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ON A BAD DESCRIPTIVE STRUCTURE OF MINKOWSKI'S SUM OF CERTAIN SMALL SETS IN A TOPOLOGICAL VECTOR SPACE

For some natural classes of topological vector spaces, we show the absolute nonmeasurability of Minkowski's sum of certain two universal measure zero sets. This result can be considered as a strong form of the classical theorem of Sierpiński [8] stating the existence of two Lebesgue measure zero subsets of the Euclidean space, whose Minkowski's sum is not Lebesgue measurable.

Let E be a topological vector space over the field of all reals, and let A, B be any two subsets of E. As usual, we denote

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

The set $A + B$ is called the algebraic (or vector, or Minkowski's) sum of the two given point-sets A and B. As known, this operation is used in many fields of mathematics. For instance, Minkowski's sums of convex subsets of E play an important role in geometry, convex analysis, and probability theory. In particular, the theory of mixed volumes of convex bodies in a finite-dimensional Euclidean space E is essentially based on the notion of Minkowski's sum.

Supposing that A and B have nice structural (or descriptive) properties, one can ask whether $A + B$ possesses the same properties. In certain cases, the answer is positive. For instance, if both A and B are convex polyhedra, then $A+B$ is a convex polyhedron, too. On the other hand, there are examples which show that the operation of algebraic summation applied to subsets of the real line $\mathbf{R}$ (more generally, to subsets of E) sometimes essentially changes the structure of summands. Among many facts of this kind, let us mention the following four ones:

(1) there exists a set $D \subset \mathbf{R}$ of the first category and the Lebesgue measure zero such that $D + D = \mathbf{R}$; this is a well-known fact of analysis which easily follows from the equality $C + C = [0, 2]$, where C denotes the Cantor set on the real line (see, e.g., [1]);

(2) there exist two Borel subsets A and B of $\mathbf{R}$ such that $A + B$ is not Borel (B.S. Sodnomov's result [3], [4]);

(3) there exist the Lebesgue measure zero subsets X and Y of $\mathbf{R}$ such that $X + Y$ is not Lebesgue measurable (W. Sierpinski's result [8]);

(4) there exist the first category subsets U and V of $\mathbf{R}$ such that $U + V$ does not have the Baire property (W. Sierpinski's dual result [8]).

In connection with the above-mentioned facts and their further generalizations, see, e.g., [1] – [9], [17].

The main goal of this article is to demonstrate that Minkowski's sum of two very small subsets of a topological vector space E can have very bad descriptive properties in E, namely, can be absolutely nonmeasurable with respect to the class of completions of all nonzero σ-finite continuous (i.e. vanishing at singletons) Borel measures on E. Also, we

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shall see below that the case of a separable topological vector space E essentially differs from the case of a nonseparable E.

First of all, let us introduce an appropriate notion of the smallness of subsets of a given topological space.

Let T be a topological space. A set $X \subset T$ is said to be universal measure zero in T if, for every σ-finite continuous Borel measure μ on T, we have the equality $\mu^*(X) = 0$, where μ^* denotes the outer measure associated with μ.

Equivalently, X is universal measure zero if there exists no nontrivial σ-finite continuous measure on the Borel σ-algebra of X. Various constructions of uncountable universal measure zero subsets of uncountable Polish topological spaces are known in the literature. One of them is due to Luzin and is based on the canonical decomposition of a proper analytic subset of a Polish space into its Borel components (see, for instance, [10], [11]). Another construction uses the properties of the so-called Marczewski characteristic function of a sequence of sets (see, e.g., [12]). Much more delicate constructions of universally small sets, in terms of abstract σ-ideals with Borel bases and corresponding σ-algebras with the c.c.c. property, are presented in [13] and [14].

It should be noticed that the existence of a universal measure zero subset of $\mathbf{R}$ having the cardinality continuum cannot be established within ZFC theory (see [11]). However, assuming the Continuum Hypothesis, we easily get the existence of such sets. The same result can be obtained by assuming Martin's Axiom which is much weaker than CH and does not bound the cardinality of the continuum (denoted below by $\mathbf{c}$) from above.

It is obvious that all members of the σ-algebra generated by a Borel σ-algebra and the class of all universal measure zero subsets of a Polish space T are absolutely measurable in T, i.e. they belong to the domain of the completion of any σ-finite Borel measure on T. However, one cannot assert that all absolutely measurable subsets of T (i.e. Radon subspaces of T) can be obtained in this manner. Moreover, in those models of set theory, where all projective sets have nice descriptive properties (in particular, where the perfect subset property is valid), any non-Borel analytic subset of T does not belong to the above-mentioned σ-algebra but, as is well known, is absolutely measurable (i.e. Radon) in T.

Let us recall the definition of generalized Luzin sets which play an essential role in our further consideration.

Let T be an uncountable Polish topological space. We say that $X \subset T$ is a generalized Luzin set if $card(X) = \mathbf{c}$ and, for each first category set $Z \subset T$, we have $card(X \cap Z) < \mathbf{c}$.

In other words, a set $X \subset T$ with $card(X) = \mathbf{c}$ is a generalized Luzin set if X almost avoids all first category subsets of T.

It is well known that, under Martin's Axiom, there are generalized Luzin sets in T, and any such set is universal measure zero. Moreover, any Luzin subset X of the real line $\mathbf{R}$ has a much stronger property, namely, X is small in the Borel sense, which means that, for every given sequence of open subintervals of the real line, there exists a covering of X by intervals which are respectively congruent to the given ones.

In fact, the construction of a generalized Luzin set imitates the classical construction of Luzin (1914), where the Continuum Hypothesis is assumed instead of Martin's Axiom (see, e.g., [2], [10], [11], [15]). By modifying both of these constructions, it can be shown the existence of generalized Luzin sets having additional purely algebraic properties. For example, it was proved (under Martin's Axiom) that there exists a generalized Luzin set on the real line which simultaneously is a vector space over the field $\mathbf{Q}$ of all rational numbers. A much stronger statement was established by Erdös, Kunen, and Mauldin [16]. Namely, by assuming Martin's Axiom and using the method of transfinite induction, they proved that there exist two sets $X \subset \mathbf{R}$ and $Y \subset \mathbf{R}$ satisfying the following relations:

(a) both X and Y are generalized Luzin sets in $\mathbf{R}$;

(b) both X and Y are vector spaces over the field $\mathbf{Q}$;

(c) $X + Y = \mathbf{R}$ and $X \cap Y = \{0\}$.

The construction of the sets X and Y presented in [16] can be generalized, without any difficulties, to the case of a Polish topological vector space $E \neq \{0\}$. In particular, we readily infer from the corresponding generalized result that the algebraic sum of two universal measure zero sets in E sometimes coincides with the whole space E. Thus, Minkowski's sum of two very small subsets of E can be maximally large.

In this connection, it makes sense to consider the question whether Minkowski's sum of two universal measure zero sets can be very bad from the descriptive (i.e. measure-theoretic) point of view. First, let us give the precise definition of the so-called "bad" sets.

Let Y be a subset of a topological space T. We say that Y is absolutely nonmeasurable in T if, for every nonzero σ-finite continuous Borel measure μ on T, we have $Y \notin dom(\mu')$, where μ' denotes the completion of μ.

It turns out that absolutely nonmeasurable sets in an uncountable Polish space T admit a purely topological characterization.

Let Z be a subset of a topological space T. Recall that Z is a Bernstein set if, for each nonempty perfect set $P \subset T$, both intersections $P \cap Z$ and $P \cap (T \setminus Z)$ are nonempty.

The well-known argument due to Bernstein and based on the method of transfinite induction establishes the existence of a Bernstein set in any uncountable Polish space T (cf. [1], [2], [10], [15]). The same construction also works in the case of a complete metric space T, where T is of cardinality continuum and has no isolated points. Moreover, any construction of a Bernstein set in such a T needs an uncountable form of the Axiom of Choice. Numerous properties of Bernstein sets are discussed in [1], [2], [9], [10], [11], [15] with their applications in general topology, measure theory, real analysis, and the theory of Boolean algebras. As a rule, Bernstein sets are used for constructing various types of counterexamples (e.g., for showing that the famous Kolmogorov extension theorem is not a purely measure-theoretic statement and requires certain topological assumptions on given finite-dimensional probability measures).

Lemma 1. *Let Z be a subset of an uncountable Polish space T. The following two assertions are equivalent:*

1) Z is absolutely nonmeasurable in T;

2) Z is a Bernstein subset of T.

Proof. 1) $\Rightarrow$ 2). Assume that, for a set $Z \subset T$, relation 1) holds, and let us show that Z is a Bernstein set in T. Supposing, on the contrary, that Z is not a Bernstein set, we claim that, for some nonempty perfect set $P \subset T$, either $P \cap Z = \emptyset$ or $P \cap (T \setminus Z) = \emptyset$. We may assume, without loss of generality, that $P \cap Z = \emptyset$. According to the well-known theorem of descriptive set theory (see, e.g., [10]), the topological spaces P and $[0, 1]$ are Borel isomorphic. Let λ_0 denote the restriction of the Lebesgue measure λ to the Borel σ-algebra of $[0, 1]$. Using any Borel isomorphism between $[0, 1]$ and P, we can transfer λ_0 to P. In this manner, we obtain a Borel probability continuous measure μ on the space T which is concentrated on P, i.e. $\mu(P) = 1$, $\mu(T \setminus P) = 0$ and hence $\mu^*(Z) = 0$. Therefore, Z turns out to be measurable with respect to the completion of μ. This contradicts 1) and establishes the implication 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Let Z be a Bernstein set in T, and let μ be an arbitrary nonzero σ-finite Borel continuous measure on T. Denote, by μ', the completion of μ. We have to verify that Z is not measurable with respect to μ'. Suppose for a while that $Z \in dom(\mu')$. Then either $\mu'(Z) > 0$ or $\mu'(T \setminus Z) > 0$. We may assume, without loss of generality, that $\mu'(Z) > 0$. Since μ' is a Radon measure, there exists a compact set $K \subset Z$ such that $\mu'(K) > 0$. Since μ' is continuous, we claim that K is uncountable, so K (hence,

Z) contains a nonempty perfect subset, which contradicts the assumption that Z is a Bernstein set. The contradiction obtained shows the absolute nonmeasurability of Z.

Lemma 1 has thus been proved.

The next auxiliary proposition is essentially more profound.

Lemma 2. *Suppose that Martin's Axiom holds. Let $E \neq \{0\}$ be a Polish topological vector space. Then there exist two generalized Luzin sets $L_1 \subset E$ and $L_2 \subset E$ such that:*

a) both L_1 and L_2 are vector spaces over $\mathbf{Q}$;

b) $L_1 + L_2$ is a Bernstein subset of E.

Consequently, under Martin's Axiom, there exist two universal measure zero sets in E, whose algebraic sum is absolutely nonmeasurable in E.

The required sets L_1 and L_2 can be constructed by using the method of transfinite induction. A detailed construction of such sets is given in [21]. Note that only the case $E = \mathbf{R}$ was considered in [21], but the same argument works for any Polish topological vector space E, whose dimension is not equal to zero.

Remark 1. It should be underlined that the additional set-theoretic assumption in the formulation of Lemma 2 cannot be omitted. Indeed, there is a model of set theory, in which the Continuum Hypothesis does not hold, and the cardinality of any universal measure zero subset of $\mathbf{R}$ does not exceed the first uncountable cardinal ω_1 (see [11]). If U_1 and U_2 are any two universal measure zero subsets of $\mathbf{R}$ in that model, then we have

$$card(U_1 + U_2) \leq \omega_1 \cdot \omega_1 = \omega_1,$$

and, consequently, $U_1 + U_2$ cannot be a Bernstein set, because the cardinality of every Bernstein subset of an uncountable Polish space is always equal to $\mathbf{c}$.

Remark 2. It would be interesting to extend Lemma 2 to a more general case where E is an arbitrary uncountable Borel subgroup (Borel vector subspace) of a Polish topological group (Polish vector space). Note that the technique of generalized Luzin sets does not work for such an E, because it can happen that E is of the first category on itself. So, a different approach is needed here. In any case, we may assert that E contains quite many uncountable universal measure zero subsets and many absolutely nonmeasurable subsets (because E is always Borel isomorphic to $\mathbf{R}$). However, at this moment, we do not know whether there are two universal measure zero sets in E, whose algebraic sum is absolutely nonmeasurable in E.

It is also natural to ask whether Lemma 2 admits a generalization to the case of complete metrizable, but nonseparable topological vector spaces of cardinality $\mathbf{c}$ (for example, among such spaces is the classical Banach space l_∞ which plays an important role in many questions of functional analysis and probability theory). It turns out that the corresponding generalization of Lemma 2 is possible under the Continuum Hypothesis. However, the argument should be essentially changed in order to obtain the desired result.

We need the following auxiliary statement which is well known in topological measure theory (see, e.g., [15]).

Lemma 3. *Let T be a metrizable topological space, whose topological weight is not a real-valued measurable cardinal (in the sense of Ulam). Let μ be an arbitrary σ-finite Borel measure on T. Then μ has a separable support, i.e. there exists a closed separable set $S \subset T$ such that $\mu(T \setminus S) = 0$.*

Remark 3. As known, Martin's Axiom implies that the cardinality of the continuum is not real-valued measurable (see, e.g., [9], [11], [12]). This fact is a direct consequence of the existence of generalized Luzin sets on the real line $\mathbf{R}$.

We also need some further generalization of the notion of a generalized Luzin subset of a Polish space.

Let E be a set of cardinality continuum, and let $\mathcal{I}$ be a $\mathbf{c}$-additive ideal of subsets of E. We say that a set $X \subset E$ is a generalized Luzin set for $\mathcal{I}$ if $card(X) = \mathbf{c}$, and, for every set $Z \in \mathcal{I}$, we have $card(X \cap Z) < \mathbf{c}$.

Obviously, if Martin's Axiom holds, E is an uncountable Polish space, and $\mathcal{I}$ is the ideal of all first-category subsets of E, then the notion of a generalized Luzin set for $\mathcal{I}$ coincides with the usual notion of a generalized Luzin set in E.

Lemma 4. *Assume Martin's Axiom. Let E be a complete metrizable topological vector space, whose topological weight is equal to $\mathbf{c}$. There exist two universal measure zero sets $U_1 \subset E$ and $U_2 \subset E$ such that each of them is a vector space over $\mathbf{Q}$, and $U_1 + U_2$ is absolutely nonmeasurable in E.*

Proof. The argument is essentially based on the fact that, under our assumptions,

$$card(E) = \mathbf{c}$$

and the family of all separable subspaces of E generates a $\mathbf{c}$-additive ideal $\mathcal{I}$ of subsets of E. If X is a generalized Luzin set for $\mathcal{I}$, then, by definition, the intersection of X with any member of $\mathcal{I}$ has cardinality strictly less than $\mathbf{c}$.

Let μ be a σ-finite continuous Borel measure on E. In view of Lemma 3, there exists a closed separable subspace S of E such that $\mu(E \setminus S) = 0$. Since $card(X \cap S) < \mathbf{c}$, we easily deduce that $\mu^*(X \cap S) = 0$, whence it follows that $\mu^*(X) = 0$. In other words, X turns out to be a universal measure zero subset of E.

Further, by using the method of transfinite induction, we can construct (cf. [16]) two sets $U \subset E$ and $V \subset E$ satisfying the following relations:

(a) both U and V are generalized Luzin sets for $\mathcal{I}$;

(b) both U and V are vector spaces over $\mathbf{Q}$;

(c) $U \cap V = \{0\}$ and $U + V = E$.

Starting with these U and V and applying again the method of transfinite induction, we can show that there exists a vector subspace V' of V such that $U + V'$ is a Bernstein set in E (the argument is very similar to that in the proof of Lemma 2; cf. also [21]). Note that the set V' being a subset of a universal measure zero set is also universal measure zero. Finally, it follows from Lemmas 1 and 3 that any Bernstein subset of E is absolutely nonmeasurable in E, so we can put $U_1 = U$ and $U_2 = V'$.

Remark 4. It should be mentioned that if the cardinal $\mathbf{c}$ is real-valued measurable (in the sense of Ulam) and E is a metrizable topological space, whose topological weight is $\mathbf{c}$, then there exists a probability measure on the σ-algebra of all subsets of E which is the completion of some Borel continuous probability measure on E. Consequently, in this case, there are no absolutely nonmeasurable subsets of E, and the assertion of Lemma 4 fails to be true.

The following statement is a straightforward consequence of Lemmas 2 and 4 formulated above.

Theorem 1. *Suppose that the Continuum Hypothesis holds. Let E be an arbitrary complete metrizable topological vector space with $card(E) = \mathbf{c}$. There exist two universal measure zero sets $U_1 \subset E$ and $U_2 \subset E$ which are vector spaces over $\mathbf{Q}$ and whose Minkowski's sum $U_1 + U_2$ is absolutely nonmeasurable in E.*

Remark 5. We do not know whether the assertion of Theorem 1 remains valid under Martin's Axiom.

Some other (relatively recent) works were devoted to algebraic sums of small sets (see, for instance, [7], [17]–[21]).

Let us mention especially that similar questions are considered in [19] for any uncountable commutative group $(G, +)$, which is not assumed to be endowed with a topology, but is equipped with a nonzero σ-finite complete G-invariant (or G-quasiinvariant) measure μ. Obviously, for this μ, the question can be posed whether there are two μ-measure zero sets in G, whose algebraic sum is nonmeasurable with respect to μ. In this context, the following open problem seems to be of interest:

Let $(G,+)$ be an uncountable commutative group, and let μ be a nonzero σ-finite complete G-invariant (or, more generally, G-quasiinvariant) measure on G such that $A+B=G$ for some two μ-measure zero sets A and B. Do there exist two μ-measure zero sets X and Y such that $X+Y$ is nonmeasurable with respect to μ?

Moreover, a notion of an absolutely nonmeasurable subset of $(G,+)$ can be introduced in the following natural way.

We say that a set $Z \subset G$ is G-absolutely nonmeasurable (in G) if Z is nonmeasurable with respect to every nonzero σ-finite G-quasiinvariant measure on G.

For instance, if G is a separable infinite-dimensional Hilbert space and Z is a unit ball in G (more generally, if Z is a bounded convex body in G), then Z turns out to be G-absolutely nonmeasurable in G. This fact can be treated as a stronger form of the well-known theorem stating the nonexistence of a nonzero σ-finite translation-quasiinvariant Borel measure on a separable infinite-dimensional Hilbert space (see, e.g., [22]).

Also, a certain analog of the universal measure zero set can be introduced for an uncountable commutative group $(G,+)$. Namely, we say that a set $X \subset G$ is G-absolutely negligible (in G) if, for every σ-finite G-quasiinvariant (G-invariant) measure μ on G, there exists a G-quasiinvariant (G-invariant) measure μ' on G which extends μ and for which the equality $\mu'(X)=0$ is satisfied.

For instance, if G is a nonseparable normed vector space and X is a ball in G, then X is G-absolutely negligible in G. Clearly, this fact implies the nonexistence of a nonzero σ-finite translation-quasiinvariant Borel measure on any nonseparable normed vector space.

For uncountable vector spaces over $\mathbf{Q}$, a statement analogous to Theorem 1 is valid in terms of absolutely negligible and absolutely nonmeasurable sets. Namely, we have

Theorem 2. *Let E be a vector space over $\mathbf{Q}$ with $card(E) \geq \mathbf{c}$. There exist two E-absolutely negligible sets $X \subset E$ and $Y \subset E$ such that $X+Y$ is E-absolutely nonmeasurable in E. Therefore, under the Continuum Hypothesis, the same fact holds for any uncountable vector space over $\mathbf{Q}$.*

Proof. The proof of Theorem 2 for the case $E=\mathbf{R}$ can be found in [18]. It should be mentioned that the argument given in [18] is based on the absolute nonmeasurability of the unit ball in a separable infinite-dimensional Hilbert space. Actually, some nice purely geometric properties of this ball are essentially used in the argument of [18].

Now, if E is an arbitrary vector space over $\mathbf{Q}$ such that $card(E) \geq \mathbf{c}$, then there exists a surjective group homomorphism $f : E \to \mathbf{R}$. It is not difficult to verify that

(i) if a set $Z \subset \mathbf{R}$ is $\mathbf{R}$-absolutely negligible, then the set $f^{-1}(Z)$ is E-absolutely negligible;

(ii) if a set $Z \subset \mathbf{R}$ is $\mathbf{R}$-absolutely nonmeasurable, then the set $f^{-1}(Z)$ is E-absolutely nonmeasurable.

Taking into account the validity of Theorem 2 for $\mathbf{R}$ and using relations (i) and (ii), we readily infer the validity of Theorem 2 for a given vector space E.

Remark 6. It would be interesting to generalize Theorem 2 to the case of an arbitrary uncountable commutative group $(G,+)$. Note that, for this purpose, it suffices to obtain the required result for all commutative groups of cardinality ω_1. In other words, if the assertion of Theorem 2 is valid for any commutative group of cardinality ω_1, then the same assertion holds true for every uncountable commutative group. The latter fact is a consequence of the well-known algebraic proposition which states that, among the homomorphic images of an uncountable commutative group $(G, +)$, there always is a group of cardinality ω_1.

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