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RESELLING OF EUROPEAN OPTION IF THE IMPLIED VOLATILITY VARIES AS COX-INGERSOLL-ROSS PROCESS

On Black and Scholes market Investor buys a European call option. At each moment of time till the maturity he is allowed to resell the option for the quoted market price. In Kukush et al. (2006) *On reselling of European option*, Theory Stoch. Process., 12(28), 75-87, a similar problem was investigated for another model of the market price. We propose a more realistic model based on Cox-Ingersoll-Ross process. Discrete approximation for this model is investigated, which is arbitrage–free. For this discrete model, a formula for penultimate optimal stopping domains is derived.

1. Introduction

In this paper we consider the European call option. For this type of option Investor is not entitled to exercise the option before the time T and should wait until the maturity. However it is known that on real financial markets he has an opportunity to resell the option before the maturity. Thus we investigate the reselling problem.

In this paper we treat the following model. On the Black-Scholes security market with an interest rate r, at the moment $t_0 = 0$ Investor buys the European call option with the strike price K and the maturity T on the stock with initial value S_0, at the price C_0 computed by the Black-Scholes formula. At any moment $t \in (0, T)$ he can resell the option for a certain market price C_t^m, which may differ from the "fair" price C_t.

The paper is organized as follows. In Section 2 we propose a model for the market price in terms of implied volatility, where the latter follows Cox-Ingersoll-Ross process. In Section 3 we describe optimal stopping domains in terms of implied volatility. Sections 4-6 focuse on discrete approximation

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of the proposed model and some properties of it are derived. The formula for penultimate stopping domain is derived in Section 7. Section 8 contains the main result that the proposed discrete model is arbitrage–free, and Section 9 concludes.

2. Model for option market price

Consider the classical Black and Scholes market in continuous time [5]:

$$\begin{cases} S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, & t \geq 0, \\ B_t = B_0 e^{rt}, & t \geq 0. \end{cases} \tag{1}$$

Here μ, σ, and r are positive parameters, S_0 and B_0 are positive and nonrandom, W_t is Wiener process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Consider a European call option with maturity T and pay-off function $g(S_T) = (S_T - K)_+ = \max\{S_T - K, 0\}$. We suppose that the Investor buys the option at fair price

$$C_0 = E^*_{S_0} e^{-rT} g(S_T) =: f(S_0, T; \sigma, r), \tag{2}$$

Here E^* is expectation w.r.t. the martingale measure P^*, and $E^*_{S_0}$ denotes the expectation (w.r.t. P^*) provided S_0 is the value of the stock price at $t = 0$. It is well known that under P^*, $\mu = r$ holds.

Now, suppose that the Investor can resell the option at any moment $t \in [0, T]$ for a certain market price C^m_t. Naturally, we assume that

$$C^m_0 = C_0, \qquad C^m_T = g(S_T). \tag{3}$$

The problem of the optimal reselling of the option is an optimization problem:

$$\Psi(\tau) := E e^{-r\tau} C^m_\tau \to \max \tag{4}$$

in the class of all (Markov) stopping times $\tau \in [0, T]$. The maximizing time is called an optimal reselling time and we denote it by τ_{opt}.

The "fair" market price at moment $t \in [0, T]$ equals

$$C_t = f(S_t, T - t; \sigma, r) := E^*[e^{-r(T-t)} g(S_T) | S_t]. \tag{5}$$

Corollary 2.3 from [2] states the following:

Corollary 1. *If an option price coincides with the Black-Scholes price, then:*

a) $\tau_{opt} = 0$ *if* $\mu < r$,

b) $\tau_{opt} = T$ *if* $\mu > r$,

c) any stopping time is optimal if $\mu = r$.

If $C_t^m = C_t$ for all $t \in [0, T]$ then Corollary 1 holds true, and the problem (4) has no practical sense. Therefore we choose a stochastic model for the market price. At any moment $t \in [0, T]$ an implied volatility σ_t is defined as a solution to the equation

$$f(S_t, T - t; \sigma_t, r) = C_t^m, \qquad \sigma_t > 0. \tag{6}$$

Under natural assumptions, see [2], the equation (6) has unique solution. Note that $C_0^m = C_0$ implies $\sigma_0 = \sigma$.

We model σ_t as a stochastic volatility. The model for σ_t as geometric Brownian motion presented in [2] is not appropriate in practical sense, because of σ_t can considerably deviate from σ, as t grows. Instead we propose the model based on Cox-Ingersoll-Ross process, see [1]:

$$d\sigma_t^2 = -\alpha(\sigma_t^2 - \sigma^2)dt + \beta\sqrt{\sigma_t^2}dW_t', \qquad \sigma_0 = \sigma, \tag{7}$$

where $\alpha, \beta > 0$ and $\beta^2 \leq 2\alpha\sigma^2$, W_t' is a Wiener process on $(\Omega, \mathcal{F}, (\widetilde{\mathcal{F}_t})_{t\geq 0}, P)$, with a new filtration $(\widetilde{\mathcal{F}_t})_{t\geq 0}$. Under imposed restrictions on α, β, and σ, the process (7) is well defined for all $t \geq 0$ [1]. Also we assume the following:

$$\begin{gathered}\text{Wiener processes } W_t \text{ and } W_t' \\ \text{are jointly Gaussian and positively correlated.}\end{gathered} \tag{8}$$

This condition can be understood as follows. If S_t grows, then so does σ_t, which makes C_t^m go beyond the "fair" price $C_t = f(S_t, T - t; \sigma, r)$. This corresponds to Investor's aim to hold an option if the stock price is growing. On the other hand, when the stock price drops, the Investor is willing to get rid of an option, which makes C_t^m go below its "fair" price.

Let $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}_t}$. In the reselling model (1), (6), and (7), the optimal stopping time τ_{opt} is defined as a maximum point of $\Psi(\tau)$ in a class of all stopping times w.r.t. the filtration $\mathcal{G}_t$.

3. Stopping sets

Similarly to Section 5.1 from [2] we have

$$\tau_{opt} = \inf\{t \in [0, T] : (S_t, C_t^m) \in G_t\}, \tag{9}$$

where nonrandom stopping sets are given by

$$G_t = \{(s, c) | s \geq 0, c = f_t(s, c)\}, \tag{10}$$

the function $f_t(s, c)$ is a reward function,

$$f_t(s, c) := \sup_{\tau\in[t,T]} E[e^{-r(\tau-t)}C_\tau^m | S_t = s, C_t^m = c], \tag{11}$$

and the upper bound is taken over all $\mathcal{G}_t$ - stopping times valued in $[t, T]$. Since f_t is jointly continuous, G_t is a closed subset of $[0,\infty) \times [0,\infty)$. By definition, $f_t(0, c) = 0$.

It is helpful to rewrite (9)-(11) in terms of S_t and σ_t:

$$\tau_{opt} = \inf\{t \in [0,T] : (S_t, \sigma_t) \in H_t\}, \tag{12}$$

$$H_t := \{(s,d) | s \geq 0, d = h_t(s,d)\}, \tag{13}$$

$$h_t(s,d) := \sup_{\tau \in [t,T]} E[e^{-r(\tau - t)} C^m_\tau | S_t = s, \sigma_t = d]. \tag{14}$$

Relations (9)-(11), as well as (12)-(14), are based on the next observation. The problem (4) is a problem of optimal realization of an American type option on two correlated assets S_t and C^m_t with pay-off function

$$g(S_t, C^m_t) = C^m_t. \tag{15}$$

4. Discrete approximation

In order to construct $\varepsilon-$optimal strategies of Investor, see [3], we deal with discrete approximations. Divide $[0,T]$ into n parts, and let $\Delta = T/n$.

We approximate the model (1) by the discrete model, which is a famous approximation of the Black-Scholes market by the Cox-Ross-Rubinstein market:

$$\begin{cases} S_n(t_{j+1}) = S_n(t_j) e^{\sigma\sqrt{\Delta}\delta_j}, \\ B_{t_j} = B_{t_{j-1}} e^{r\Delta}, \quad j = \overline{1,n}, \quad t_j = \frac{j}{n}T, \quad \text{where} \\ \delta_j = \delta_{nj} = \pm 1 \quad \text{- i.i.d. with distribution:} \\ P(\delta_j = 1) = \frac{e^{\mu\Delta} - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}} =: p_n, \\ P(\delta_j = -1) = 1 - p_n =: q_n, \end{cases} \tag{16}$$

Consider the reselling problem for the binomial market $(S(t_j), B_{t_j}, j = \overline{0,n})$. The fair value of the European option with the pay-off from Section 2 equals [4]:

$$C_{0n} = E^*_{S_0} e^{-rT} g(S_T) =: f_n(S_0, T; \sigma, r).$$

Here E^* corresponds to the martingale measure P^*_n, for which instead of (16) we have

$$P^*(\delta_j = 1) = \frac{e^{r\Delta} - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}} =: p^*_n.$$

The fair value at the moment $t = \frac{k}{n}T$ equals

$$C_{tn} = E^*[e^{-r(T-t)} g(S_T) | S_t] =: f_n(S_t, T-t; \sigma, r).$$

Introduce a market option price C^m_{tn} with $C^m_{0n} = C_{0n}$, $C^m_{Tn} = C_{Tn} = g(S_T)$.

For $t \in \{\frac{k}{n}T, k = \overline{0, n-1}\}$, the implied volatility σ_{nt} is defined as a solution to the equation

$$C_{tn}^{m} = f_n(S_t, T-t; \sigma_{nt}, r). \tag{17}$$

We model the implied volatility as a stochastic volatility in such a way, that the process $\sigma_n(t)$, which is a linear interpolant of $\{\sigma_{nt}, t = 0, \frac{T}{n}, \dots, T\}$, converges to $\{\sigma_t, t \in [0, T]\}$ from (7) in the sense of the first two moments. The $\sigma_n(t)$ is presented in more detail in Section 5.

5. Approximation of implied volatility

In this section we derive the discrete approximation of the implied volatility described by model (7).

Return back to (7). Let

$$z_t = \sigma_t^2, \quad t \geq 0; \qquad z_0 = \sigma^2. \tag{18}$$

Then by (7) we have

$$dz_t = -\alpha(z_t - \sigma^2)dt + \beta\sqrt{z_t}dW_t^{'}, \qquad z_0 = \sigma^2. \tag{19}$$

The stochastic differential equation (19) describes the Cox-Ingersoll-Ross process. Its first two moments and covariance function are as follows, see [6]:

$$Ez_t = \sigma^2, \tag{20}$$

$$Var z_t = \frac{\sigma^2\beta^2}{2\alpha}(1 - e^{-2\alpha t}), \tag{21}$$

$$Cov(z_t, z_s) = \frac{\sigma^2\beta^2}{2\alpha} \cdot e^{-\alpha(s+t)}(e^{2\alpha(s\wedge t)} - 1), \tag{22}$$

where $s \wedge t := \min(s, t)$.

Based on (19) we propose an approximation scheme for z_t. From (19) we have

$$z_{t+h} - z_t = -\alpha \int_t^{t+h} (z_t - \sigma^2)dt + \beta \int_t^{t+h} \sqrt{z_t}dW_t^{'}.$$

Then,

$$z_{t+h} - z_t \approx -\alpha(z_t - \sigma^2)h + \beta\sqrt{hz_t}\gamma_{th}, \qquad \gamma_{th} \sim N(0,1). \tag{23}$$

We need an approximation with Bernoulli variables. Therefore instead of γ_{th} we use ε_{th} which equals ± 1 with equal probabilities. Then,

$$z_{t+h} \approx z_t(1 - \alpha h) + \alpha h\sigma^2 + \beta\sqrt{hz_t}\varepsilon_{th}. \tag{24}$$

Now we use the relation (24) for a uniform partition of $[0, T]$ with step $\Delta = \frac{T}{n}$. We write $z_j = z_n(\frac{j}{n}T)$, $j = \overline{0, n}$. Here is the approximation scheme:

$$z_{j+1} = z_j(1 - \alpha\Delta) + \alpha\Delta\sigma^2 + \beta\sqrt{z_j\Delta}\varepsilon_{nj}, \tag{25}$$

where $\varepsilon_{nj} = \pm 1$ with equal probabilities, and $\{\varepsilon_{nj}, j = \overline{0, n}\}$ is i.i.d. sequence. Then we set $z_n(t) = z_j$, $t \in [\frac{j}{n}T, \frac{j+1}{n}T)$, $j = \overline{0, n-1}$; $z_n(T) = z_n$.

Next we find the upper and lower bounds for all z_j, $j = \overline{0, n}$.

Lemma 1. *For $\beta^2 < 2\alpha\sigma^2$ and $\Delta \leq \frac{4\alpha\sigma^2 - 2\beta^2}{\alpha(8\alpha\sigma^2 - \beta^2)} < \frac{1}{\alpha}$ we have the lower bound for z_j:*

$$z_j > Z_{(L)} := \left(\frac{-\beta\sqrt{\Delta} + \sqrt{\beta^2\Delta + 4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta} \right)^2, \text{ for all } j = \overline{0, n}. \tag{26}$$

Proof. We prove by induction. For all $\Delta > 0$, the base of induction $z_0 > Z_{(L)}$ holds true, indeed

$$z_0 = \sigma^2 = \left(\frac{-\beta\sqrt{\Delta} + \beta\sqrt{\Delta} + \sqrt{4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta} \right)^2 > Z_{(L)}.$$

For fixed $j \leq n - 1$ we assume that $z_j > Z_{(L)}$ and want to prove that $z_{j+1} > Z_{(L)}$. For $\Delta > 0$ we have that $Z_{(L)}$ from (26) satisfies

$$Z_{(L)}(1 - \alpha\Delta) + \alpha\Delta\sigma^2 - \beta\sqrt{Z_{(L)}\Delta} = Z_{(L)}. \tag{27}$$

Now we will prove the next inequality for $\beta^2 < 2\alpha\sigma^2$ and $\Delta \leq \frac{4\alpha\sigma^2 - 2\beta^2}{\alpha(8\alpha\sigma^2 - \beta^2)}$:

$$z_j(1 - \alpha\Delta) + \alpha\Delta\sigma^2 - \beta\sqrt{\Delta z_j} > Z_{(L)}(1 - \alpha\Delta) + \alpha\Delta\sigma^2 - \beta\sqrt{\Delta Z_{(L)}}. \tag{28}$$

We can rewrite (28) as follows:

$$(z_j - Z_{(L)})(1 - \alpha\Delta) > \beta\sqrt{\Delta}(\sqrt{z_j} - \sqrt{Z_{(L)}}).$$

Since $z_j > Z_{(L)}$ and $\Delta < \frac{1}{\alpha}$ we have

$$\sqrt{z_j} + \sqrt{Z_{(L)}} > \frac{\beta\sqrt{\Delta}}{(1 - \alpha\Delta)}, \qquad \sqrt{z_j} + \sqrt{Z_{(L)}} > 2\sqrt{Z_{(L)}},$$

and we can prove that:

$$\begin{gathered}
\sqrt{Z_{(L)}} = \tfrac{-\beta\sqrt{\Delta} + \sqrt{\beta^2\Delta + 4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta} \geq \tfrac{\beta\sqrt{\Delta}}{2(1-\alpha\Delta)} \Leftrightarrow \\
\Leftrightarrow (1 - \alpha\Delta)\sqrt{\beta^2\Delta + 4\alpha^2\sigma^2\Delta^2} \geq \beta\sqrt{\Delta} \Leftrightarrow \\
\Leftrightarrow 2\alpha(2\alpha\sigma^2 - \beta^2) + \alpha\Delta(\beta^2 - 8\alpha\sigma^2) + 4\alpha^4\sigma^2\Delta^2 \geq 0 \Leftarrow \\
\Leftarrow 2\alpha(2\alpha\sigma^2 - \beta^2) + \alpha\Delta(\beta^2 - 8\alpha\sigma^2) \geq 0 \Leftrightarrow 0 < \Delta \leq \tfrac{4\alpha\sigma^2 - 2\beta^2}{\alpha(8\alpha\sigma^2 - \beta^2)}.
\end{gathered}$$

Then by (27) and (28) we have for $\beta^2 < 2\alpha\sigma^2$ and $\Delta < \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$ next relations:

$$\begin{aligned} z_{j+1} &\geq z_j(1-\alpha\Delta)+\alpha\Delta\sigma^2-\beta\sqrt{\Delta z_j} > \\ &> Z_{(L)}(1-\alpha\Delta)+\alpha\Delta\sigma^2-\beta\sqrt{\Delta Z_{(L)}} = Z_{(L)}. \end{aligned}$$

Lemma 1 is proven.

Note that from Lemma 1, for $\beta^2 < 2\alpha\sigma^2$ and $\Delta \leq \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$, it follows immediately that $z_j > 0$ for all $j=\overline{0,n}$, and relation (25) is well defined. But it is easy to show that for the positivity of z_j, the condition $\beta^2 < 2\alpha\sigma^2$ can be subsided.

Lemma 2. *For $\Delta < \frac{1}{\alpha}$ we have the upper bound for z_j:*

$$z_j < Z_{(U)} := \left(\frac{\beta\sqrt{\Delta}+\sqrt{\beta^2\Delta+4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta}\right)^2, \text{ for all } j=\overline{0,n}. \tag{29}$$

Proof. We prove by induction. For all $\Delta > 0$, the base of induction $z_0 < Z_{(U)}$ holds true, indeed

$$Z_{(U)} = \left(\frac{\beta\sqrt{\Delta}+\sqrt{\beta^2\Delta+4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta}\right)^2 > \sigma^2 = z_0.$$

For fixed $j \leq n-1$ we assume that $z_j < Z_{(U)}$ and want to prove that $z_{j+1} < Z_{(U)}$. For $\Delta > 0$ we have that $Z_{(U)}$ from (29) satisfies

$$Z_{(U)}(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sqrt{\Delta Z_{(U)}} = Z_{(U)}. \tag{30}$$

Now, we have the next inequalities for $\Delta < \frac{1}{\alpha}$:

$$\begin{aligned} z_{j+1} &\leq z_j(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sqrt{\Delta z_j} < \\ &< Z_{(U)}(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sqrt{\Delta Z_{(U)}}. \end{aligned} \tag{31}$$

Then (30) and (31) imply

$$\begin{aligned} z_{j+1} &\leq z_j(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sqrt{\Delta z_j} < \\ &< Z_{(U)}(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sqrt{\Delta Z_{(U)}} = Z_{(U)}. \end{aligned}$$

Lemma 2 is proven.

In the next two lemmas we prove the convergence

$$z_n(t) \to z_t, \text{ as } n \to \infty \tag{32}$$

in the sense of convergence for the first two moments of finite-dimensional distributions. Denote $u_j = z_j - \sigma^2$. From (25) we get

$$u_{j+1} = u_j(1-\alpha\Delta) + \beta\sqrt{\Delta(u_j+\sigma^2)}\varepsilon_{nj}, \qquad u_0 = 0. \tag{33}$$

Lemma 3. *Convergence (32) holds true in the sense of the first moments, that is*

$$\lim_{n\to\infty} Ez_n(t) = Ez_t.$$

Proof. First we consider $t = T$. From (33) we have

$$Eu_{j+1} = Eu_j(1-\alpha\Delta), \qquad Eu_0 = 0,$$

therefore

$$Eu_j = 0, \qquad j = \overline{0,n}. \tag{34}$$

Then

$$\lim_{n\to\infty} Ez_n = \lim_{n\to\infty} E(u_n+\sigma^2) = \sigma^2 = Ez_T.$$

Next, verify (32) for all $t \in (0,T)$. We have $z_n(t) = u_{\lambda n} + \sigma^2$, $\lambda n = \left[\frac{t}{\Delta}\right]$; $\lambda \to \frac{t}{T}$, as $n \to \infty$. Then by (33) we have

$$Eu_{\lambda n} = 0, \tag{35}$$

and

$$\lim_{n\to\infty} Ez_{\lambda n} = \lim_{n\to\infty} E(u_{\lambda n}+\sigma^2) = \sigma^2 = Ez_t.$$

Lemma 3 is proven.

Lemma 4. *Convergence (32) holds true in the sense of the second moments, that is*

$$\lim_{n\to\infty} Cov(z_n(s), z_n(t)) = Cov(z_s, z_t).$$

Proof. Similarly to Lemma 1, first we consider $t = T$. From (33) we get

$$Var(u_{j+1}) = Var(u_j(1-\alpha\Delta)) + \beta^2 Var(\varepsilon_{nj}\cdot\sqrt{u_j+\sigma^2})\Delta.$$

We have

$$\begin{gathered} Var(\varepsilon_{nj}\sqrt{u_j+\sigma^2}) = Var\sqrt{u_j+\sigma^2}Var\varepsilon_{nj} + (E\sqrt{u_j+\sigma^2})^2 Var\varepsilon_{nj} + \\ +(E\varepsilon_{nj})^2 Var\sqrt{u_j+\sigma^2} = Var\sqrt{u_j+\sigma^2} + (E\sqrt{u_j+\sigma^2})^2 = \\ = E(u_j+\sigma^2) = \sigma^2, \end{gathered}$$

and

$$Var(u_{j+1}) = Var(u_j(1-\alpha\Delta)) + \beta^2\sigma^2\Delta, \qquad Var(u_0) = 0.$$

Then by induction we obtain the variance of u_j:

$$\begin{aligned} Var(u_j) &= \sigma^2\beta^2\Delta\sum_{i=0}^{j-1}(1-\alpha\Delta)^{2i} = \sigma^2\beta^2\frac{1-(1-\alpha\Delta)^{2j}}{2\alpha-\alpha^2\Delta}, j=\overline{1,n}; \\ Var(u_0) &= 0, \end{aligned} \tag{36}$$

moreover

$$Var(u_n) = \sigma^2\beta^2\frac{1-(1-\alpha\Delta)^{2n}}{2\alpha-\alpha^2\Delta}. \tag{37}$$

Now,

$$(1-\alpha\Delta)^{2n} = (1-\alpha\Delta)^{\frac{1}{\alpha\Delta}\cdot 2n\alpha\Delta} = (1-\alpha\Delta)^{\frac{1}{\alpha\Delta}\cdot 2\alpha T} \to e^{-2\alpha T}, \text{as } n\to\infty.$$

Then from (37) we have

$$\lim_{n\to\infty} Var(z_n) = \lim_{n\to\infty} Var(u_n) = \frac{\sigma^2\beta^2}{2\alpha}(1-e^{-2\alpha T}) = Var(z_T).$$

Next, similarly to Lemma 1, verify (32) for all $t\in(0,T)$. We have $z_n(t) = u_{\lambda n}+\sigma^2$, $\lambda n = \left[\frac{t}{\Delta}\right]$; $\lambda\to\frac{t}{T}$, as $n\to\infty$.

By (36) we get

$$Var(u_{\lambda n}) = \sigma^2\beta^2\frac{1-(1-\alpha\Delta)^{2\lambda n}}{2\alpha-\alpha^2\Delta}. \tag{38}$$

Next,

$$(1-\alpha\Delta)^{2\lambda n} = (1-\alpha\Delta)^{\frac{1}{\alpha\Delta}\cdot 2\alpha n\lambda\Delta} = (1-\alpha\Delta)^{\frac{1}{\alpha\Delta}\cdot 2\alpha T\lambda} \to e^{-2\alpha t}, \text{as } n\to\infty.$$

Then from (38) we have the desirable convergence of variances:

$$\lim_{n\to\infty} Var(z_{\lambda n}) = \lim_{n\to\infty} Var(u_{\lambda n}) = \frac{\sigma^2\beta^2}{2\alpha}(1-e^{-2\alpha t}) = Var(z_t).$$

Next we have to verify the convergence of the covariance functions. First we show that the covariance functions of u_j, u_i and z_j, z_i are equal. We assume $j>i$ (the case $j=i$ was considered earlier). We have

$$\begin{gathered} Cov(z_i,z_j) = Cov(u_i,u_j), \\ Cov(u_i,u_j) = E\left[u_i(u_{j-1}(1-\alpha\Delta)+\beta\sqrt{(u_{j-1}+\sigma^2)\Delta}\varepsilon_{nj-1})\right] = \\ = (1-\alpha\Delta)E(u_iu_{j-1}) + \beta\sqrt{\Delta}E(\sqrt{u_{j-1}+\sigma^2}\varepsilon_{nj-1}u_i) = \\ = (1-\alpha\Delta)Cov(u_i,u_{j-1}) + \beta\sqrt{\Delta}E\varepsilon_{nj-1}\cdot E(\sqrt{u_{j-1}+\sigma^2}u_i) = \\ = (1-\alpha\Delta)Cov(u_i,u_{j-1}). \end{gathered}$$

Then by induction for $j \geq i$, using (36) we have:

$$Cov(z_i, z_j) = (1-\alpha\Delta)^{j-i} Var(z_i) = \sigma^2\beta^2(1-\alpha\Delta)^{i+j}\frac{(1-\alpha\Delta)^{-2i}-1}{2\alpha-\alpha^2\Delta}. \quad (39)$$

Similarly to the convergence of variances, we have the convergence of covariance functions:

$$Cov(u_{\lambda_1 n}, u_n) \to Cov(u_t, u_T),$$

$$Cov(u_{\lambda_2 n}, u_{\lambda_1 n}) \to Cov(u_s, u_t), \quad \text{as } n \to \infty,$$

where for all s, t such that $0 < s < t < T$, $\lambda_1 n = [\frac{t}{\Delta}]$, $\lambda_2 n = [\frac{s}{\Delta}]$ and $\lambda_1 \to \frac{t}{T}$, $\lambda_2 \to \frac{s}{T}$, as $n \to \infty$.

Lemma 4 is proven.

6. Correlation

Now due to condition (8) we want to derive a correlation between ε_{nj} and δ_{nj}. Let

$$EW_tW_t^{'} = 2\rho_1 t, \quad 0 < \rho_1 < \frac{1}{2}. \quad (40)$$

Here $2\rho_1$ is the correlation coefficient between the two processes. Now, we introduce a correlation between ε_{nj} and δ_{nj} due to the table of joint probabilities:

$\delta_n \diagdown \varepsilon_n$	-1	1
-1	$(1-p_n) - \frac{1-\rho}{2}$	$\frac{1-\rho}{2}$
1	$p_n - \frac{\rho}{2}$	$\frac{\rho}{2}$

Then $E\varepsilon_{nj} = 0$, $E\delta_{nj} = p_n - (1-p_n) = 2p_n - 1$, and $E\delta_{nj} \to 0$, as $n \to \infty$. Next we find the covariance between ε_{nj} and δ_{nj}.

$$E(\varepsilon_{nj} \cdot \delta_{nj}) = \left(\frac{\rho}{2} + (1-p_n) - \frac{1-\rho}{2}\right) - \left(p_n - \frac{\rho}{2} + \frac{1-\rho}{2}\right) = 2\rho - 2p_n,$$

and we investigate its convergence:

$$Cov(\varepsilon_{nj}, \delta_{nj}) = E(\varepsilon_{nj} \cdot \delta_{nj}) - E(\varepsilon_{nj})E(\delta_{nj}) \to 2\rho - 1, \text{ as } n \to \infty.$$

Then we set $\rho = \rho_1 + \frac{1}{2}$ which corresponds to (40). Moreover from (40) we have that $\frac{1}{2} < \rho < 1$.

7. Structure of stopping domain in discrete time

For fixed Δ, we have two sequences

$$S_j = S_n(t_j), \qquad j = \overline{0,n}, \qquad n = \frac{T}{\Delta}, \tag{41}$$

$$\sigma_j = \sigma_n(t_j), \qquad j = \overline{0,n}, \qquad \sigma_0 = \sigma, \tag{42}$$

where σ is historical volatility, which describes the behavior of the stock price S_t in continuous time. In discrete time we have

$$\tau_{opt} = \tau_{opt,n} = \min\{t_j : (\sigma_j, s_j) \in \Gamma_j\}, \tag{43}$$

where $\Gamma_j \subset [0,\infty) \times [0,\infty)$.

Consider European call option with pay-off function from Section 2. If at the moment t_j the relation $e^{\sigma\sqrt{\Delta}(n-j)} \cdot S_j \le K$ holds true, then $S_n(T) \le K$, and the market option price equals zero at moments $t \ge t_j$. Therefore we have the next relation:

$$[0,\infty) \times [0, Ke^{-\sigma\sqrt{\Delta}(n-j)}] \subset \Gamma_j, \quad j \le n-1.$$

Due to Section 3 we can write the stopping sets as follows:

$$\begin{aligned} \Gamma_j \quad = \quad & \Big\{(v,s) \subset [0,\infty) \times [0,\infty) : C^m_{t_j}\Big|_{S_j=s,\sigma_j=v} \ge \\ & \ge \sup_{t_{j+1} \le \tau \le T} E\Big[e^{-r(\tau - t_j)} \cdot C^m_\tau \big| S_j = s, \sigma_j = v\Big]\Big\} \cup \\ & \cup \Big([0,\infty) \times [0, Ke^{-\sigma\sqrt{\Delta}(n-j)}]\Big), \quad j \le n-1; \\ & \Gamma_n = [0,\infty) \times [0,\infty). \end{aligned} \tag{44}$$

We can simplify this formula for the case $j = n-1$. Introduce a function

$$p^*_n(v) = \frac{e^{r\Delta} - e^{-v\sqrt{\Delta}}}{e^{v\sqrt{\Delta}} - e^{-v\sqrt{\Delta}}}. \tag{45}$$

First, we find the LHS of inequality in (44):

$$\begin{aligned} & C^m_{t_j}\Big|_{S_j=s,\sigma_j=v} = f_n(s, \Delta(n-j), v) = E^*_{\sigma_j=v}\Big[e^{-r\Delta(n-j)} g(S_n) \big| S_j = s\Big] = \\ & = e^{-r\Delta(n-j)} \sum_{l=0}^{n-j} \binom{n-j}{l} g(se^{v\sqrt{\Delta}l} e^{-v\sqrt{\Delta}(n-j-l)}) (p^*_n(v))^l (1 - p^*_n(v))^{n-j-l}, \end{aligned}$$

and for $j = n-1$ we get:

$$C^m_{t_{n-1}}\Big|_{S_{n-1}=s,\sigma_{n-1}=v} = e^{-r\Delta}\big[g(se^{v\sqrt{\Delta}}) p^*_n(v) + g(se^{-v\sqrt{\Delta}})(1 - p^*_n(v))\big]. \tag{46}$$

Then, we can rewrite the RHS of inequality in (44) for $j = n-1$:

$$E\Big[e^{-r\Delta}C_T^m\big|S_{n-1}=s,\sigma_{n-1}=v\Big] = e^{-r\Delta}E\Big[g(S_n)\big|S_{n-1}=s,\sigma_{n-1}=v\Big] = \\ = e^{-r\Delta}\big[g(se^{\sigma\sqrt{\Delta}})p_n + g(se^{-\sigma\sqrt{\Delta}})(1-p_n)\big], \tag{47}$$

where p_n is defined in (16).

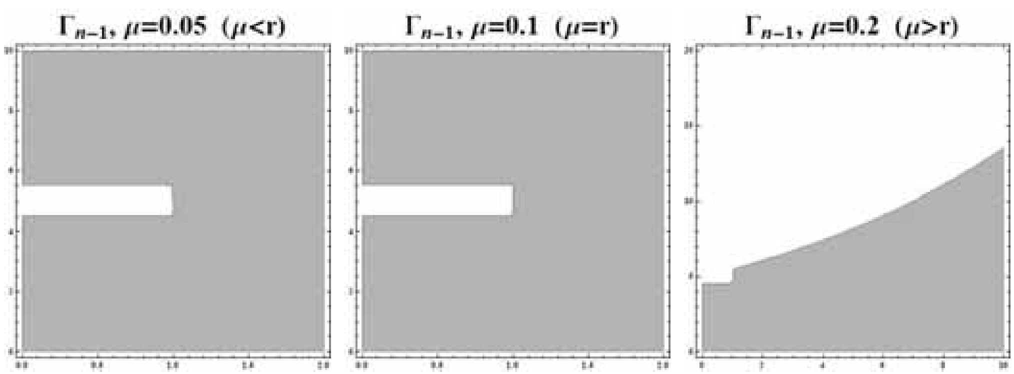
Then from relations (44), (46), and (47) we can write the formula for Γ_{n-1}:

$$\Gamma_{n-1} = \Big\{(v,s) \subset [0,\infty)\times[0,\infty) : g(se^{v\sqrt{\Delta}})p_n^*(v) + \\ +g(se^{-v\sqrt{\Delta}})(1-p_n^*(v)) \geq \ g(se^{\sigma\sqrt{\Delta}})p_n + g(se^{-\sigma\sqrt{\Delta}})(1-p_n)\Big\} \cup \\ \cup\Big([0,\infty)\times[0, Ke^{-\sigma\sqrt{\Delta}}]\Big). \tag{48}$$

Now, based on (48) we plot some stopping domains (all plots are in the coordinate plane (v,s)). Parameters of our model are the following:

$K = 5, n = 100, \alpha = 2, \sigma = 1, \beta = \sqrt{2}, \rho = 0.7, r = 0.1,$
and μ takes one of three values: 0.05, 0.1, or 0.2.

Stopping sets Γ_{n-1} for three relations between μ and r on the plane (v,s):



8. Absence of arbitrage

In this section we consider Markov stopping times with discrete values $\{t_0, t_1, \ldots, t_n\} = \{0, \Delta, \ldots, n\Delta\}$. We start with a standard definition of arbitrage.

Definition 1. *In a model (16), (17), (18), and (25) a stopping time τ provides an arbitrage possibility if:*

a) $P(e^{-r\tau}C_\tau^m \geq C_0) = 1,$

b) $P(e^{-r\tau}C_\tau^m > C_0) > 0$.

Lemma 5. *For* $\beta^2 < 2\alpha\sigma^2$ *and* $\Delta \le \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$ *we have the next inequality:*

$$\sigma_{j+1*}(\sigma_j) < \sigma_j < \sigma_j^*(\sigma_{j+1}), \text{ for all } j = \overline{0, n-1}, \tag{49}$$

where

$$\begin{aligned}\sigma_{j+1*}(\sigma_j) &:= \sqrt{\sigma_j^2(1-\alpha\Delta)+\alpha\Delta\sigma^2-\beta\sigma_j\sqrt{\Delta}},\\ \sigma_{j+1}^*(\sigma_j) &:= \sqrt{\sigma_j^2(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sigma_j\sqrt{\Delta}}.\end{aligned}$$

Proof. First we prove that $\sigma_{j+1*}(\sigma_j) < \sigma_j$. We have to verify that:

$$\sigma_j^2(1-\alpha\Delta)+\alpha\Delta\sigma^2-\beta\sigma_j\sqrt{\Delta} < \sigma_j^2 \Leftrightarrow \alpha\Delta\sigma_j^2+\beta\sigma_j\sqrt{\Delta}-\alpha\Delta\sigma^2 > 0.$$

It is easy to solve the latter inequality:

$$\sigma_j > \frac{-\beta\sqrt{\Delta}+\sqrt{\beta^2\Delta+4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta},$$

but this holds true for $\beta^2 < 2\alpha\sigma^2$ and $\Delta \le \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$ due to Lemma 1.

Next, we prove $\sigma_{j+1}^*(\sigma_j) > \sigma_j$. We must verify that:

$$\sigma_j^2(1-\alpha\Delta)+\alpha\Delta\sigma^2+\beta\sigma_j\sqrt{\Delta} > \sigma_j^2 \Leftrightarrow \alpha\Delta\sigma_j^2-\beta\sigma_j\sqrt{\Delta}-\alpha\Delta\sigma^2 < 0.$$

It is easy to solve the last inequality:

$$0 < \sigma_j < \frac{\beta\sqrt{\Delta}+\sqrt{\beta^2\Delta+4\alpha^2\sigma^2\Delta^2}}{2\alpha\Delta},$$

which holds true for $\Delta \le \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)} < \frac{1}{\alpha}$ due to Lemma 2. Lemma 5 is proven.

Theorem 1. *For* $\beta^2 < 2\alpha\sigma^2$ *and* $\Delta \le \frac{4\alpha\sigma^2-2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$ *there is no arbitrage possibility in the discrete model (16), (17), (18), and (25).*

Proof. We construct a martingale measure $P^{**} = P_n^{**}$ under which $e^{-rt}C_t^m$ is a martingale w.r.t. the filtration $\mathcal{G}_t$ (if such a measure exists then there is no arbitrage possibility), which means:

$$\begin{aligned} E^{**}\Big(e^{-r\Delta(j+1)}C_{\Delta(j+1)}^m\Big|\mathcal{G}_j\Big) &= e^{-r\Delta j}C_{\Delta j}^m \Leftrightarrow\\ \Leftrightarrow E^{**}\Big(e^{-r\Delta(j+1)}C_{\Delta(j+1)}^m\Big|S_j,\sigma_j\Big) &= e^{-r\Delta j}C_{\Delta j}^m \Leftrightarrow\\ \Leftrightarrow E^{**}\Big(f_n(S_{j+1},T-\Delta(j+1);\sigma_{j+1},r)\Big|S_j,\sigma_j\Big) &=\\ &= f_n(S_j,T-\Delta j;\sigma_j,r). \end{aligned} \tag{50}$$

Further in this section we write $f^j(S_j,\sigma_j) = f_n(S_j, T-\Delta j; \sigma_j, r)$.

We choose a measure P^{**} such that $\varepsilon_{nj}, j = 0,1,...,n$ have the same distribution as under P_n^*, and ε_{nj} and δ_{nj} are independent with

$$P^{**}(\varepsilon_{nj} = 1) = h(S_j), \quad P^{**}(\varepsilon_{nj} = -1) = 1 - h(S_j).$$

Then by (25) and (49) we have

$$E^{**}\Big(f^{j+1}(S_{j+1},\sigma_{j+1})\Big|S_j,\sigma_j\Big) =$$
$$= E^{**}\Big(f^{j+1}(S_{j+1},\sigma^*_{j+1}(\sigma_j))h(S_j) + f^{j+1}(S_{j+1},\sigma_{j+1*}(\sigma_j))(1-h(S_j))\Big|S_j,\sigma_j\Big) =$$
$$= f^j(S_j,\sigma^*_{j+1}(\sigma_j))h(S_j) + f^j(S_j,\sigma_{j+1*}(\sigma_j))(1-h(S_j)).$$

Now we can rewrite (50):

$$f^j(S_j,\sigma^*_{j+1}(\sigma_j))h(S_j) + f^j(S_j,\sigma_{j+1*}(\sigma_j))(1-h(S_j)) = f^j(S_j,\sigma_j).$$

And we can choose $h(S_j)$ as follows:

$$h(S_j) = \frac{f^j(S_j,\sigma_j) - f^j(S_j,\sigma_{j+1*}(\sigma_j))}{f^j(S_j,\sigma^*_{j+1}(\sigma_j)) - f^j(S_j,\sigma_{j+1*}(\sigma_j))}, \tag{51}$$

and $0 < h(S_j) < 1$ for $\beta^2 < 2\alpha\sigma^2$ and $\Delta \le \frac{4\alpha\sigma^2 - 2\beta^2}{\alpha(8\alpha\sigma^2-\beta^2)}$ by the strict monotonicity of f^j in the second argument and by Lemma 5.

Thus P^{**} is equivalent to P, and P^{**} is a martingale measure. Theorem 1 is proven.

9. Conclusion

We considered the reselling problem for European option and proposed a stochastic model for the market option price. For this model, we constructed the discrete approximation and proved the absence of arbitrage in it. Optimal strategy of the Investor in this discrete model is described by nonrandom stopping sets in the phase space of possible implied volatility and stock price. We derived the formula for penultimate stopping domain. The structure of this domain is illustrated by plots for some numerical values of parameters.

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