$ by approximately 60%, 6%, and 54%, respectively. The tangential stresses τ_{rz} for $z = 0.2l$ reach values approximately 5% smaller than for $z = 0.5l$. For $z = 0.2l$, the deformations along the radial direction ε_r reach values approximately 75% smaller than for $z = 0.5l$. For $z = 0.2l$, the deformation along the axial direction ε_z reaches a value approximately 50% smaller than for $z = 0.5l$.

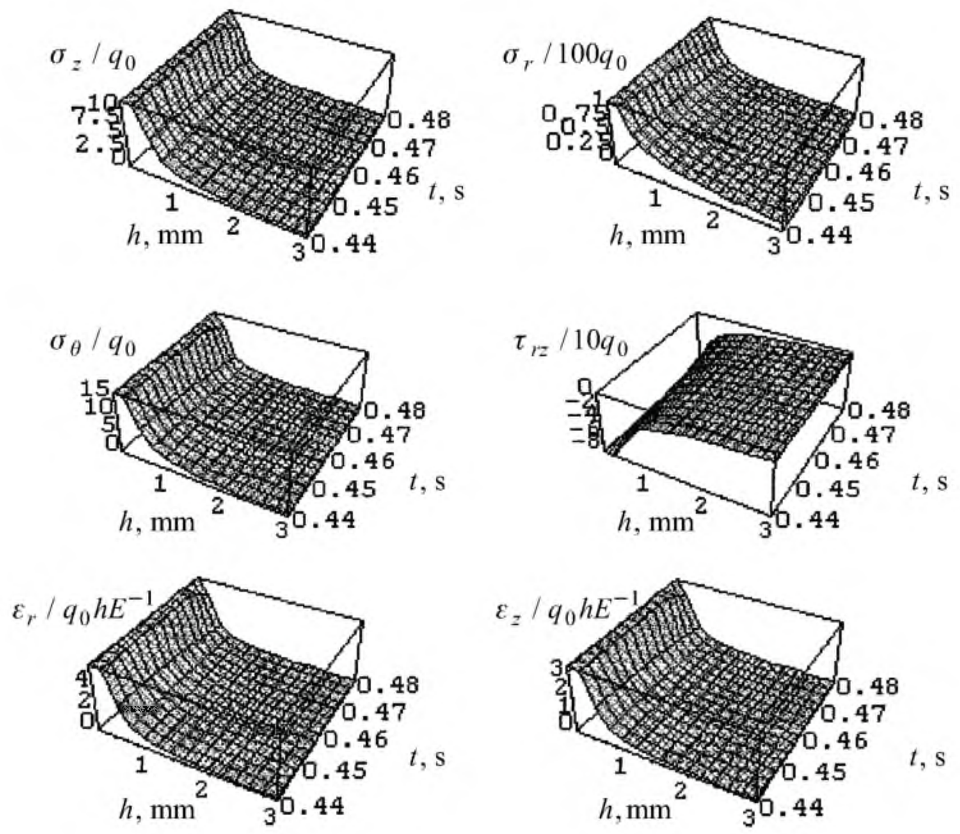

Fig. 3. Distribution of stresses $\sigma_z, \sigma_r, \sigma_\theta, \tau_{rz}$ and deformations $\varepsilon_r, \varepsilon_z$ for cylindrical shells for $z = 0.5\,l$ across the thickness $-h/2 < r < h/2$.

The distributions of the radial displacement $w(r,t)$ across the thickness h, $z = 0.5l$ at time t are shown in Fig. 4. The diagrams marked with "a" represent the distribution according to the Timoshenko model, and the diagrams marked with "b" describe the distribution according to the Kirchhoff–Love model. In the case of the Timoshenko model, the radial displacements vary across the thickness and decrease with time. In the case of the Kirchhoff–Love model, the radial displacements are described by a constant function across the thickness and also decrease with time.

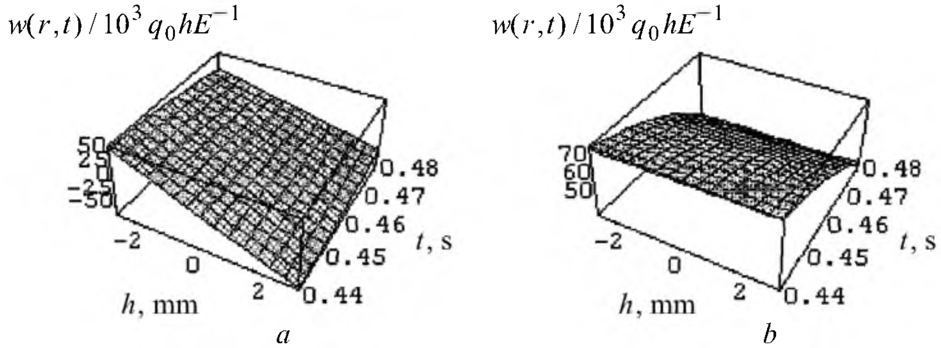

Fig. 4. Distribution of the radial displacement $w(r,t)$ across the thickness h, $z = 0.5l$ at time t.

For cylindrical shells with $h / R \geq 1/10$ and $\delta \leq 3$ subjected to nonuniform loading, it is necessary to use the Timoshenko model. The application of the classical Kirchhoff–Love model in these cases may lead to significant errors.

Sandwich Cylindrical Shell. In the case of a system of two cylindrical shells coupled by a viscoelastic interlayer, the mathematical model of the problem corresponding to the Timoshenko model is represented by the following system of coupled partial differential equations describing small transverse vibrations of the system:

$$
\begin{aligned}
& D_1 \frac{d^2\psi_1}{dz^2} + k'G_1h_1\left[\frac{dw_1}{dz} - \psi_1\right] - \frac{\rho_1 h_1^3}{12}\frac{d^2\psi_1}{dt^2} = 0, \\
& \rho_1 h_1 \frac{d^2 w_1}{dt^2} - \frac{1}{k'G_1h_1}\left[\frac{d^2 w_1}{dz^2} - \frac{d\psi_1}{dz}\right] + \frac{E_1 h_1}{(1-\nu^2)R_1}\left(\frac{w_1}{R_1} + \nu\frac{du_1}{dz}\right) + \\
& + (w_1 - w_2)\left(k + c\frac{d}{dt}\right) = 0, \\
& D_2 \frac{d^2\psi_2}{dz^2} + k'G_2h_2\left[\frac{dw_2}{dz} - \psi_2\right] - \frac{\rho_2 h_2^3}{12}\frac{d^2\psi_2}{dt^2} = 0, \\
& \rho_2 h_2 \frac{d^2 w_2}{dt^2} - \frac{1}{k'G_2h_2}\left[\frac{d^2 w_2}{dz^2} - \frac{d\psi_2}{dz}\right] + \frac{E_2 h_2}{(1-\nu^2)R_2}\left(\frac{w_2}{R_2} + \nu\frac{du_2}{dz}\right) - \\
& - (w_1 - w_2)\left(k + c\frac{d}{dt}\right) = 0,
\end{aligned}
\qquad (7)
$$

where R_1 is the radius of the internal shell I, R_2 is the radius of the interlayer, R_3 is the radius of the external shell II, h_1 and h_2 are the heights of shells I and II, ρ_1 and ρ_2 are the mass densities of the materials of shells I and II, w_{1n}, w_{2n}, u_{1n}, and u_{2n} are the displacements of shells I and II, ψ_{1n} and ψ_{2n} are the angles of rotation of cross sections of shells I and II, E_1 and E_2 are Young's moduli of the materials of shells I and II, k' is the shear coefficient, k is the coefficient of elasticity of the interlayer, and c is the coefficient of viscosity of the interlayer.

The free vibration of a system of two cylindrical shells coupled by a viscoelastic interlayer was determined with the use of the Timoshenko model as follows:

$$
\begin{aligned}
w_{1n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{1n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \chi_{1n}),\\
u_{1n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |U_{1n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \vartheta_{1n}),\\
\psi_{1n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Psi_{1n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \theta_{1n}),\\
w_{2n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |W_{2n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \chi_{2n}),\\
u_{2n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |U_{2n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \vartheta_{2n}),\\
\psi_{2n} &= \sum_{n=1}^{\infty} e^{-\eta_n t} |\Psi_{2n}||\Phi_n| [\cos(\omega_n t + \varphi_n + \theta_{2n}),
\end{aligned}
\tag{8}
$$

where

$$
\Phi_n = \left[\left(2\left(W_{1n}^2 + U_{1n}^2 + \frac{h_1^2}{12R_1^2}\Psi_{1n}^2 + W_{2n}^2 + U_{2n}^2 + \frac{h_2^2}{12R_2^2}\Psi_{2n}^2\right) + \right.\right.
$$

$$
\left.\left. + c(W_{1n} - W_{2n})^2\right)dx\right]^{-1} \int_0^l \left(W_{1n}^2 w_{10} + U_{1n}^2 u_{10} + \frac{h_1^2}{12R_1^2}\Psi_{1n}^2 \psi_{10} + \right.
$$

$$
\left. + W_{2n}^2 w_{20} + U_{2n}^2 u_{20} + \frac{h_2^2}{12R_2^2}\Psi_{2n}^2 \psi_{20} + c(W_{1n} - W_{2n})(w_{10} - w_{20})\right)dx,
$$

$$
\chi_{1n} = \arg W_{1n}, \quad \chi_{2n} = \arg W_{2n}, \quad \vartheta_{1n} = \arg U_{1n}, \quad \vartheta_{2n} = \arg U_{2n},
$$

$$
\theta_{1n} = \arg \psi_{1n}, \quad \theta_{2n} = \arg \psi_{2n}, \quad \varphi_n = \arg \Phi_n,
$$

and $v_n = i\eta_n \pm \omega_n$ are the complex natural frequencies.

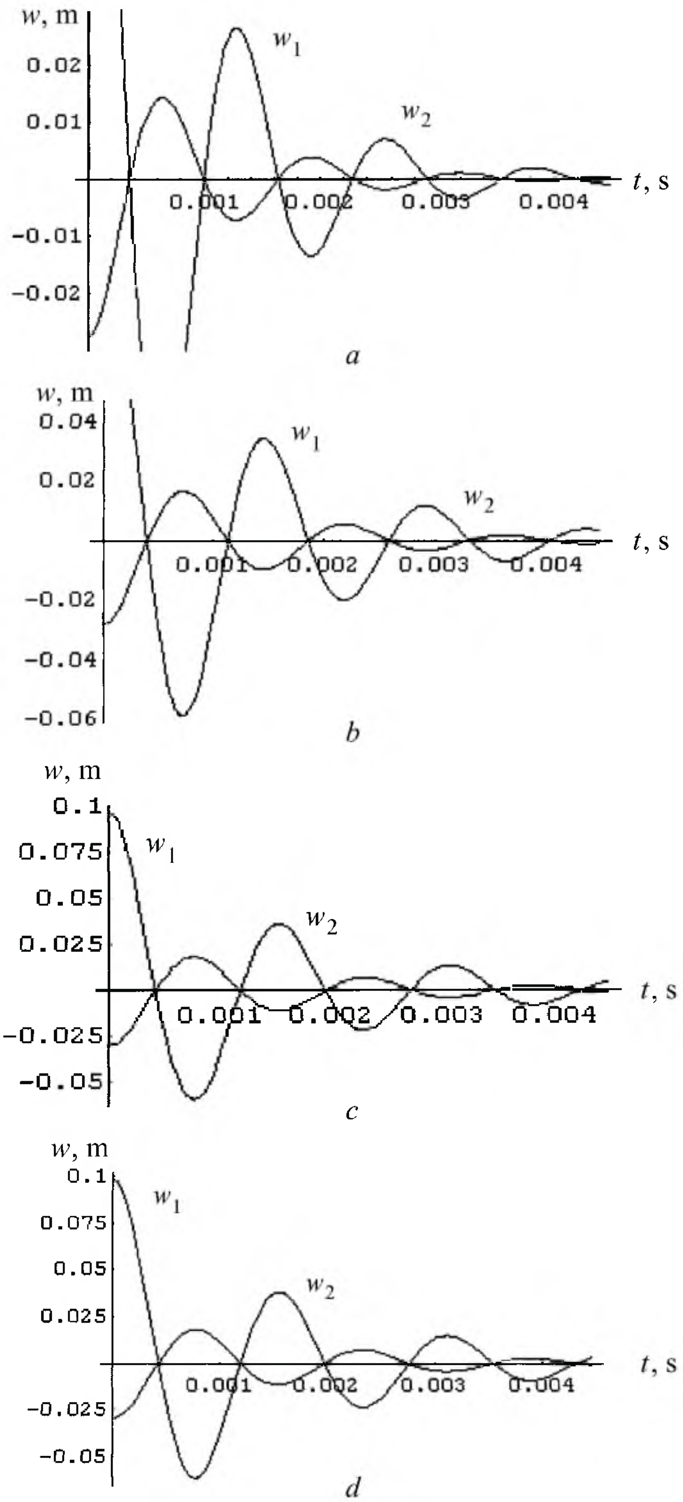

Fig. 5. Free vibration of a system of two cylindrical shells coupled by a viscoelastic interlayer for $z = 0.05\,l$ and different thicknesses of internal layers: *a*) $R_1 = 0.02$ m, $R_2 = 0.03$ m, $R_3 = 0.04$ m, and $h = 0.01$ m; *b*) $R_1 = 0.02$ m, $R_2 = 0.04$ m, $R_3 = 0.05$ m, and $h = 0.02$ m; *c*) $R_1 = 0.02$ m, $R_2 = 0.05$ m, $R_3 = 0.06$ m, and $h = 0.03$ m; *d*) $R_1 = 0.02$ m, $R_2 = 0.06$ m, $R_3 = 0.07$ m, and $h = 0.04$ m.

According to the Kirchhoff–Love model, the free vibration is described by (2).

Some results concerning the time dependence of the distributions of displacements w_1 and w_2 for the external layers I and II in the system of two cylindrical shells coupled together by a viscoelastic interlayer for different thicknesses of the internal layers in the case $z = 0.5l$ are presented in Fig. 5. The thicknesses of the external layers do not change. The internal layers are made of the soft material with $E = 10^8$ N/mm^2, $\nu = 0.3$ and are described by the Voigt–Kelvin model. The external layers of the cylindrical shell are made of an elastic material with $E_1 = E_2 = 2.1 \cdot 10^{11}$ N/mm^2, and $\nu = 0.3$. The displacements of the internal layer for the thickness $h = 0{,}01$ m are shown in a Fig. 5a and reach values approximately 15% smaller than those in Fig. 5b for the thickness $h = 0{,}02$ m. The displacements in Fig. 5b reach values approximately 8.5% smaller than those in Fig. 5c for $h = 0{,}03$ m. The displacements in Fig. 5c reach values approximately 5.3% smaller than those in Fig. 5d for $h = 0{,}04$ m.

In conclusion, it may be noted that, as the thickness R_3 of the interlayer increases, the free vibration of a sandwich system decays more slowly with time *t*.

Резюме

Проведено динамічний аналіз циліндричних оболонок та розглянуто нові механічні ефекти в розподіленні напружень, деформацій і переміщень. Циліндричні оболонки розглядаються на основі гіпотези Кірхгофа–Лява й уточненої моделі Тимошенка.

1. *Гринченко В. Т., Мелешко В. В.* Гармонические колебания и волны в упругих телах. – Киев: Наук. думка, 1981. – 284 с.
2. *Гринченко В. Т.* Равновесие и установившиеся колебания упругих тел конечных размеров. – Киев: Наук. думка, 1978. – 264 с.
3. *Borisyuk A.* Modelling of the acoustic properties of the larger human blood vessel // J. Acoustic Wisnik. – 1998. – **1**, N 3. – P. 3 – 13 (Institute of Hydromechanics, National Academy of Sciences of Ukraine).
4. *Скучик Е.* Простые и сложные колебательные системы. – М.: Мир, 1971. – 558 с.
5. *Карнаухов В. Г., Сенченков И. К., Гуменюк Б. П.* Термомеханическое поведение вязкоупругих тел при гармоническом нагружении. – Киев: Наук. думка, 1981. – 288 с.
6. *Pankratova N. D., Nikolaev B., Switoncski E.* Nonaxisymmetrical deformation of flexible rotational shell in classical and improved statements // J. Eng. Mech. – 1996. – **3**, N 2. – P. 89 – 96.
7. *Григоренко Я. М., Беспалова Е. И., Китайгородский А. Б., Шинкарь А. И.* Свободные колебания элементов оболочечных конструкций. – Киев: Наук. думка, 1981. – 172 с.

Received 02. 10. 99