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A LIMIT THEOREM FOR THE NUMBER OF SIGN CHANGES FOR A SEQUENCE OF ONE-DIMENSIONAL DIFFUSIONS

The (normalized) number of sign changes for a weakly convergent sequence of one-dimensional diffusion processes is considered. The limit theorem for this number is established.

1. Introduction

In this paper, we consider a sequence of one-dimensional diffusion processes satisfying SDE's

$$dX_n(t) = a_n(X_n(t))\,dt + \sigma_n(X_n(t))\,dW(t), \quad t \in \mathbb{R}^+, n \in \mathbb{N}, \tag{1}$$

weakly convergent to a diffusion process satisfying SDE

$$dX(t) = a(X(t))\,dt + \sigma(X(t))\,dW(t), \quad t \in \mathbb{R}^+. \tag{2}$$

We take some *discretization parameter* $\alpha > 0$ and consider the processes

$$\phi_n^t = n^{-\frac{\alpha}{2}} \sum_{0 \le k < tn^\alpha} \mathbb{1}_{X_n(kn^{-\alpha}) \cdot X_n((k+1)n^{-\alpha}) < 0}, \quad t \in \mathbb{R}^+, n \in \mathbb{N}. \tag{3}$$

We call ϕ_n^t *the (normalized) number of sign changes* for the diffusion X_n, corresponding to the time discretization $\{kn^{-\alpha}, k \in \mathbb{Z}_+\}$.

The question under discussion is whether the sequence $\{\phi_n^\cdot\}$ converges weakly and what is the structure of the limiting process. This question has a long history. In the 1950s, I.I.Gikhman proposed a general method for investigation of the limiting behavior of functionals of such a type, based on the asymptotic study of difference equations for the family of corresponding characteristic functions ([1],[2]). Later on, this method was developed further and used widely by M.I.Portenko and his pupils (see the discussion and overview in [3]). This method appears to be powerful enough to provide the limit theorems for $\{\phi_n^\cdot\}$ in quite delicate situations where σ_n converge to σ only in a weak L_∞-sense (i.e., where $\{X_n\}$ is a sequence of diffusions with oscillating coefficients), see the recent preprint [4].

We investigate the limiting behavior of $\{\phi_n^\cdot\}$ using another approach introduced in [5]. In [5], the general limiting theorem was proved, being in fact a generalization of the Dynkin's criterion for the L_2-convergence of W-functionals of a given Markov process in terms of their *characterictics* (i.e., expectations). In this paper, we show that this theorem can be applied in order to prove the limit theorem for $\{\phi_n^\cdot\}$ in a situation where $\{X_n\}$ is a sequence of diffusions with oscillating coefficients. Our results differ from those obtained in [4], since we do not suppose, in general, the coefficients of (1) to have the

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form $a_n(x) = n^\alpha \hat a(n^\alpha x), \sigma_n(x) = \hat\sigma(n^\alpha x)$. On the other hand, some assumptions of [4] are less restrictive than those made in the current paper. For instance, in [4], non-uniform partitions of the time axis are allowed.

2. The main result

We formulate the main statement for processes (1),(2) under supposition that $a_n \equiv 0, a \equiv 0$. This allows us to simplify notation but does not restrict generality, since one can reduce the general case to the one indicated before, using the following standard trick. If X_n, X are given by (1),(2) with non-trivial a_n, a, then the processes $\tilde X_n(t) = S_n(X_n(t)), \tilde X(t) = S(X(t))$ with

$$S_n(x) = \int_0^x e^{-\int\limits_0^y \frac{2a_n(z)}{\sigma_n^2(z)}\,dz}\,dy, \quad S(x) = \int_0^x e^{-\int\limits_0^y \frac{2a(z)}{\sigma^2(z)}\,dz}\,dy, \quad x\in\mathbb{R}$$

satisfy SDE's with the coefficients $\tilde a_n \equiv 0, \tilde a \equiv 0$ and

$$\tilde\sigma_n(x) = S_n'(S_n^{-1}(x))\sigma_n(S^{-1}(x)), \quad \tilde\sigma(x) = S'(S^{-1}(x))\sigma(S^{-1}(x)), \quad x\in\mathbb{R},$$

respectively. Since the mappings $x \mapsto S_n(x)$ preserve the sign, the functionals ϕ_n given by (3) coincide for the processes X_n and $\tilde X_n$.

Together with the processes X_n, X, we consider the re-scaled processes

$$Z_n(t) = n^{\frac{\alpha}{2}} X_n(tn^{-\alpha}), \quad Z^n(t) = n^{\frac{\alpha}{2}} X(tn^{-\alpha}), \quad t\in\mathbb{R}^+, \quad n\in\mathbb{N}.$$

One can easily see that if

$$X_n(t) = X_n(0) + \int_0^t \sigma_n(X_n(s))\,dW(s), \quad X(t) = X(0) + \int_0^t \sigma(X(s))\,dW(s), \quad t\in\mathbb{R}^+,$$

then

$$Z_n(t) = Z_n(0) + \int_0^t \varrho_n(Z_n(s))\,dW_n(s), \quad Z^n(t) = Z^n(0) + \int_0^t \varrho^n(Z^n(s))\,dW_n(s),$$
$$t\in\mathbb{R}^+,$$

with $\varrho_n(z) = \sigma_n(n^{-\frac{\alpha}{2}}z), \varrho^n(z) = \sigma(n^{-\frac{\alpha}{2}}z), W_n(t) = n^{\frac{\alpha}{2}}W(tn^{-\alpha})$.

Denote, by Σ, the class of measurable functions $b:\mathbb{R}\to\mathbb{R}$ that are globally bounded and separated from 0 on every finite interval. If σ_n, σ belong to Σ, then Eqs. (1),(2) (with $a_n \equiv 0, a\equiv 0$) uniquely define Feller Markov processes (see [7], Chapter 6 §3). For $R\geq 1$, denote, by Σ_R, the class of functions $b\in\Sigma$ such that $R^{-1}\leq b^2(x)\leq R, x\in\mathbb{R}$.

For the process X, its *local time* at the point x is defined via the Tanaka formula:

$$L_X(t,x) = |X(t)-x| - |X(0)-x| - \int_0^t \operatorname{sign}(X(s)-x)\,dX(s), \quad t\in\mathbb{R}^+.$$

Denote $K_t(x,y) = \int_0^t \frac{1}{\sqrt{2\pi s}} e^{-\frac{(y-x)^2}{2s}}\,ds, t\in\mathbb{R}^+, x,y\in\mathbb{R}$. Define the weak L_∞-convergence of a sequence $\{f_n\}\subset L_\infty(\mathbb{R})$ by the relation

$$f_n \xrightarrow{w} f \overset{df}{\Leftrightarrow} \int_{\mathbb{R}} f_n(y)g(y)\,dy \to \int_{\mathbb{R}} f(y)g(y)\,dy, \quad g\in L_1(\mathbb{R}).$$

The main statement of the paper is given in the following theorem.

Theorem 1. *Suppose that the following conditions hold true:*

(A) $\sigma \in \Sigma_R, \sigma_n \in \Sigma_R$ *for some* $R > 0$,

(B) Equation (2) possesses the path-wise uniqueness property,

(C) For every $T > 0$,

$$\sup_{x\in\mathbb{R}, t\leq T} \left| \int_{\mathbb{R}} [\sigma_n^{-2}(y) - \sigma^{-2}(y)] K_t(x,y)\, dy \right| \to 0, \quad n \to \infty,$$

(D)

$$(D) \qquad \sup_{x\in\mathbb{R}, t\leq T} \frac{1}{T} \left| \int_{\mathbb{R}} [\varrho_n^{-2}(y) - (\varrho^n)^{-2}(y)] K_t(x,y)\, dy \right| \to 0, \quad n, T \to +\infty,$$

(E) There exists $\varrho \in \Sigma$ *such that* $\varrho_n^{-2} \xrightarrow{w} \varrho^{-2}$.

Then the sequence X_n *converges weakly to the process* X, *and the sequence* $\phi^{\cdot}_n$ *converges weakly to the process* $\phi^{\cdot} = cL_X(\cdot, 0)$ *w.r.t. topologies of uniform convergence on compacts in* $C(\mathbb{R}^+)$ *and* $\mathbb{D}(\mathbb{R}^+)$, *respectively. The constant c is equal to*

$$c = \int_{\mathbb{R}} \varrho^{-2}(z)\, P\Big(Z(1)\cdot z < 0 \Big| Z(0) = z\Big)\, dz$$

with the diffusion process Z *defined by SDE*

$$dZ(t) = \varrho(Z(t))\, dW(t).$$

One can interpret the statement of Theorem 1 in the following way. The process X represents the "macroscopic" behavior of the sequence X_n, while the process Z represents the "microscopic" behavior of the same sequence at the vicinity of the point 0. The "shape" of the limiting functional ϕ is completely defined by the macroscopic behavior of the sequence: up to a some constant, nothing but the local time of X can occur at the limit. But this constant, having a natural interpretation as the "intensity" of ϕ, depends essentially on the microscopic behavior of the sequence. Examples given in Section 6 below demonstrate that the macro- and microscopic descriptions for the sequence X_n may differ essentially.

3. Weak convergence of additive functionals of a sequence of Markov chains

Our proof of Theorem 1 is based on the general theorem on the convergence in distribution of a sequence of additive functionals of Markov chains given in [5]. In this section, this theorem is formulated, and the necessary auxiliary notions are introduced.

Suppose the processes $X_n(\cdot), X(\cdot)$ to be defined on $\mathbb{R}^+$ and to take their values in a locally compact metric space $(\mathcal{X}, \rho)$. We say that the process X possesses the Markov property at the time moment $s \in \mathbb{R}^+$ w.r.t. filtration $\{\mathcal{G}_t, t \in \mathbb{R}^+\}$, if X is adapted with this filtration and, for every $k \in \mathbb{N}, t_1, \dots, t_k > s$, there exists a probability kernel $\{P_{st_1\dots t_k}(x, A), x \in \mathcal{X}, A \in \mathcal{B}(\mathcal{X}^k)\}$ such that

$$E[\mathbf{1}_A((X(t_1), \dots, X(t_k)))|\mathcal{G}_s] = P_{st_1\dots t_k}(X(s), A) \quad \text{a.s.}, \quad A \in \mathcal{B}(\mathcal{X}^k).$$

The measure $P_{st_1\dots t_k}(x, \cdot)$ has a natural interpretation as the conditional finite-dimensional distribution of X at the points $t_1, \dots, t_k$ under condition $\{X(s) = x\}$.

Below we suppose the discretization parameter $\alpha > 0$ to be fixed and claim the process X to possess the Markov property (w.r.t. its canonical filtration) at every $s \in \mathbb{R}^+$, and every process X_n to possess this property (w.r.t. its canonical filtration) at the points of the type $in^{-\alpha}, i \in \mathbb{Z}_+$; this means that every process X_n is, in fact, a Markov chain with the time scale proportional to $n^{-\alpha}$.

Consider a sequence of non-negative additive functionals $\{\phi_n^{s,t}, 0 \le s \le t\}, n \ge 1$ of the processes X_n of the form

$$\phi_n^{s,t} = \sum_{k: s \le kn^{-\alpha} < t} F_n\left(X_n(kn^{-\alpha}), \dots, X_n((k+L)n^{-\alpha})\right), \quad 0 \le s \le t, \tag{4}$$

where $L \in \mathbb{Z}_+$ and F_n are non-negative measurable functions on $\mathcal{X}^{L+1}$. For the functional ϕ_n, its characteristic f_n (the analogue of the characteristic of a W-functional) is defined by the formula

$$f_n^{s,t}(x) = E[\phi_n^{s,t} | X_n(s) = x], \quad s = in^{-\alpha}, i \in \mathbb{Z}_+, t \ge s, x \in \mathcal{X}. \tag{5}$$

The process X_n possesses the Markov property w.r.t. its canonical filtration at the time moments $s = in^{-\alpha}, i \in \mathbb{Z}_+$, and functional (4) is a function of the values of X_n at the finite family of such time moments. Therefore, the mean value in (5) is well defined as the integral w.r.t. family of the conditional finite-dimensional distributions $\{P_{st_1 \dots t_k}(x, \cdot), t_1, \dots, t_k > s, k \in \mathbb{N}\}$ of the process X_n.

The following result ([5], Theorem 1) is an analogue of the well-known theorem by E.B.Dynkin that describes the convergence of W-functionals in the terms of their characteristics ([7], Theorem 6.4). Denote $\mathbb{T} = \{(s,t) : 0 \le s \le t\} \subset \mathbb{R}^2$ and define the random broken lines ψ_n corresponding to ϕ_n by

$$\psi_n^{s,t} = \phi_n^{(j-1)n^{-\alpha},(k-1)n^{-\alpha}} + (n^\alpha s - j + 1)\phi_n^{(j-1)n^{-\alpha}, jn^{-\alpha}} + (n^\alpha t - k + 1)\phi_n^{(k-1)n^{-\alpha}, kn^{-\alpha}},$$

$$s \in \left[(j-1)n^{-\alpha}, jn^{-\alpha}\right), t \in \left[(k-1)n^{-\alpha}, kn^{-\alpha}\right).$$

Theorem 2. *Let the sequence of the processes X_n be given, providing a Markov approximation for the homogeneous Markov process X (see Definition 1 below), and let the sequence $\{\phi_n\}$ be defined by (4). Suppose that the following conditions hold true:*

1. The functions $F_n(\cdot)$ are non-negative, bounded on $\mathcal{X}^{L+1}$, and uniformly converge to zero:

$$\delta(F_n) = \sup_{x_0,\dots,x_L \in \mathcal{X}} F_n(x_0, \dots, x_L) \to 0, \quad n \to \infty.$$

2. There exists a function f, that is the characteristic of a certain W-functional ϕ of the limiting process X, such that, for every $T \in \mathbb{R}^+$,

$$\sup_{s = in^{-\alpha}, t \in (s,T)} \sup_{x \in \mathcal{X}} |f_n^{s,t}(x) - f^{t-s}(x)| \to 0, \quad n \to \infty.$$

3. The limiting function f is continuous w.r.t. variable x, locally uniformly w.r.t. time variable, i.e., for every $T \in \mathbb{R}^+$,

$$\sup_{t \le T} \left| f^t(x) - f^t(y) \right| \to 0, \quad \|x - y\| \to 0.$$

Then

$$\psi_n \equiv \{\psi_n^{s,t}, (s,t) \in \mathbb{T}\} \Rightarrow \phi \equiv \{\phi^{s,t}, (s,t) \in \mathbb{T}\}$$

in a sense of weak convergence in $C(\mathbb{T}, \mathbb{R}^+)$.

The notion of Markov approximation introduced in [8] is the key one in Theorem 2. Below we give the slightly modified definition, taking into account that, in the current considerations, the time discretization points have the form $in^{-\alpha}, i \in \mathbb{Z}_+$.

Definition 1. *The sequence of the processes $\{X_n\}$ provides the Markov approximation for the Markov process X, if for every $\gamma > 0, S < +\infty$ there exist a constant $K(\gamma, S) \in \mathbb{N}$ and a sequence of two-component processes $\{\hat{Y}_n = (\hat{X}_n, \hat{X}^n)\}$, possibly defined on another probability space, such that*

(i) $\hat{X}_n \stackrel{d}{=} X_n, \hat{X}^n \stackrel{d}{=} X$;

(ii) the processes $\hat{Y}_n, \hat{X}_n, \hat{X}^n$ *possess the Markov property at the points* $iK(\gamma, S)n^{-\alpha}$, $i \in \mathbb{N}$ *w.r.t. the filtration* $\{\hat{\mathcal{F}}_t^n = \sigma(\hat{Y}_n(s), s \leq t)\}$;

(iii)

$$\lim \sup_{n \to +\infty} P\left(\sup_{i \leq \frac{Sn^\alpha}{K(\gamma,S)}} \rho\left(\hat{X}_n\left(iK(\gamma,S)n^{-\alpha}\right), \hat{X}^n\left(iK(\gamma,S)n^{-\alpha}\right)\right) > \gamma \right) < \gamma.$$

4. Weak convergence and Markov approximation

In this section, we prove that, under conditions of Theorem 1, the processes X_n both converge weakly to X and provide the Markov approximation for X.

Lemma 1. *Under conditions (A) and (C), X_n converge to X weakly in $C(\mathbb{R}^+)$.*

Proof. Let $\hat{W}$ be a Wiener process. Consider the processes of the type

$$\hat{X}_n(t) = x + \hat{W}(\zeta_{n,t}), \quad \hat{X}(t) = x + \hat{W}(\zeta_t),$$

where $\zeta_{n,\cdot}, \zeta.$ are inverse functions to the functions $\eta_{n,\cdot}, \eta.$ defined by

$$\eta_{n,t} = \int_0^t \sigma_n^{-2}(x + \hat{W}(s))\, ds, \quad \eta_t = \int_0^t \sigma^{-2}(x + \hat{W}(s))\, ds.$$

Then $\hat{X}_n$ has the same distribution with X_n, and $\hat{X}$ has the same distribution with X. The processes $\eta_{n,\cdot}, \eta.$ are W-functionals of the Wiener process with their characteristics equal to

$$g_n^t(x) = E_x \eta_{n,t} = \int_{\mathbb{R}} K_t(x,y)\sigma_n^{-2}(y)\, dy, \quad g^t(x) = E_x \eta_t = \int_{\mathbb{R}} K_t(x,y)\sigma^{-2}(y)\, dy.$$

Thus, condition (C) provides that $g_n \to g$ uniformly on $\mathbb{R} \times [0,T]$. Now, the Dynkin's theorem ([7], Theorem 6.4) provides that, for every T, $\sup_{t \leq T} |\eta_{n,t} - \eta_t| \to 0, n \to +\infty$ in probability. Since $0 \leq \left[\frac{d}{dt}\eta_{n,t}\right]^{-1} = \sigma_n^2(x + \hat{W}(t)) \leq R^2$, the convergence in probability $\sup_{t \leq T} |\eta_{n,t} - \eta_t| \to 0, T > 0$ implies the convergence in probability $\sup_{t \leq T} |\zeta_{n,t} - \zeta_t| \to 0, T > 0$ and, therefore, the convergence in probability $\sup_{t \leq T} |\hat{X}_n(t) - \hat{X}(t)| \to 0, T > 0$. The lemma is proved.

In order to prove that X_n provide the Markov approximation for X, we need some auxiliary estimates and constructions. Denote

$$d_2(\xi, \eta) = \left[\inf_{(\hat{\xi},\hat{\eta}),\, \hat{\xi} \stackrel{d}{=} \xi, \hat{\eta} \stackrel{d}{=} \eta} E(\hat{\xi} - \hat{\eta})^2 \right]^{\frac{1}{2}}, \quad \xi, \eta \in L_2,$$

which is the Wasserstein–Kantorovich–Rubinshtein distance between the distributions of ξ and η. Denote, by $X_n(t,x), X(t,x), t \in \mathbb{R}^+$, the diffusion processes satisfying (1) and (2), respectively, with the initial conditions $X_n(0,x) = x, X(0,x) = x$.

Lemma 2. *Under conditions (A) and (D), for every $\varepsilon > 0$, there exist $T = T_\varepsilon \in \mathbb{N}$ and $N = N_\varepsilon \in \mathbb{N}$ such that*

$$d_2(X_n(Tn^{-\alpha}, x), X(Tn^{-\alpha}, x)) \le \varepsilon\sqrt{Tn^{-\alpha}}, \quad n \ge N, x \in \mathbb{R}. \tag{6}$$

Proof. One can easily see that (6) is equivalent to the following estimate for the re-scaled processes Z_n, Z^n:

$$d_2(Z_n(T, xn^{\frac{\alpha}{2}}), Z^n(T, xn^{\frac{\alpha}{2}})) \le \varepsilon\sqrt{T}, \quad n \ge N, x \in \mathbb{R},$$

that, in turn, follows from the estimate

$$\sup_{z\in\mathbb{R}^d} d_2(Z_n(T, z), Z^n(T, z)) \le \varepsilon\sqrt{T}, \quad n \ge N. \tag{7}$$

Let us prove (7). Let z be fixed, and let $\hat{W}$ be a Wiener process. Consider the processes of the type

$$\hat{Z}_n(t, z) = z + \hat{W}(\theta_{n,t}), \quad \hat{Z}^n(t, z) = z + \hat{W}(\theta^n_t), \tag{8}$$

where $\theta_{n,\cdot}, \theta^n_\cdot$ are inverse functions to the functions $\vartheta_{n,\cdot}, \vartheta^n_\cdot$ defined by

$$\vartheta_{n,t} = \int_0^t \varrho_n^{-2}(z + \hat{W}(s))\, ds, \quad \vartheta^n_t = \int_0^t (\varrho^n)^{-2}(z + \hat{W}(s))\, ds.$$

Since $\hat{Z}_n(t, z) \stackrel{d}{=} Z_n(t, z)$, $\hat{Z}^n(t, z) \stackrel{d}{=} Z^n(t, z)$ and $\theta_{n,t}, \theta^n_t$ are a stopping times w.r.t. filtration generated by $\hat{W}$, we have

$$\begin{aligned} d_2^2(Z_n(T, z), Z^n(T, z)) &\le E(\hat{Z}_n(T, z) - Z^n(T, z))^2 \\ &= E(\hat{W}(\theta_{n,T}) - \hat{W}(\theta^n_T))^2 = E|\theta_{n,T} - \theta^n_T|. \end{aligned}$$

Condition (A) provides that $|\theta_{n,T} - \theta^n_T| \le R \sup_{s\le TR} |\vartheta_{n,s} - \vartheta^n_s|$. The processes $\vartheta_{n,\cdot}, \vartheta^n_\cdot$ are W-functionals of the process $\hat{W}$ with their characteristics f_n, f^n equal to

$$f_n^t(z) = \int_{\mathbb{R}} \varrho_n^2(y) K_t(z, y)\, dy, \quad f^{n,t}(z) = \int_{\mathbb{R}} (\varrho^n)^2(y) K_t(z, y)\, dy.$$

Since $\int_{\mathbb{R}} K_t(z, y)\, dy = t, z \in \mathbb{R}, t \in \mathbb{R}^+$,

$$\|f_n^T\| \le R^2 T, \quad \|f^{n,T}\| \le R^2 T$$

by condition (A) (here and below, we denote $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$). Then Lemma 3 below provides the estimate

$$E(\sup_{s\le TR} |\vartheta_{n,s} - \vartheta^n_s|)^2 \le 8(\sqrt{2} + \sqrt{3})^2 T^2 R^2 \frac{\sup_{s\le TR} \|f_n^s - f^{n,s}\|}{T}. \tag{9}$$

By condition (D), $\frac{1}{T} \sup_{s\le TR} \|f_n^s - f^{n,s}\| \to 0, n, T \to +\infty$. This provides inequality (7) for N, T large enough and completes the proof of the lemma.

Estimate (9) in the proof above is provided by the following result, that is a generalization of Lemma 6.5 [7].

Lemma 3. *Let Y be a homogeneous Markov process with its phase space $\mathcal{Y}$ being a locally compact metric space. Let ϕ, ψ be W-functionals of Y, and let f, g be their characteristics, respectively. Then*

$$E\,[\sup_{s\le t} (\phi^{0,s} - \psi^{0,s})^2 | Y(0) = y] \le 8(\sqrt{2} + \sqrt{3})^2 (\|f^t\| + \|g^t\|) \sup_{s\le t} \|f^s - g^s\|, \quad t \in \mathbb{R}^+, y \in \mathcal{Y}.$$

Proof. The proof is based on the idea of the proof of Doob's maximal martingale inequality ([9], Chapter 7, §3). Since the functions $\phi^{0,\cdot}, \psi^{0,\cdot}$ are continuous, it is sufficient to prove that, for every $k \in \mathbb{Z}_+, m \in \mathbb{N}$,

$$E\,[\max_{j\le k}(\phi^{0,j2^{-m}} - \psi^{0,j2^{-m}})^2 | Y(0) = y] \le 8(\sqrt{2}+\sqrt{3})^2(\|f^{k2^{-m}}\| + \|g^{k2^{-m}}\|) \sup_{s\le k2^{-m}} \|f^s - g^s\|.$$

We suppose $y \in \mathcal{Y}, m \in \mathbb{N}$ to be fixed and omit them in the notation. Denote $M_k = \max_{j\le k}(\phi^{0,j2^{-m}} - \psi^{0,j2^{-m}}), k \in \mathbb{Z}_+$. We have that $M_k \ge 0$ since $\phi^{0,0} = \psi^{0,0} = 0$. For $u > 0$, denote $\tau_u = \min\{k : M_k \ge u\}$ and write

$$uP(M_k \ge u) = uP(\tau_u \le k) \le E(\phi^{0,(\tau_u\wedge k)2^{-m}} - \psi^{0,(\tau_u\wedge k)2^{-m}})\mathbb{1}_{\tau_u\le k} =$$

$$= E(\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}})\mathbb{1}_{\tau_u\le k} - E(\phi^{(\tau_u\wedge k)2^{-m},k2^{-m}} - \psi^{(\tau_u\wedge k)2^{-m},k2^{-m}})\mathbb{1}_{\tau_u\le k}. \quad (10)$$

The sequence $\{(Y(k2^{-m}), \phi^{0,k2^{-m}}, \psi^{0,k2^{-m}}), k \in \mathbb{Z}_+\}$ is a Markov chain, and therefore it is strongly Markov. Denote, by $\mathbb{G} = \{\mathcal{G}_k\}$, the corresponding filtration and write

$$-E[(\phi^{(\tau_u\wedge k)2^{-m},k2^{-m}} - \psi^{(\tau_u\wedge k)2^{-m},k2^{-m}}|\mathcal{G}_{\tau_u\wedge k}] =$$

$$= g^{(k-\tau_u\wedge k)2^{-m}}(Y((\tau_u\wedge k)2^m)) - f^{(k-\tau_u\wedge k)2^{-m}}(Y((\tau_u\wedge k)2^m)) \le \sup_{j\le k}\|f^{j2^{-m}} - g^{j2^{-m}}\| \quad (11)$$

(here, we have used the strong Markov property and the fact that τ_u is a stopping time w.r.t. $\mathbb{G}$). One can easily verify that $\{\tau_u \le k\} \in \mathcal{G}_{\tau_u\wedge k}$, and thus (11) shows that the second summand on the right-hand side of (10) is estimated by

$$\max_{j\le k}\|f^{j2^{-m}} - g^{j2^{-m}}\|P(\tau_u \le k) = \max_{j\le k}\|f^{j2^{-m}} - g^{j2^{-m}}\|P(M_k \ge u).$$

Denote

$$d = (\|f^{k2^{-m}}\| + \|g^{k2^{-m}}\|)^{\frac12} \sup_{s\le k2^{-m}} \|f^s - g^s\|^{\frac12};$$

then $d \ge \max_{j\le k}\|f^{j2^{-m}} - g^{j2^{-m}}\|$. Inequalities (10), (11) provide the estimate

$$(u-d)P(M_k \ge u) \le E(\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}})\mathbb{1}_{M_k\ge u}, \quad u \ge d.$$

Then

$$EM_k^2 = 2\int_0^\infty uP(M_k \ge u)\,du \le 2\int_0^{2d} u\,du + 4\int_{2d}^\infty (u-d)P(M_k\ge u)\,du \le$$

$$\le 4d^2 + 4E\Big|\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}}\Big|\int_{2d}^\infty \mathbb{1}_{u\le M_k}\,du \le 4d^2 + 4E|\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}}|M_k \le$$

$$\le 4d^2 + 4\Big[E(\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}})^2\Big]^{\frac12} \cdot [EM_k^2]^{\frac12}.$$

By Lemma 6.5 [7], $E(\phi^{0,k2^{-m}} - \psi^{0,k2^{-m}})^2 \le 2d^2$. Thus, $\varkappa \equiv [EM_k^2]^{\frac12}$ satisfies the inequality

$$\varkappa^2 \le 4d^2 + 4\sqrt{2}d\varkappa,$$

that means that

$$EM_k^2 \le 4(\sqrt{2}+\sqrt{3})^2 d^2. \quad (12)$$

Completely analogously, one can prove that

$$E[\min_{j\le k}(\phi^{0,j2^{-m}} - \psi^{0,j2^{-m}})]^2 \le 4(\sqrt{2}+\sqrt{3})^2 d^2. \quad (13)$$

Since $\max_{j\le k}(\phi^{0,j2^{-m}}-\psi^{0,j2^{-m}})^2\le[\max_{j\le k}(\phi^{0,j2^{-m}}-\psi^{0,j2^{-m}})]^2+[\min_{j\le k}(\phi^{0,j2^{-m}}-\psi^{0,j2^{-m}})]^2$, inequalities (12) and (13) provide the required estimate. The lemma is proved.

Now we are ready to prove the main statement of this section.

Theorem 3. *Under conditions of Theorem 1, the sequence $\{X_n(\cdot,x)\}$ provides the Markov approximation for $X(\cdot,x)$.*

Proof. For every $\varepsilon>0, n\in\mathbb{N}$, we fix $T=T_\varepsilon$ from Lemma 2 and construct iteratively the process $Q^n(t)=(\hat X_n(t),\hat R_n(t),\hat X^n(t))$ and the sequence $(\kappa_k,\varsigma_k,\chi_k), k\ge 1$ in the following way. For $t\in[0,Tn^{-\alpha})$, put

$$\hat X_n(t)=n^{-\frac{\alpha}{2}}\hat Z_n(tn^\alpha,xn^{\frac{\alpha}{2}}),\quad \hat R_n(t)=\hat X^n(t)=\hat Z^n(tn^\alpha,xn^{\frac{\alpha}{2}}),$$

where $\hat Z_n\hat Z^n$ are defined by (9). Put

$$\kappa_1=n^{-\frac{\alpha}{2}}\hat Z_n(n^\alpha,xn^{\frac{\alpha}{2}}),\quad \varsigma_1=\chi_1=n^{-\frac{\alpha}{2}}\hat Z^n(n^\alpha,xn^{\frac{\alpha}{2}}).$$

Next, suppose that $Q^n(t)$ is already defined for $t\in[0,mTn^{-\alpha})$, and $(\kappa_k,\varsigma_k,\chi_k)$ is already defined for $k\le m$. Consider some Wiener process $\hat W_m$ independent of the values of the process $Q^n(t)$ on $t\in[0,mTn^{-\alpha})$ and consider the processes $\hat Z_{n,m}(\cdot,z),\hat Z_n^m(\cdot,z),\ z\in\mathbb{R}$ defined by (9) with $\hat W$ replaced by $\hat W_m$. For $t\in[mTn^{-\alpha},(m+1)Tn^{-\alpha})$, put

$$\hat X_n(t)=n^{-\frac{\alpha}{2}}\hat Z_{n,m}(tn^\alpha-m,\kappa_m n^{\frac{\alpha}{2}}),\quad \hat R_n(t)=\hat Z_m^n(tn^\alpha-m,\kappa_m n^{\frac{\alpha}{2}}).$$

The process $\hat R_n$ satisfies SDE (2) on $[mTn^{-\alpha},(m+1)Tn^{-\alpha})$ with a certain Wiener process $\check W$. Define the process $\hat X^n$ on $[mTn^{-\alpha},(m+1)Tn^{-\alpha})$ as the solution to SDE (2) with the same Wiener process $\check W$ and $\hat X^n(Tn^{-\alpha})=\chi_m$. Such a definition is correct since (2) has a weak solution, possesses the path-wise uniqueness property, and therefore, by the Yamada–Watanabe theorem, possesses the unique strong solution. At last, put

$$\kappa_{m+1}=\hat X_n((m+1)Tn^{-\alpha}-),\quad \varsigma_{m+1}=\hat R_n((m+1)Tn^{-\alpha}-),$$
$$\chi_{m+1}=\hat X^n((m+1)Tn^{-\alpha}-).$$

Repeating this construction, we obtain the processes Q^n which are defined on $\mathbb{R}^+$ and possess the following properties:

(i) $\hat X_n\overset{d}{=}X_n,\hat X^n\overset{d}{=}X$;

(ii) the processes $Q_n,\hat X_n,\hat X^n$ possess the Markov property at the points $iT_\varepsilon n^{-\alpha}, i\in\mathbb{N}$ w.r.t. the filtration $\{\mathcal{F}_t^n=\sigma(\hat Q_n(s),s\le t)\}$.

Now we are going to prove that, for every $\gamma>0,S<+\infty$, there exists $\varepsilon>0$ such that

$$\lim\sup_{n\to+\infty}P\left(\sup_{i\le\frac{Sn^\alpha}{T_\varepsilon}}\left|\hat X_n\left(iT_\varepsilon n^{-\alpha}\right)-\hat X^n\left(iT_\varepsilon n^{-\alpha}\right)\right|>\gamma\right)<\gamma. \tag{14}$$

This will mean that conditions (i) – (iii) of Definition 1 hold true with $K(\gamma,S)=T_\varepsilon$.

Using the fact that the coefficients σ_n,σ are uniformly bounded, one can verify that, for every $a>0$,

$$\sup_n P(w_S(\hat X_n,\delta)>a)+\sup_n P(w_S(\hat X^n,\delta)>a)\to 0,\quad \delta\to 0+,$$

where $w_S(X,\delta)\equiv\sup_{|s-t|\le\delta,s,t\in[0,S]}|X(t)-X(s)|$ (the proof is standard and omitted). Since $\hat R_n(iT_\varepsilon n^{-\alpha})=\hat X_n(iT_\varepsilon n^{-\alpha}), i\in\mathbb{N}$, this implies that

$$\sup_{t\le S}|\hat X_n(t)-\hat R_n(t)|\to 0,\quad n\to+\infty \tag{15}$$

in probability. The process $\hat{X}^n$ satisfies SDE

$$\hat{X}^n(t) = x + \int_0^t \sigma(\hat{X}^n(s))\, dW^n(s), \quad t \in \mathbb{R}^+.$$

On the other hand, the process $\hat{R}_n$, by construction, satisfies SDE

$$\hat{R}^n(t) = x + \int_0^t \sigma(\hat{R}^n(s))\, dW^n(s) + \Delta_n(t), \quad t \in \mathbb{R}^+$$

with $\Delta_n(t) = \sum_{k \le tT_\varepsilon^{-1} n^\alpha} (\kappa_k - \varsigma_k)$. By construction, Δ_n is a martingale (here, we make use of the supposition made at the beginning of Section 2 that the coefficients a_n, a in Eqs. (1),(2) are equal to 0). By Lemma 2, $E(\kappa_k - \varsigma_k)^2 \le \varepsilon^2 T n^{-\alpha}, k \ge 1$. Therefore,

$$E \max_{t \le S} \Delta_n^2(t) \le 2E\Delta_n^2(S) = 2 \sum_{k \le ST_\varepsilon^{-1} n^\alpha} (\kappa_k - \varsigma_k)^2 \le 2\varepsilon^2 S. \tag{16}$$

Now suppose that, for every $\varepsilon > 0$, (14) fails. This means that there exists some subsequence $\{n_r\}$ such that

$$P\left(\sup_{i \le \frac{S n_r^\alpha}{T_\varepsilon}} \left| \hat{X}_{n_r}\left(iT_\varepsilon {n_r}^{-\alpha}\right) - \hat{X}^{n_r}\left(iT_\varepsilon n_r^{-\alpha}\right) \right| > \gamma \right) \ge \gamma. \tag{17}$$

The families $\{\hat{X}_n\}, \{\hat{X}^n\}$ are weakly compact in $C(\mathbb{R}^+)$, and therefore one can suppose that the 3-component processes $(\hat{X}_{n_r}, \hat{X}^{n_r}, W^{n_r})$ converge weakly in $C(\mathbb{R}^+, \mathbb{R}^3)$ to some process $(\hat{X}_*, \hat{X}^*, W^*)$. Relation (15) implies that $(\hat{X}_{n_r}, \hat{X}^{n_r}, W^{n_r}, \hat{R}_{n_r}) \Rightarrow (\hat{X}_*, \hat{X}^*, W^*, \hat{X}_*)$ in $C(\mathbb{R}^+, \mathbb{R}^3) \times D(\mathbb{R}^+)$ with $D(\mathbb{R}^+)$ endowed with the topology of uniform convergence on every compact. Then

$$\left(\hat{X}_{n_r}, \hat{X}^{n_r}, W^{n_r}, \hat{R}_{n_r}, \int_0^\cdot \sigma(\hat{R}^{n_r}(s)) dW^{n_r}(s)\right) \Rightarrow \left(\hat{X}_*, \hat{X}^*, W^*, \hat{X}_*, \int_0^\cdot \sigma(\hat{X}_*(s)) dW^*(s)\right)$$

in $C(\mathbb{R}^+, \mathbb{R}^3) \times D(\mathbb{R}^+) \times C(\mathbb{R}^+)$, and consequently

$$\left(\hat{X}_{n_r}, \hat{X}^{n_r}, W^{n_r}, \hat{R}_{n_r}, \int_0^\cdot \sigma(\hat{R}^{n_r}(s)) dW^{n_r}(s), \Delta_{n_r}\right) \Rightarrow \left(\hat{X}_*, \hat{X}^*, W^*, \hat{X}_*, \int_0^\cdot \sigma(\hat{X}_*(s)) dW^*(s), \Delta_*\right)$$

in $C(\mathbb{R}^+, \mathbb{R}^3) \times D(\mathbb{R}^+) \times C(\mathbb{R}^+) \times D(\mathbb{R}^+)$. The statement analogous to this one was given in the proof of the Theorem 1 [6, Chapter 5.3], (see also the discussion after Lemma 2.3 in [10]).

Now we conclude that, for every $\varepsilon > 0$, there exists the 4-component process $H_* = (\hat{X}_*, \hat{X}^*, W^*, \Delta_*)$ such that W^* is a Wiener process w.r.t. filtration generated by H_*, the processes X_*, X^* satisfy the relations

$$X_*(t) = x + \int_0^t \sigma(X_*(s))\, ds + \Delta_*(t), \quad X^*(t) = x + \int_0^t \sigma(X^*(s))\, ds,$$

and the estimates

$$E \sup_{s \le S} \Delta_*^2(s) \le 2\varepsilon^2 S, \tag{18}$$

$$P(\sup_{s \le S} |X_*(s) - X^*(s)| \ge \gamma) \ge \gamma \tag{19}$$

hold true [(18) follows from (16), and (19) follows from (17)]. Once more passing to the limit as $\varepsilon \to 0+$, we obtain the 3-component process $(X_\diamondsuit, X^\diamondsuit, W^\diamondsuit)$ such that both $X_\diamondsuit$ and $X^\diamondsuit$ satisfy (2) with the same initial condition x and the same Wiener process $W^\diamondsuit$, but $X_\diamondsuit \not\equiv X^\diamondsuit$ due to (19). This contradicts the condition on (2) to possess the path-wise uniqueness property. Consequently, our supposition that (14) fails for every $\varepsilon > 0$ is false, and the conditions of Definition 1 hold true with $K(\gamma, S) = T_\varepsilon$ with some $\varepsilon > 0$. The theorem is proved.

5. Proof of Theorem 1

We reduce the proof of Theorem 1 to the verification of the conditions of Theorem 2. The sequence X_n provides the Markov approximation for the process X due to Theorem 3. Condition 1 of Theorem 2 holds true since $\delta_n = n^{-\frac{\alpha}{2}}$. In this section, we prove that conditions 2 and 3 hold true. Recall that $f_n, n \in \mathbb{N}$ denote characteristics for the functionals $\phi_n, n \in \mathbb{N}$ (see (5)).

Lemma 4. *Denote*

$$L_{Z_n}(t,z) = |Z_n(t) - z| - |Z_n(0) - z| - \int_0^t \operatorname{sign}(Z_n(s) - z)\, dZ_n(s), \quad t \in \mathbb{R}^+.$$

Then, for every bounded measurable function Φ with bounded support,

$$\int_0^t \Phi(Z_n(s))\, ds = \int_{\mathbb{R}} \Phi(z) L_{Z_n}(t,z) \varrho_n^{-2}(z)\, dz. \tag{20}$$

Proof. The statement of the lemma is known to hold true for the Wiener process (i.e., for $\varrho_n \equiv 1$, see [11], Chapter 3, §4). The process Z_n can be represented in the form (9), and then $L_{Z_n}(t,z) = L_{\hat{W}}(\theta_{n,t}, z)$, where $\hat{W}, \theta_{n,\cdot}$ denote the Wiener process and the time change from this representation (the proof is easy and omitted). Now (20) follows from the analogous equality for $\tilde{\Phi} \equiv \Phi \varrho_n^{-2}$ for the Wiener process by changing the variables in the integral w.r.t. ds. The lemma is proved.

Lemma 5. *Under conditions (A), (C) – (E),*

$$f_n^{s,t}(x) \to cE\Big(L_X(t-s,0)\Big| X(0) = x\Big), \quad n \to +\infty,$$

uniformly on $\mathbb{R} \times \{0 \le s \le t \le T\}$ for every T, with the constant c defined in the formulation of Theorem 1.

Proof. We have

$$\begin{aligned} f_n^{s,t}(x) &= n^{-\frac{\alpha}{2}} \sum_{k<(t-s)n^\alpha} E\Big(\mathbf{1}_{Z_n(k)Z(k+1)<0}\Big| Z_n(0) = xn^{\frac{\alpha}{2}}\Big) \\ &= n^{-\frac{\alpha}{2}} \sum_{k<tn^\alpha} E\Big(\Phi_n(Z_n(k))\Big| Z_n(0) = xn^{\frac{\alpha}{2}}\Big), \end{aligned}$$

where $\Phi_n(z) = P\Big(Z_n(1) \cdot z < 0 \Big| Z_n(0) = z\Big)$. Denote, by $G_n(t,x,dy)$, the transition probability for the process Z_n. Due to Theorem 1.2 in [12], there exists a constant $\mu > 0$ depending on R only, such that, for every g,

$$\Big|\frac{d}{dt} \int_{\mathbb{R}} g(y) G_n(t,z,dy)\Big| \le \mu t^{-\frac{3}{2}} \int_{\mathbb{R}} e^{-\frac{\mu(y-z)^2}{t}} |g(y)|\, dy. \tag{21}$$

Since the diffusion coefficients for Z_n are uniformly bounded, for every $m \ge 1$, there exists a constant C_m such that $\Phi_n(z) \le C_m(1 \wedge |z|)^{-m}, n \in \mathbb{N}$. This, together with (21), implies that

$$\sup_{0 \le s \le t \le T, x \in \mathbb{R}} \left| f_n^{s,t}(x) - n^{-\frac{\alpha}{2}} \int_0^{(t-s)n^\alpha} E\Big(\Phi_n(Z_n(s)) \Big| Z_n(0) = xn^{\frac{\alpha}{2}}\Big)\, ds \right| \to 0, \quad n \to +\infty.$$

The function Φ_n does not have a compact support, and Lemma 4 can not be applied to it straightforwardly. But, applying Lemma 4 to the functions $\Phi_{n,A} = \Phi_n \mathbb{1}_{[-A,A]}$, using the estimate of Φ_n given before, and then passing to the limit as $A \to +\infty$, one can obtain the estimate

$$\begin{aligned} n^{-\frac{\alpha}{2}} & \int_0^{(t-s)n^\alpha} E\Big(\Phi_n(Z_n(s)) \Big| Z_n(0) = xn^{\frac{\alpha}{2}}\Big)\, ds \\ &= n^{-\frac{\alpha}{2}} \int_{\mathbb{R}} \Phi_n(z) \varrho_n^{-2}(z) E\Big(L_{Z_n}((t-s)n^\alpha, z) \Big| Z_n(0) = xn^{\frac{\alpha}{2}}\Big)\, dz \\ &= \int_{\mathbb{R}} \Phi_n(z) \varrho_n^{-2}(z) E\Big(|X_n(t-s,x) - n^{-\frac{\alpha}{2}} z| - |x - n^{-\frac{\alpha}{2}} z|\Big)\, dz. \end{aligned}$$

The estimates given in the proof of Lemma 1 provide that, for every $A > 0$,

$$\sup_{x \in \mathbb{R}, |z| \le A} \left| \Phi_n(z) E\Big(|X_n(t-s,x) - n^{-\frac{\alpha}{2}} z| - |x - n^{-\frac{\alpha}{2}} z|\Big) - \Phi(z) E\Big(|X(t,x)| - |x|\Big) \right| \to 0,$$
$$n \to +\infty,$$

with $\Phi(z) = P\Big(Z(1) \cdot z < 0 \Big| Z(0) = z\Big)$. Then the estimate on Φ_n given before provides that

$$\sup_{x \in \mathbb{R}, t-s \le T} \int_{\mathbb{R}} \varrho_n^{-2}(z) \Big| \Phi_n(z) E\left(|X_n(t-s,x) - n^{-\frac{\alpha}{2}} z| - |x - n^{-\frac{\alpha}{2}} z|\right) - \Phi(z) E\left(|X(t,x)| - |x|\right) \Big|\, dz \to 0,$$

$n \to +\infty$. Since ϱ_n^{-2} are uniformly bounded and converge weakly to ϱ^{-2}, this implies that

$$\begin{aligned} f_n^{s,t}(x) \to & \lim_{N \to +\infty} \int_{\mathbb{R}} \Phi(z) \varrho_N^{-2}(z)\, dz \cdot E\Big(|X(t-s,x)| - |x|\Big) \\ &= \int_{\mathbb{R}} \Phi(z) \varrho^{-2}(z)\, dz \cdot E\Big(L_X(t-s,0) \Big| X(0) = x\Big), \end{aligned}$$

$n \to +\infty$, uniformly for $x \in \mathbb{R}, 0 \le s \le t \le T$. The lemma is proved.

The function $(x,t) \mapsto L_X(t,x)$ is continuous in mean square. In order to prove this, one should write $L_X(t,z) = L_W(\theta_t, z)$ (as in the proof of Lemma 4) and then use the same property for the local time of the Wiener process. Then

$$L_X(t,0) = L_2 - \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} L_X(t,x)\, dx = L_2 - \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|X(s)| < \varepsilon} \sigma^2(X(s))\, ds,$$

and, therefore (see [7] Chapter 6, §2,3), the process $\phi^{s,t} = L_X(t,0) - L_X(s,0)$ is a W-functional of the process X. Hence, condition 2 of Theorem 2 holds true. At last, one can use the arguments from the proof of Lemma 2 in order to prove that $E(X(t,x) - X(t,y))^2 \to 0, |x-y| \to 0$ uniformly for $t \le T$. Since

$$\begin{aligned} |f^t(x) - f^t(y)| &= |E\left(L_X(t,0) \mid X(0) = x\right) - E\left(L_X(t,0) \mid X(0) = x\right)| \\ &\le |E|X(t,x)| - |X(t,y)| - |x| + |y|| \le |x-y| + \Big(E(X(t,x) - X(t,y))^2\Big)^{\frac{1}{2}}, \end{aligned}$$

this provides condition 3 of Theorem 3. Therefore, all conditions of Theorem 3 hold true. Applying this theorem to functionals (3), we obtain the statement of Theorem 1.

6. Examples

Example 1. Let $\sigma_n(x) = \varrho(n^\alpha x)$, and let the function $\varrho \in \Sigma$ satisfy the condition $R^{-1} \le \varrho \le R$. Suppose that

$$\frac{1}{v-u}\int_u^v \varrho^{-2}(z)\,dz \to \varkappa_+, \quad \frac{1}{v-u}\int_{-v}^{-u} \varrho^{-2}(z)\,dz \to \varkappa_-, \quad u,v\to+\infty,\, v-u\to+\infty, \tag{22}$$

and put $\sigma = \sigma_- 1\!\!{\rm I}_{\mathbb{R}^-} + \sigma_+ 1\!\!{\rm I}_{\mathbb{R}^+}$ with $\sigma_\pm = \varkappa_\pm^{-\frac12}$. Then $\varrho_n = \varrho, \varrho^n = \sigma$ for every n, and conditions (C) and (D) are, in fact, equivalent one to another and follow from condition (22) (we omit the detailed exposition here, since analogous estimates are given, in a more delicate situation, in the next example). Condition (E) is trivial. At last, condition (B) holds true by the Nakao theorem [13].

Suppose that $\sigma_- + \sigma_+ = 1$ (one can reduce the general case to this one by making the time change $t \mapsto (\sigma_- + \sigma_+)^{-\frac12} t$), and put $q = \sigma_- - \sigma_+$. Then (see [14]) the process X defined by SDE (2) is the image of the *skew Brownian motion* W^q with the skewing parameter q under the phase transformation

$$x \mapsto x\frac{1+q}{2} 1\!\!{\rm I}_{\mathbb{R}^-} + x\frac{1-q}{2} 1\!\!{\rm I}_{\mathbb{R}^+}.$$

Moreover, the local time of X at the point 0 is equal to the local time of W^q at the same point. Thus, in the example under consideration, the number of sign changes (3) converges weakly to the local time of the skew Brownian motion $W^{\sigma_- - \sigma_+}$ at the point 0 multiplied by $c = \int_{\mathbb{R}} \varrho^{-2}(z)\, P\Big(X_1(1)\cdot z < 0 \Big| X_1(0) = z\Big)\, dz$.

Example 2. Let $\sigma_n(x) = \varsigma(x)\exp\cos(n^\alpha x), x \in \mathbb{R}$, with the function $\varsigma \in \Sigma_R$ that is uniformly continuous on $\mathbb{R}$. We state that conditions (C) and (D) hold true with $\sigma = C^{-\frac12}\cdot\varsigma$, where $C = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\cos y}\,dy$. Let us prove (D); the proof for (C) is analogous and more simple. After the change of variables $\hat{x} = \frac{x}{\sqrt{s}}, \hat{y} = \frac{y}{\sqrt{s}}$, we have

$$\int_{\mathbb{R}} [\varrho_n^{-2}(y) - (\varrho^n)^{-2}(y)] K_t(x,y)\, ds = \int_0^t \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} (e^{-2\cos y} - C)\varsigma(yn^{-\alpha}) e^{-\frac{(y-x)^2}{2s}}\, dy\, ds =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^t \int_{\mathbb{R}} (e^{-2\cos(\hat{y}\sqrt{s})} - C)\varsigma(\hat{y} n^{-\alpha}\sqrt{s}) e^{-\frac{(\hat{y}-\hat{x})^2}{2}}\, d\hat{y}\, ds.$$

Therefore, in order to prove (D), it is enough to show that

$$\sup_x \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} (e^{-2\cos(y\sqrt{t})} - C)\varsigma(yn^{-\alpha}\sqrt{t}) e^{-\frac{(y-x)^2}{2}}\, dy \right| \to 0, \quad n,t\to+\infty. \tag{23}$$

Denote $w_{\varsigma^{-2}}(z) = \sup_{|x-y|\le z} |\varsigma^{-2}(x) - \varsigma^{-2}(y)|$. We have that $w_{\varsigma^{-2}}(z) \to 0, z\to 0+$. Let $\varepsilon > 0$ be fixed; consider $D, \delta > 0$ such that

$$\int_{-D}^{D} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \ge 1-\varepsilon, \qquad \sup_{|x-y|\le\delta} |e^{-\frac{x^2}{2}} - e^{-\frac{y^2}{2}}| \le \frac{\varepsilon}{2D+1}.$$

For a given $x \in \mathbb{R}, t\in\mathbb{R}^+$, put

$$k_-(x,t) = \max\left\{k\in\mathbb{Z} : \frac{2\pi k}{\sqrt{t}} \le x - D\right\}, \quad k_+(x,t) = \min\left\{k\in\mathbb{Z} : \frac{2\pi k}{\sqrt{t}} \ge x + D\right\},$$

then $k_+(x,t) - k_-(x,t) \le 2D + \frac{4\pi}{\sqrt{t}}$. Now we have

$$\frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}}(e^{-2\cos(y\sqrt{t})} - C)\varsigma^{-2}(yn^{-\alpha}\sqrt{t})e^{-\frac{(y-x)^2}{2}}\,dy\right|$$

$$\le \varepsilon(e^2+C)\sup_{z\in\mathbb{R}}\varsigma^{-2}(z) + \frac{1}{\sqrt{2\pi}}\left|\int\limits_{\frac{2k_-(x,t)\pi}{\sqrt{t}}}^{\frac{2k_+(x,t)\pi}{\sqrt{t}}}(e^{-2\cos(y\sqrt{t})} - C)\varsigma^{-2}(yn^{-\alpha}\sqrt{t})e^{-\frac{(y-x)^2}{2}}\,dy\right|$$

$$\le \varepsilon(e^2+C)(\sup_{z\in\mathbb{R}}\varsigma^{-2}(z) + w_{\varsigma^{-2}}\left(\left(D+\frac{2\pi}{\sqrt{t}}\right)n^{-\alpha}\right)$$

$$+\frac{\varsigma^{-2}(x)}{\sqrt{2\pi}}\left|\int\limits_{\frac{2k_-(x,t)\pi}{\sqrt{t}}}^{\frac{2k_+(x,t)\pi}{\sqrt{t}}}(e^{-2\cos(y\sqrt{t})} - C)e^{-\frac{(y-x)^2}{2}}\,dy\right|$$

$$\le \varepsilon(e^2+C)(\sup_{z\in\mathbb{R}}\varsigma^{-2}(z) + w_{\varsigma^{-2}}\left(\left(D+\frac{2\pi}{\sqrt{t}}\right)n^{-\alpha}\right)$$

$$+\frac{\varsigma^{-2}(x)}{\sqrt{2\pi}}\sum_{k=k_-(x,t)}^{k_+(x,t)-1}\left|\int\limits_{\frac{2k\pi}{\sqrt{t}}}^{\frac{2(k+1)\pi}{\sqrt{t}}}(e^{-2\cos(y\sqrt{t})} - C)(e^{-\frac{(y-x)^2}{2}} - e^{-\frac{(\frac{2k\pi}{\sqrt{t}}-x)^2}{2}})\,dy\right|.$$

Here, in the last inequality, we have used the fact that $\int_{\frac{2k\pi}{\sqrt{t}}}^{\frac{2(k+1)\pi}{\sqrt{t}}}(e^{-2\cos(y\sqrt{t})} - C)\,dy = 0$, $k\in\mathbb{Z}$. If t is such that $\frac{2\pi}{\sqrt{t}} < \delta$ and $\frac{4\pi}{\sqrt{t}} < 1$, then

$$\sup_x \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}}(e^{-2\cos(y\sqrt{t})} - C)\varsigma(yn^{-\alpha}\sqrt{t})e^{-\frac{(y-x)^2}{2}}\,dy\right| \le$$

$$\le \varepsilon(e^2+C)\left(1+\frac{1}{\sqrt{2\pi}}\right)\left(\sup_{z\in\mathbb{R}}\varsigma^{-2}(z) + \varepsilon(e^2+C)w_{\varsigma^{-2}}\left(\left(D+\frac{1}{2}\right)n^{-\alpha}\right)\right). \tag{24}$$

Now we obtain (23) by passing to the limit in (24) first as $n\to\infty$ and then as $\varepsilon\to 0+$.

Condition (E) holds true with $\varrho(x) = \varsigma(0)\exp\cos x$. Condition (A) holds true obviously. At last, condition (B) holds true provided that the SDE

$$dX(t) = C^{-\frac{1}{2}}\varsigma(X(t))\,dW(t) \tag{25}$$

possesses the path-wise uniqueness property. Thus, the number of sign changes (3) converges weakly to the local time of the process X, defined by SDE (25), at the point 0, multiplied by

$$\varsigma^{-2}(0)\int_{\mathbb{R}}\exp[-2\cos z]\,P\Big(Z(1)\cdot z < 0\Big|Z(0) = z\Big)\,dz,$$

where the process Z is defined by the SDE

$$dZ(t) = \varsigma(0)e^{\cos Z(t)}\,dW(t).$$

Bibliography

1. I.I.Gikhman, *Some limit theorems for the number of intersections of the boundary of a domain by a random function*, Sci. Notes of Kiev Univ. **16** (1957), no. 10, 149 – 164. (Ukrainian)
2. I.I.Gikhman, *Asymptotic distributions for the number of intersections of the boundary of a domain by a random function*, Visnyk Kiev Univ., Ser. Astronomy, Mathematics and Mechanics **1** (1958), no. 1, 25 – 46. (Ukrainian)

3. N.I.Portenko, *The development of I.I.Gikhman's idea concerning the methods for investigating local behavior of diffusion processes and their weakly convergent sequences*, Theor. Probability and Math. Statist. **50** (1994), 7 – 22.
4. H.Al Farah, M.I.Portenko, *Limit theorem for the number of intersections of the fixed level by weakly convergent sequence of diffusion processes*, Preprint 2007.6 (2007), Institute of Math., Kiev, 24. (Ukrainian)
5. Yu.N.Kartashov, A.M.Kulik, *Invariance principle for additive functionals of Markov chain*, submitted (2006). (Russian; preprint, in English translation, available at arXiv:0704.0508v1)
6. I.I.Gikhman, A.V.Skorokhod, *Stochastic Differential Equations and Their Applications, 2-nd ed*, Kiev, Nauk. Dumka, 1982. (Russian)
7. E.B.Dynkin, *Markov Processes*, Moscow, Fizmatgiz, 1963. (Russian)
8. A.M.Kulik, *Markov approximation of stable processes by random walks*, Theory of Stochastic Processes **12(28)** (2006), no. 1-2, 87 – 93.
9. J.L.Doob, *Stochastic Processes*, NY, Wiley, 1953.
10. A.M.Kulik, *The optimal coupling property and its applications: limit theorems for non-elliptic diffusions and construction of canonical stochastic flow*, Theory of Stochastic Processes **9(25)** (2003), no. 1-2, 82 – 98.
11. N.Ikeda, S.Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, 1981.
12. O.F.Porper, S.D.Eidelman, *Asymptotic behavior of the classical and generalized one-dimensional parabolic equations of second order*, Trudy Mosk. Mat. Ob., vol. 36, Mosc. Univ. publ., 1978, pp. 85 – 130. (Russian)
13. S.Nakao, *On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations*, Osaka J. Math **9** (1972), 513 – 518.
14. J.M.Harrison, L.A.Shepp, *On skew Brownian motion*, Annals of Probability **9** (1981), no. 2, 309 – 313.

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