Theory of Stochastic Processes
Vol. 14 (30), no. 2, 2008, pp. 108–115

UDC 519.21

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NECESSARY CONDITION FOR SOME SINGULAR STOCHASTIC CONTROL SYSTEMS WITH VARIABLE DELAY

The purpose of this paper is to study conditions for the optimality of singular stochastic control systems with variable delay and constraint on the endpoint of state. The necessary condition of optimality for singular systems is obtained.

Introduction

The necessary conditions of optimality for systems which are described by stochastic differential equations with constant delay are obtained in [1,2] and with variable delay in [3,4]. In the present paper we consider optimal control problem for stochastic delay systems with constraint on the state variable. It is obtained new necessary condition of optimality for a singular systems [5,6].

Statement of problem

Let (Ω, F, P) be a complete probability space with the filtration $\{F^t : \ t_0 \le t \le t_1\}$ generated by Wiener process w_t and $\mathcal{F}^t = \overline{\sigma}(w_s;\ t_0 \le s \le t)$. $L_F^2(t_0, t_1, R^n)$ – space of predictable processes $x_t(\omega)$ such that:

$$E\int_{t_0}^{t_1} |x_t|^2 dt < +\infty.$$

Consider the following stochastic system with variable delay on state:

$$dx_t = g(x_t, x_{t-h(t)}, u_t, t)dt + f(x_t, t)dw_t, \quad t \in (t_0, t_1]; \tag{1}$$

$$x_{t_0} = x_0; \tag{2}$$

$$x_t = \Phi(t),\ t \in [t_0 - h(t_0), t_0); \tag{3}$$

$$u_t(\omega) \in U_\partial \equiv \{u(\cdot) \in L_F^2(t_0, t_1; R^m) \,|\, u(\omega) \in U \subset R^m, \text{a.c.}\}, \tag{4}$$

where U – nonempty bounded set, $\Phi(t)$ – piecewise continuous non-random function. $h(t) \ge 0$ is a continuously differentiable, non-random function such that $\frac{dh(t)}{dt} < 1$. Let it is required to minimize the functional in set of admissible controls:

$$J(u) = E\int_{t_0}^{t_1} l(x_t, u_t, t)dt \tag{5}$$

at condition

$$Eq(x_{t_1}) = 0. \tag{6}$$

Let assume that the following requirements are satisfied:

A1. Functions l, g, f and their derivatives are continuous in (x, u, t):

$$l(x, u, t) :\ R^n \times R^m \times [t_0, t_1] \to R^1;\ f(x, t) :\ R^n \times [t_0, t_1] \to R^{n\times n}$$

2000 *AMS Mathematics Subject Classification.* Primary 93E20.

Key words and phrases. Stochastic differential equations with variable delay, optimal control problem, stochastic control problem, necessary conditions of optimality, admissible controls, singular controls.

$$g(x, y, u, t): \ R^n \times R^n \times R^m \times [t_0, t_1] \to R^n;$$

A2. l, g, f functions are twice continuously differentiable with respect to (x, y) and hold the condition of linear growth:

$$(1 + |x| + |y|)^{-1}(|g(x, y, u, t)| + |g_x(x, y, u, t)| + |g_y(x, y, u, t)| \\ +|l(x, u, t)| + |l_x(x, u, t)| + |f(x, t)| + |f_x(x, t)|) \le N;$$

A3. Function $q(x): \ R^m \to R^k$ is twice continuously differentiable and

$$|q(x)| + |q_x(x)| \le N(1 + |x|); \quad |q_{xx}(x)| \le N.$$

Introduce following set:

$$B(x_t, u_t) = \{(k_t, u_t^*): \ dk_t = [g_x(x_t, x_{t-h(t)}, u_t, t)k_t + \\ + g_y(x_t, x_{t-h(t)}, u_t, t)k_{t-h(t)} + g(x_t, x_{t-h(t)}, u_t^*, t) \\ g(x_t, x_{t-h(t)}, u_t, t)]dt + [f_x(x_t, t)k_t]dw_t, \ t \in (t_0, t_1]; \\ Eq_x(x_{t_1})k_{t_1} = 0; \ k_t = 0, \ t \in [t_0 - h(t_0), t_0]\}$$

At the same time we will consider following additional problem:

$$\widetilde{J}(u) = E\int_0^{t_1} l(x_t, u_t, t)dt \to \min \tag{7}$$

(8)

$$\begin{cases} dx_t = g(x_t, x_{t-h(t)}, u_t, t)dt + f(x_t, t)dw_t; \ t \in (t_0, t_1] \\ x_{t_0} = x_0 \\ x_t = \Phi(t), \ t \in [t_0 - h(t_0), t_0) \\ dk_t = \{[g_x(x_t, x_{t-h(t)}, u_t, t)k_1 + g_y(x_t, x_{t-h(t)}, u_t, t)k_{t-h(t)} + g(x_t, x_{t-h(t)}, u_t^*, t) - \\ -g(x_t, x_{t-h(t)}, u_t, t)]dt + [f_x(x_t, t)k_t]dw_t, \ t \in (t_0, t_1]; \\ k_t = 0, \ t \in [t_0 - h(t_0), t_0]\} \\ Eq(x_{t_1}) = 0 \\ Eq_x(x_{t_1})k_{t_1} = 0 \\ u_t(\omega) \in U_\partial \equiv \{u(\cdot) \in L_F^2(t_0, t_1; \ R^m) \,|\, u(\omega) \in U \subset R^m, \ \text{a.c.}\}. \end{cases}$$

Note that

$$\min_{u \in U} J(u) = \min_{u \in U} \widetilde{J}(u)$$

Theorem. *Let conditions A1-A3 hold and (x_t^0, u_t^0) is a solution of the problem (1) – (6). Then exist random processes $(\psi_t^1, \beta_t^1) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$ and $(\psi_t^2, \beta_t^2) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$ which are the solutions of the following adjoint systems:*

$$\begin{cases} d\psi_t^1 = -\{\overline{H}_x(\psi_t^1, \psi_t^2, x_t^0, y_t^0, u_t^0, k_t, t) + \overline{H}_y(\psi_z^1, \psi_z^2, x_z^0, y_z^0, u_z^0, k_z, z)\big|_{z=s(t)} + \\ +[g_x^*(x_t^0, y_t^0, u_t^*, t) - g_x^*(x_t^0, y_t^0, u_t^0, t)]\psi_t^2 + \\ +[g_y^*(x_z^0, y_z^0, u_z^*, z) - g_y^*(x_z^0, y_z^0, u_z^0, z)]\psi_z^2\big|_{z=s(t)} s'(t) + \}dt + \\ +\beta_t^1 dw_t, \ t_0 \le t < t_1 - h(t_1), \\ d\psi_t^1 = -\{\overline{H}_x(\psi_t^1, \psi_t^2, x_t^0, y_t^0, u_t^0, k_t, t) + \\ +[g_x^*(x_t^0, y_t^0, u_t^*, t) - g_x^*(x_t^0, y_t^0, u_t^0, t)]\psi_t^2\}dt + \beta_t^1 dw_t, \ t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1}^1 = -\lambda_0 q_x(x_{t_1}^0) - \lambda_1 q_{xx}(x_{t_1}^0)k_{t_1}. \end{cases} \tag{9}$$

$$\begin{cases} d\psi_t^2 = -\{g_x^*(x_t^0, y_t^0, u_t^0, t)\psi_t^2 + f_x^*(x_t^0, t)\beta_t^2 + g_y^*(x_z^0, y_z^0, u_z^0, z)\psi_z^2\big|_{z=s(t)} s'(t)\}dt + \\ +\beta_t^2 dw_t, \ t_0 \le t < t_1 - h(t_1), \\ d\psi_t^2 = -[g_x^*(x_t^0, y_t^0, u_t^0, t)\psi_t^2 + f_x^*(x_t^0, t)\beta_t^2]dt + \beta_t^2 dw_t, \ t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1}^2 = -\lambda_0 q_x(x_{t_1}^0), \end{cases} \tag{10}$$

here $\lambda_0, \lambda_1 \in R^n$ *such that* $|\lambda_0|^2 + |\lambda_1|^2$, *then for any* $(k_1, u_t^*) \in B(x_t^0, u_t^0)$ *a.e.* $t \in [t_0, t_1)$ *fulfills the following:*

$$\overline{H}_x(\psi_t^1, \psi_t^2, x_t^0, y_t^0, u_t^*, k_t, t) - \overline{H}_y(\psi_t^1, \psi_t^2, x_t^0, y_t^0, u_t^0, k_t, t) \le 0, \ a.c., \tag{11}$$

were

$$\begin{gathered}\overline{H}(\psi_t^1, \psi_t^2, x_t, y_t, \nu, k_t, t) = \psi_t^{1*} g(x_t, y_t, \nu, t) + \beta_t^{1*} f(x_t, t) + \psi_t^{2*} g_x(x_t, y_t, \nu, t) k_t + \\ + \beta_t^{2*} f_x(x_t, t) k_t + \psi_t^{2*} g_y(x_t, y_t, \nu, t) k_{t-h(t)} - l(x_t, u_t, t).\end{gathered} \tag{12}$$

Proof. For any natural j let's introduce the following approximating functional:

$$J_j(u) = S_j\left(E\int_{t_0}^{t_1} l(x_t, u_t, t)dt, Eq(x_{t_1})\right) =$$

$$= \min_{c\in\mathcal{E}} \sqrt{|c - 1/j - E\int_{t_0}^{t_1} l(x_t, u_t, t)dt|^2 + |Eq(x_{t_1})|^2 + |Eq_x(x_{t_1})k_{t_1}|^2}$$

$\mathcal{E} = \{x : c \le J^0\}$, J_0 minimal value of the functional in (1) – (6). $V \equiv (U_\partial, d)$ – space of controls obtained by means of introducing of the following metric:

$$d(u, \nu) = (l \otimes P)\{(t, \omega) \in [t_0, t_1] \times \Omega : \ \nu_t \ne u_t\},$$

V – complete metric space [8].

We will used following result:

Lemma 1. *Let's assume, that conditions* **A1** – **A3** *hold,* u_t^n – *sequence of admissible controls from* V, $z_t^n = (x_t^n, k_t^n)$ – *sequence of corresponding trajectories of the system (8). If* $d(u_t^n, u_t) \to 0, \ n \to \infty$, *then* $\lim_{n\to\infty}\left\{\sup_{t_0\le t\le t_1} E|z_t^n - z_t|^2\right\} = 0$, *where* $z_t = (x_t, k_t)$ *is a trajectory corresponding to an admissible control* u_t.

Due to Lemma 1 and assumptions **A1** – **A3** implies continuity of the functional $J_j : V \to R^n$, then according to variation principle of Ekeland we have that it exists a control $u_t^j : \ d(u_t^j, u_t^0) \le \sqrt{\varepsilon_j}$ and $\forall \ u \in V$ it is fulfilled: $J_j(u^j) \le J_j(u) + \sqrt{\varepsilon_j} d(u^j, u), \ \varepsilon_j = \frac{1}{j}$.

This inequality means that (u_t^j, x_t^j, k_t^j) is a solution of the following problem:

$$\begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} E\int_{t_0}^{t_1} \delta(u_t, u_t^j)dt \to \min \\ dx_t = g(x_t, y_t, \nu_t, t)dt + f(x_t, t)dw_t, \quad t \in (t_0, t_1] \\ x_t = \Phi(t), \quad t \in [t_0 - h(t_0), t_0] \\ dk_t = [g_x(x_t, y_t, u_t, t)k_t + g_y(x_t, y_t, u_t, t)k_{t-h(t)} + g(x_t, y_t, u_t^*, t) - \\ -g(x_t, y_t, u_t, t)]dt + [f_x(x_t, t)k_t + f_y(x_t, t)k_{t-h(t)}]dw_t, \ t \in (t_0, t_1]; \\ k_t = 0, \quad t \in [t_0 - h(t_0), t_0]; \\ u_t \in U. \end{cases} \tag{13}$$

Function $\delta(u, \nu)$ is determined in the following way: $\delta(u, \nu) = \begin{cases} 0, \ u = \nu \\ 1, \ u \ne \nu. \end{cases}$

Let $\overline{u}_t^j = u_t^j + \Delta u_t^j$ – some admissible control, $\overline{x}_t^j = x_t^j + \Delta x_t^j$ and $\overline{k}_t^j = k_t^j + \Delta k_t^j$ corresponding solution of problem (13).

Let's use the following identities:

$$\begin{gathered} d\Delta x_t^j = d(\overline{x}_t^j - x_t^j) = \\ \{\Delta_{\overline{u}^j} g(x_t^j, y_t^j, u_t^j, t) + g_x(x_t^j, y_t^j, u_t^j, t)\Delta x_t^j + g_y(x_t^j, y_t^j, u_t^j, t)\Delta y_t^j\}dt + \\ + f_x(x_t^j, t)\Delta x_t^j dw_t + \eta_t^1, \ t \in (t_0, t_1], \end{gathered} \tag{14}$$

$$
\begin{aligned}
d\Delta k_t^j &= d(\overline{k_t^j} - k_t^j) = \{\Delta_{\overline{u}^j} g(x_t^j, y_t^j, u_t^j, t) + g_x(x_t^j, y_t^j, u_t^j, t)\Delta k_t^j + \\
&+ g_y(x_t^j, y_t^j, u_t^j, t)\Delta k_{t-h(t)}^j + g_{xx}(x_t^j, y_t^j, u_t^j, t)\Delta k_t^j \Delta x_t^j + \\
&g_{xy}(x_t^j, y_t^j, u_t^j, t)\Delta k_t^j \Delta y_t^j + g_{yx}(x_t^j, y_t^j, u_t^j, t)\Delta k_{t-h(t)}^j \Delta x_t^j + \\
&+ g_{yy}(x_t^j, y_t^j, u_t^j, t)\Delta k_{t-h(t)}^j \Delta y_t^j\}dt + \{f_x(x_t^j, t)\Delta k_t^j + \\
&+ f_{xx}(x_t^j, t)\Delta k_t^j \Delta x_t^j\}dw_t + \eta_t^2, \ t \in (t_0, t_1],
\end{aligned} \tag{15}
$$

where

$$
\Delta_{\overline{u}^j} g(x, y_t, u_t, t) = g(x, y_t, \overline{u}_t^i, t) - g(x, y_t, u_t^j, t);
$$

$$
\begin{aligned}
\eta_t^1 = \Big\{ &\int_0^1 [g_x^*(x_t^j + \mu\Delta x_t^j, \overline{y}_t^j, \overline{u}_t^j, t) - g_x^*(x_t^j, \overline{y}_t^j, \overline{u}_t^j, t)]\Delta x_t d\mu + \\
&+ \int_0^1 [g_y^*(x_t^j, y_t^j + \mu\Delta y_t^j, \overline{u}_t^j, t) - g_y^*(x_t^j, y_t^j, \overline{u}_t^j, t)]\Delta y_t^j d\mu \Big\} dt + \\
&+ \int_0^1 [f_x^*(x_t^j + \mu\Delta x_t^j, t) - f_x^*(x_t^j, t)]\Delta x_t^j d\mu dw_t;
\end{aligned}
$$

$$
\begin{aligned}
\eta_t^2 = \Big\{ \eta_t^1 &+ \int_0^1 [g_{xx}^*(x_t^j + \mu\Delta x_t^j, \overline{y}_t^j, \overline{u}_t^j, t) - g_{xx}^*(x_t^j, \overline{y}_t^j, \overline{u}_t^j, t)] k_t^j \Delta x_t^j d\mu + \\
&+ \int_0^1 [g_{xy}^*(x_t^j, y_t^j + \mu\Delta y_t^j, \overline{u}_t^j, t) - g_{xy}^*(x_t^j, y_t^j, \overline{u}_t^j, t)] k_t^j \Delta y_t^j d\mu \Big\} dt + \\
&+ \int_0^1 [g_{yx}^*(x_t^j + \mu\Delta x_t^j, \overline{y}_t^j, \overline{u}_t^j, t) - g_{yx}^*(x_t^j, \overline{y}_t^j, \overline{u}_t^j, t)] k_{t-h(t)}^j \Delta x_t^j d\mu + \\
&+ \int_0^1 [g_{yy}^*(x_t^j, y_t^j + \mu\Delta y_t^j, \overline{u}_t^j, t) - g_{yy}^*(x_t^j, y_t^j, \overline{u}_t^j, t)] k_{t-h(t)}^j \Delta y_t^j d\mu \Big\} dt + \\
&+ \int_0^1 [f_{xx}^*(x_t^j + \mu\Delta x_t^j, \overline{y}_t^j, t) - f_{xx}^*(x_t^j, \overline{y}_t^j, t)] k_t^j \Delta x_t^j d\mu dw_t.
\end{aligned}
$$

Assume that stochastic differential the vector $\psi_t = (\psi_t^1, \psi_t^2)$ has following form: $d\psi_t = (\alpha_1 dt + \beta_1 dw_t, \alpha_2 dt + \beta_2 dw_t)$. So that the state of system described by (8), then the vector of increment will be have stochastic differential of form: $d\Delta z_t^j = (d\Delta x_t^j, d\Delta k_t^j)$, where $d\Delta x_t^j$ and $d\Delta k_t^j$ satisfy to equation (14), (15) accordingly.

Then by Ito's formula we have:

$$
\begin{aligned}
d(\psi_t^{j*}\Delta z_t^j) &= d\psi_t^{1j*}\Delta x_t^j + \psi_t^{1j*} d\Delta x_t^j + d\psi_t^{2j*}\Delta k_t^j + \psi_t^{2*}\Delta k_t^j + \\
&+ \{\beta_t^{1j*} f_x(x_t^j, t)\Delta x_t + \beta_t^{2j*}[f_x(x_t^j, t)\Delta k_t^j - f_{xx}(x_t^j, t)\Delta k_t^j \Delta x_t^j]\}dt.
\end{aligned}
$$

Increment for functional (7) across admissible control defined following way

$$
\begin{aligned}
\Delta_{\overline{u}^j} J(u) &= J(\overline{u}^j) - J(u^j) = E\int_{t_0}^{t_1} [l(\overline{x}_t^j, \overline{u}_t^j, t) - l(x_t^j, u_t^j, t)]dt = \\
&= E\int_{t_0}^{t_1} [\Delta_{\overline{u}^j} l(x_t^j, u_t^j, t) + l_x(x_t^j, \overline{u}_t^j, t)\Delta x_t^j + l_{xx}(x_t^j, \overline{u}_t^j, t)\Delta k_t^j \Delta x_t^j]dt + \eta^2,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\eta^2 = E\int_{t_0}^{t_1} \Big\{ &\sqrt{\varepsilon_j}\delta(\overline{u}_t^j, u_t^j) + \int_0^1 [l_x^*(x_t^j + \mu\Delta x_t^j, \overline{u}_t^j, t) - l_x^*(x_t^j, u_t^j, t)]\Delta x_t^j d\mu + \\
&+ \int_0^1 [l_{xx}^*(x_t^j + \mu\Delta x_t^j, \overline{u}_t^j, t) - l_{xx}^*(x_t^j, u_t^j, t)]\Delta k_t^j \Delta x_t^j d\mu \Big\} dt.
\end{aligned}
$$

Let defined stochastic process ψ_t^j at the point t_1 following way:

$$
\psi_{t_1}^j = (\psi_{t_1}^{1j}, \psi_{t_1}^{2j}) = (-\lambda_0^j q_x(x_{t_1}^j) - \lambda_1^j q_{xx}(x_{t_1}^j) k_{t_1}, -\lambda_0^j q_x(x_{t_1}^j)).
$$

Assume that, the random processes $(\psi_t^{1j}, \beta_t^{1j}) \in L_F^2(t_0,t_1;R^n) \times L_F^2(t_0,t_1;R^{n\times n})$ and $(\psi_t^{2j}, \beta_t^{2j}) \in L_F^2(t_0,t_1;R^n) \times L_F^2(t_0,t_1;R^{n\times n})$ are almost certainly unique solutions of the following adjoin systems [7]:

$$\text{(17)}\quad \begin{cases} d\psi_t^{1j} = -\{\overline{H}_x(\psi_t^{1j}, \psi_t^{2j}, x_t^j, y_t^j, u_t^j, k_t^j, t)+ \\ +\{\overline{H}_y(\psi_z^{1j}, \psi_z^{2j}, x_z^j, y_z^j, u_z^j, k_z^j, z)\Big|_{z=s(t)} s'(t)+ \\ +[g_x^*(x_t^j, y_t^j, u_t^*, t) - g_x^*(x_t^j, y_t^j, u_t^j, t)]\psi_t^{2j} + [g_y^*(x_z^j, y_z^j, u_z^*, z)- \\ -g_y^*(x_z^j, y_z^j, u_z^j, z)]\psi_z^{2j}\Big|_{z=s(t)} s'(t)\}dt + \beta_t^{1j}dw_t,\ t_0 \le t < t_1 - h(t_1), \\ d\psi_t^{1j} = -\{\overline{H}_x(\psi_t^{1j}, \psi_t^{2j}, x_t^j, y_t^j, u_t^j, k_t^j, t) + [g_x^*(x_t^j, y_t^j, u_t^*, t)- \\ -g_x^*(x_t^j, y_t^j, u_t^j, t)]\psi_t^2\}dt + \beta_t^{1j}dw_t, \\ t_1 - h(t_1) \le t < t_0 \\ \psi_{t_1}^{1j} = -\lambda_0^j q_x(x_{t_1}^j) - \lambda_1^j q_{xx}(x_{t_1}^j)k_{t_1}. \end{cases}$$

$$\text{(18)}\quad \begin{cases} d\psi_t^{2j} = -\{g_x^*(x_t^j, y_t^j, u_t^j, t)\psi_t^{2j} + f_x^*(x_t^j, t)\beta_t^{2j} + g_y^*(x_z^j, y_z^j, u_z^j, z)\psi_z^{2j}\Big|_{z=s(t)} \times \\ s'(t)\}dt + \beta_t^{2j}dw_t,\ t_0 \le t < t_1 - h(t_1), \\ d\psi_t^{2j} = -[g_x^*(x_t^j, y_t^j, u_t^j, t)\psi_t^{2j} + f_x^*(x_t^j, t)\beta_t^{2j}]dt + \beta_t^{2j}dw_t,\ t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1}^{2j} = -\lambda_0^j q_x(x_{t_1}^j), \end{cases}$$

where meet the following requirement:

$$\text{(19)}\quad (\lambda_0^j, \lambda_1^j) = (-Eq(x_{t_1}^j)/J_j^0, -Eq_x(x_{t_1}^j)k_{t_1}^j/J_j^0)$$

and

$$J_j^0 = \sqrt{|c_j - 1/j - E\int_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt|^2 + |Eq(x_{t_1}^j)|^2}.$$

Since $\|(\lambda_0^j, \lambda_1^j)\| = 1$, then we can think that $(\lambda_0^j, \lambda_1^j) \to (\lambda_0, \lambda_1)$. It is known that S_j is a convex function which is Gato-differentiable at point: $\left(E\int\limits_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt, Eq(x_{t_1}^j)\right)$. Then for all $c \in \mathcal{E}$:

$$\left(\lambda_0^j, c - \frac{1}{j} - E\int\limits_{t_0}^{t_1} l(x_t^j, u_t^j, t)dt\right) + (\lambda_0^j, Eq(x_{t_1}^j)+)(\lambda_1^j, Eq_x(x_{t_1}^j)k_{t_1}^j) \le \frac{1}{j}.$$

Since

$$\text{(20)}\quad \psi_{t_1}^{1j} = -\lambda_0^j q_x(x_{t_1}^j) - \lambda_1^j q_{xx}(x_{t_1}^j),\ \text{then}\ \ \psi_{t_1}^{1j} \to \psi_{t_1}^1\ \text{in}\ L_F^2(t_0,t_1;R^n)$$

and

$$\text{(21)}\quad \psi_{t_1}^{2j} = -\lambda_0^j q_x(x_{t_1}^j),\ \text{then}\ \ \psi_{t_1}^{2j} \to \psi_{t_1}^2\ \text{in}\ L_F^2(t_0,t_1;R^n).$$

Due to (14) and (15) expression (16) may be writing following way:

$$\Delta J(u^j) = -E\int_{t_0}^{t_1} d\psi_t^{1j^*}\Delta x_t - \int_{t_0}^{t_1} d\psi_t^{2j^*}\Delta k_t +$$

$$\text{(22)}\quad +E\int_{t_0}^{t_1} [\Delta_{\overline{u}^j} l(x_t^j, u_t^j, t) + l_x(x_t^j, u_t^j, t)\Delta x_t^j + l_{xx}(x_t^j, u_t^j, t)\Delta k_t^j \Delta x_t^j]dt-$$

$$-E\int_{t_0}^{t_1}\beta_t^{1j^*}f_x(x_t^j,t)\Delta x_t^j dt-$$
$$-E\int_{t_0}^{t_1}\beta_t^{2j^*}[f_x(x_t^j,t)\Delta k_t^j+f_{xx}(x_t^j,t)\Delta k_t^j\Delta x_t^j]dt+\eta_{t_0,t_1},$$

where

$$\eta_{t_0,t_1}=\eta^2+E\int_{t_0}^{t_1}\left\{\int_0^1\psi_t^{1j^*}(g_x(x_t^j+\mu\Delta x_t^j,\overline{y}_t^j,u_t^j,t)-\right.$$
$$-g_x(x_t^j,\overline{y}_t^j,u_t^j,t))\Delta x_t^j d\mu+\int_0^1\psi_t^{1j^*}(g_y(x_t^j,y_t^j+\mu\Delta y_t^j,u_t^j,t)-$$
$$\left.-g_y(x_t^j,y_t^j,u_t^j,t))\Delta y_t^j d\mu\right\}dt+$$
$$+E\int_{t_0}^{t_1}\left\{\int_0^1\beta_t^{1j^*}(f_x(x_t^j+\mu\Delta x_t^j,t)-f_x(x_t^j,t))\Delta x_t^j d\mu\right\}dt+$$
$$+E\int_{t_0}^{t_1}\psi_t^{2j^*}[(g_x(x_t^j+\mu\Delta x_t^j,\overline{y}_t^j,u_t^j,t)-g_x(x_t^j,\overline{y}_t^j,u_t^j,t))\Delta k_t^j+$$
$$+(g_y(x_t^j,y_t^j+\mu\Delta y_t^j,u_t^j,t)-g_y(x_t^j,y_t^j,u_t^j,t))\Delta k_{t-h(t)}^j+$$
$$+(g_{xx}(x_t^j+\mu\Delta x_t^j,\overline{y}_t^j,u_t^j,t)-g_{xx}(x_t^j,\overline{y}_t^j,u_t^j,t))\Delta k_t^j\Delta x_t^j+$$
$$+(g_{xy}(x_t^j,y_t^j+\mu\Delta y_t^j,u_t^j,t)-g_{xx}(x_t^j,\overline{y}_t^j,u_t^j,t))\Delta k_t^j\Delta y_t^j+$$
$$+(g_{yx}(x_t^j+\mu\Delta x_t^j,\overline{y}_t^j,u_t^j,t)-g_{yx}(x_t^j,\overline{y}_t^j,u_t^j,t))\Delta k_{t-h(t)}^j\Delta x_t^j+$$
$$+(g_{yy}(x_t^j,y_t^j+\mu\Delta y_t^j,u_t^j,t)-g_{yy}(x_t^j,y_t^j,u_t^j,t))\Delta k_{t-h(t)}^j\Delta y_t^j]dt+$$
$$+E\int_{t_0}^{t_1}\beta_t^{2j^*}[(f_x(x_t^j+\mu\Delta x_t^j)-f_x(x_t^j,t))\Delta k_t^j+$$
$$+(f_{xx}(x_t^j+\mu\Delta x_t^j,t)-f_{xx}(x_t^j,t))\Delta k_t^j\Delta x_t^j]dt. \tag{23}$$

Helping simple transformation (22) has following form:

$$\Delta J(u^j)=-E\int_{t_0}^{t_1}[\psi_t^{1j^*}\Delta_{\overline{u}^j}g(x_t^j,x_{t-h(t)}^j,u_t^j,t)+\psi_t^{2j^*}\Delta_{\overline{u}^j}g_x(x_t^j,x_{t-h(t)}^j,u_t^j,t)k_t^j+$$
$$+\psi_t^{2j^*}\Delta_{\overline{u}^j}g_y(x_t^j,x_{t-h(t)}^j,u_t^j,t)k_t^j-\Delta_{\overline{u}^j}l(x_t^j,u^j)t,t)]dt+\eta_{t_0,t_1}. \tag{24}$$

Let

$$\Delta u_t^j=\Delta u_{t,\varepsilon}^{j\theta}=\begin{cases}0,\ t\overline{\in}[\theta,theta+\varepsilon),\ \varepsilon>0,\ \theta\in[t_0,t_1)\\ \nu^j-u_t^j,\ t\in[\theta,\theta+\varepsilon),\ \nu^j\in L^2(\Omega,F^\theta,P;R^m)\end{cases}$$

some special admissible control, $\varepsilon>0$ enough smaller number . Let $x_{t,\varepsilon}^{j\theta}$ – trajectory corresponding to control $u_{t,\varepsilon}^{j\theta}=u_t^j+\Delta u_{t,\varepsilon}^{j\theta}$.Then expression (24) may be represented following way:

$$\Delta J(u^j)=-E\int_{\theta}^{\theta+\varepsilon}[\psi_t^{1j^*}\Delta_{\nu^j}g(x_t^j,x_{t-h(t)}^j,u_t^j,t)+\psi_t^{2j^*}\Delta_{\nu^j}g_x(x_t^j,x_{t-h(t)}^j,u_t^j,t)k_t^j+$$
$$+\psi_t^{2j^*}\Delta_{\nu^j}g_y(x_t^j,x_{t-h(t)}^j,u_t^j,t)k_{t-h(t)}^j-\Delta_{\nu^j}l(x_t^j,u_t^j,t)]dt+\eta_{\theta,\theta+\varepsilon}.$$

We prove the following lemmas by scheme [4]:

Lemma 2. *Let's assume, that conditions* **A1** – **A3** *hold. If* $\varepsilon\to 0$, *then* $E\left|\frac{x_{t,\varepsilon}^{j\theta}-x_t^j}{\varepsilon}\right|^2\leq N$, *where* x_t^j *is a trajectory of system (9),* $x_{t,\varepsilon}^{j\theta}$ *– a trajectory corresponding to an admissible control* $u_{t,\varepsilon}^{j\theta}=u_t^j+\Delta u_{t,\varepsilon}^{j\theta}$.

Lemma 3. *Let's assume, that conditions* **A1** – **A3** *hold. If* $\varepsilon \to 0$, *then* $E|k^{j\theta}_{t,\varepsilon} - k^j_t|^2 \le N\varepsilon$, *where* k^j_t *is a trajectory of system (9),* $k^{j\theta}_{t,\varepsilon}$ *– a trajectory corresponding to an admissible control* $u^{j\theta}_{t,\varepsilon} = u^j_t + \Delta u^{j\theta}_{t,\varepsilon}$.

According to lemma 2 and lemma 3 we have: $E|z^{j\theta}_{t,\varepsilon} - z^j_t|^2 \le N\varepsilon$, and from expression (23) we get: $\eta_{\theta,\theta+\varepsilon} = o(\varepsilon)$. Then for (22) obtained:

$$\begin{aligned}\Delta_\theta J(u^j) = -E[\psi^{1j^*}\Delta_{\nu^j} g(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta) + \psi^{2^*}\Delta_{\nu^j} g_x(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta)k_\theta + \\ +\psi^{2^*}\Delta_{\nu^j} g_y(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta)k_{\theta-h(\theta)} - \Delta_{\nu^j} l(x^j_\theta, u^j_\theta, \theta)]\varepsilon + o(\varepsilon) \ge 0.\end{aligned} \tag{25}$$

Due to smallest of ε :

$$\begin{aligned}E[\psi^{1j^*}_\theta \Delta_{\nu'} g(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta) + \psi^{2^*}_\theta \Delta_{\nu'} g_x(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta)k_\theta + \\ +\psi^{2^*}_\theta \Delta_{\nu'} g_y(x^j_\theta, x^j_{\theta-h(\theta)}, u^j_\theta, \theta)k_{\theta-h(\theta)} - \Delta_{\nu^j} l(x^j_\theta, u^j_\theta, \theta)] \le 0,\end{aligned}$$

and arbitrarily of $\theta \in [t_0, t_1)$:

$$\overline{H}(\psi^{1j}_t, \psi^{2j}_t, x^j_t, y^j_t, u^*_t, k^j_t, t) - \overline{H}(\psi^{1j}_t, \psi^{2j}_t, x^j_t, y^j_t, u^j_t, k^j_t, t) \le 0 \text{ a.c.} \tag{26}$$

Lemma 4. *Let* $(\psi^{1j}_t, \psi^{2j}_t)$ *are solutions of systems (17), (18) and* (ψ^1_t, ψ^2_t) *are solution of systems (9), (10). Then*

$$E\int_{t_0}^{t_1} |\psi^{1j}_t - \psi^1_t|^2 dt + E\int_{t_0}^{t_1} |\beta^{1j}_t - \beta^1_t|^2 dt \to 0,$$

$$E\int_{t_0}^{t_1} |\psi^{2j}_t - \psi^2_t|^2 dt + E\int_{t_0}^{t_1} |\beta^{2j}_t - \beta^2_t|^2 dt \to 0, \text{ if } d(u^j_t, u_t) \to 0, \ j \to \infty.$$

Proof. According to Ito formula, for $s \in [t_1 - h(t), t_1]$,

$$\begin{aligned}E|\psi^{2j}_{t_1} - \psi^2_{t_1}|^2 - E|\psi^{2j}_s - \psi^2_s|^2 = 2E\int_s^{t_1}[\psi^{2j}_t - \psi^2_t][(g^*_x(x^j_t, y^j_t, u^j_t, t) - g^*_x(x^0_t, y^0_t, u^0_t, t))\psi^{2j}_t + \\ +g^*_x(x^0_t, y^0_t, u^0_t, t)(\psi^{2j}_t - \psi^2_t) + (f^*_x(x^j_t, t) - f^*_x(x^0_t, t))\beta^{2j}_t + f^*_x(x^0_t, t)\times \\ \times(\beta^{2j}_t - \beta^2_t)]dt + E\int_s^{t_1} |\beta^{2j}_t - \beta^2_t|^2 dt.\end{aligned}$$

Due to assumptions **A1** – **A2** and using simple transformations for arbitrary positive we obtain:

$$\begin{aligned}E\int_s^{t_1} |\beta^{2j}_t - \beta^2_t|^2 dt + E|\psi^{2j}_t - \psi^2_t|^2 \le \\ \le EN\int_s^{t_1} |\psi^{2j}_t - \psi^2_t|^2 dt + EN\alpha\int_s^{t_1} |\beta^{2j}_t - \beta^2_t|^2 dt + E|\psi^{2j}_{t_1} - \psi^2_{t_1}|^2.\end{aligned}$$

Hence, according to Gronwall inequality we have:

$$E|\psi^{2j}_s - \psi^2_s|^2 \le De^{N(t_1 - s)} \text{ a.e. in } [t_1 - h(t), t_1], \tag{27}$$

where constant D is determined in the following way: $D = E|\psi^{2j}_{t_1} - \psi^{2j}_{t_1}|^2 \to 0$.

According to Ito formula, for $s \in [t_0, t_1 - h(t_1))$,

$$\begin{aligned}E|\psi^{2j}_{t_1-h(t_1)} - \psi^2_{t_1-h(t_1)}|^2 - E|\psi^{2j}_s - \psi^2_s|^2 = 2E\int_s^{t_1-h(t_1)} (\psi^{2j}_t - \psi^2_t)[(g^*_x(x^j_t, y^j_t, u^j_t, t) + \\ +g^*_x(x^0_t, y^0_t, u^0_t, t))(\psi^{2j}_t - \psi^2_t) + (f^*_x(x^j_t) - (f^*_x(x^0_t))\beta^{2j}_t + f^*_x(x^0_t, t)\times \\ \times f^*_x(x^0_t, t))\beta^{2j}_t + f^*_x(x^0_t, t)(\beta^{2j}_t - \beta^2_t) + g^*_y(x^j_z, y^j_z, u^j_z, z) - \\ -g^*_y(x^0_z, y^0_z, u^0_z, z))\psi^{2j}_z s'(t) + g^*_y(x^0_z, y^0_z, u^0_z, z)(\psi^{2j}_z - \psi^2_z)s'(t) + E\int_s^{t_1-h(t_1)} |\beta^{2j}_t - \beta^2_t|^2 dt.\end{aligned}$$

In view of assumptions **A1** – **A2** and the expression (21) we obtain:

$$E\int_s^{t_1-h(t_1)}|\beta_t^{2j}-\beta_t^2|^2dt+E|\psi_s^{2j}-\psi_s^2|^2\leq$$

$$\leq EN\int_s^{t_1-h(t_1)}|\psi_t^{2j}-\psi_t^2|^2dt+EN\alpha\int_s^{t_1-h(t_1)}|\psi_z^{2j}-\psi_z^2|^2dt+$$

$$+EN\alpha\int_s^{t_1-h(t_1)}|\beta_t^{2j}-\beta_t^2|^2dt+EN\alpha\int_s^{t_1-h(t_1)}|\beta_z^{2j}-\beta_t^2|^2dt+E|\psi_{t_1-h(t_1)}^{2j}-\psi_{t_1-h(t_1)}^2|^2.$$

Hence using simple transformations we have:

$$E(1-2N\alpha)\int_s^{t_1-h(t_1)}|\beta_t^{2j}-\beta_t^2|^2dt+E|\psi_s^{2j}-\psi_s^2|^2\leq E(N+N\alpha)\int_s^{t_1-h(t_1)}|\psi_t^{2j}-\psi_t^2|^2dt+$$

$$+EN\alpha\int_{t_1-h(t_1)}^{t_1}|\psi_t^{2j}-\psi_t^2|^2dt+EN\alpha\int_{t_1-h(t_1)}^{t_1}|\beta_t^{2j}-\beta_t^2|^2dt+E|\psi_{t_1-h(t_1)}^{2j}-\psi_{t_1-h(t_1)}^2|^2.$$

According to Gronwall inequality:

$$E|\psi_s^{2j}-\psi_s^2|^2\leq D\exp[(N+N\alpha)(t_1-h(t_1))]\ \text{ a.e. in } [t_0,t_1-h(t_1)),$$

where

$$D=E|\psi_{t_1-h(t_1)}^{2j}-\psi_{t_1-h(t_1)}^2|^2+EN\alpha\int_{t_1-h(t_1)}^{t_1}|\psi_t^{2j}-\psi_t^2|^2dt+EN\alpha\int_{t_1-h(t_1)}^{t_1}|\beta_t^{2j}-\beta_t^2|^2dt.$$

Due to sufficient smallness of α and from inequality (27) we receive that $D\to 0$.

Thus,

$$\psi_t^{2j}\to\psi_t^2 \text{ in } L_F^2(t_0,t_1;R^n),\ \beta_t^{2j}\to\beta_t^2 \text{ in } L_F^2(t_0,t_1;R^{n\times n}. \tag{28}$$

Then, similarly way according to assumptions **A1** – **A2** and expression (28) using simple transformations we obtain: Thus,

$$\psi_t^{j1}\to\psi_t^1 \text{ in } L_F^2(t_0,t_1;R^n),\ \beta_t^{j1}\to\beta_t^1 \text{ in } L_F^2(t_0,t_1;R^{n\times n}. \tag{29}$$

Lemma 4 is proved.

It is follows from lemma 4 and assumptions **A1** – **A3** that we can proceed to the limit in systems (17),(18),(26) and get fulfillment of (9),(10) and (11). Theorem is proved.

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