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DISTRIBUTION OF THE MAXIMUM OF THE CHENTSOV RANDOM FIELD

Let $D = [0, 1]^2$ and $X(s, t)$, $(s, t) \in D$, be a two-parameter Chentsov random field. The aim of this paper is to find the probability distribution of the maximum of $X(s, t)$ on a class of polygonal lines.

1. Introduction

Let $\{X(s, t) : s, t \geq 0\}$ be a standard Chentsov field of two parameters that is a separable real Gaussian stochastic process such that

1) $X(0, t) = X(s, 0) = 0$ for all $s, t \in [0, 1]$;
2) $E[X(s, t)] = 0$ for all $s, t \geq 0$;
3) $E[X(s, t)X(s_1, t_1)] = \min\{s, s_1\} \min\{t, t_1\}$ for all (s, t) and $(s_1, t_1) \in D$.

This definition is given by Yeh [6] in 1960. Another (equivalent) definition is given by Chentsov [7] in 1955 in terms of the probability density of $X(s, t)$. Yeh showed that the sample paths of this field are continuous with probability one and $X(s, t)$ has independent stationary increments in the plane.

The probability distributions of functionals of a Chentsov random field like $M = \max_{(s,t)\in D} X(s, t)$ are not yet known. Some trivial probability distribution theory for $X(s, t)$ can be obtained by using the known results about the standard Wiener process.

The distribution of the supremum of a Chentsov random field on the curve $f(s)$, where $f(s)$ is a non-decreasing function of s, can be obtained, since a transformation of $X(s, f(s))$ is equivalent to a one-dimensional standard Wiener process.

The probability distribution of the supremum of $X(s, t)$ on the boundary of a unit square is obtained by Paranjape and Park [1]. This probability is of its own interest, and it gives a nice lower bound for the probability distribution of the supremum of $X(s, t)$ over the whole unit square D, which is unknown yet.

Park and Skoug [5] have found the probability that $X(s, t)$ crosses a barrier of the type $ast + bs + ct + d$ on the boundary $\partial\Lambda$, where $\Lambda = [0, S] \times [0, T]$ is a rectangle. Later on, I. Klesov [3] considered a probability of the form

$$P(L, g) = P\left\{\sup_L X(s, t) - g(s, t) < 0\right\}, \tag{1}$$

where X is a Chentsov random field on $D = [0, 1]^2$, $L \subset D$, and g is an almost everywhere Lebesgue continuous function on D. He presented results, where $g(s, t)$ is a linear function and L is a polygonal line with one point of break. Klesov and Kruglova [8] considered a probability of the form (1), where L is a polygonal line with two points of break.

The main purpose of this paper is to evaluate the probability distribution of the form (1), where $g(s, t) = \lambda$ and L is a polygonal line with several points of break. we can

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express this distribution in a very useful form: as an expression of the "tail" of the two-dimensional Gauss process.

2. Auxiliary results

Lemma 1. *(Doob's Transformation Theorem) [2]. Let $X(t)$ be any Gaussian process with covariance function $R(s,t) = u(s)v(t)$, $s \le t$, if the ratio $a(t) = u(t)/v(t)$ is continuous and strictly increasing with inverse $a_1(t)$, then $w(t)$ and $Y(a_1(t))/v(a_1(t))$ are stochastically equivalent processes.*

Lemma 2. *(Malmquist's Theorem 1) [4]. For a standard Wiener process $w(t)$ and for $b > 0$, $a \ge 0$, $s_1 \le at' + b$,*

$$P\left\{w(t) \le at + b, 0 < t < t' | w(t') = s_1\right\} =$$

$$= P\left\{w(t) \le bt + (at' + b - s_1)/t', 0 < t < \infty\right\} =$$

$$= 1 - \exp\left\{-2b(at' + b - s_1/t'\right\}.$$

Lemma 3. *(Malmquist's Theorem 2) [4]. For a standard Wiener process $w(t)$ and for $b > 0$, $a \ge 0$,*

$$P\left\{w(t) \le at + b, x < t \le y | w(x) = s_1, w(y) = s_2\right\} =$$

$$= 1 - \exp\left\{-\frac{2R}{1-R^2} \cdot \frac{P_1 - s_1}{\sqrt{x}} \cdot \frac{P_2 - s_2}{\sqrt{y}}\right\},$$

where $R = \sqrt{\frac{x}{y}}$, $s_1 \le P_1 = ax + b$, $s_2 \le P_2 = ay + b$.

Let L be a line as shown in Fig. 1 and given by the formula

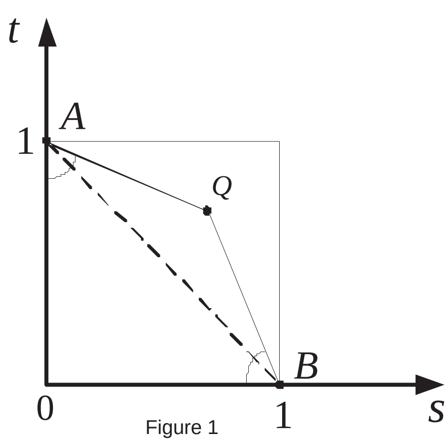

Figure 1

(2) $$L = \left\{(s,t) : sa^{-1} + t = 1, s \le k; s + tb^{-1} = 1, s > k, (s,t) \in D\right\},$$

where $\tan\alpha = a$, $\tan\beta = b$, $k = \frac{a(b-1)}{ab-1}$, $\alpha, \beta > \frac{\pi}{4}$.

Theorem 1. *(Paranjape and Park)[1]. Let $\{X(s,t) : s,t \geq 0\}$ be a standard Chentsov field. Then*

$$
\begin{aligned}
&P\left\{\sup_{(s,t)\in L} X(s,t) \leq \lambda\right\} \\
&\quad = \Phi\left(\frac{\lambda(a+c)}{a\sqrt{c}}\right) - \exp\left\{\frac{-2\lambda^2}{a}\right\}\Phi\left(\frac{\lambda(c-a)}{a\sqrt{c}}\right) - \\
&\quad - \exp\left\{\frac{-2\lambda^2}{b}\right\}\Phi\left\{\frac{\lambda(1-bc)}{b\sqrt{c}}\right\} + \exp\left\{-2\lambda^2(a^{-1}+b^{-1}-2)\right\} \\
&\qquad \times \Phi\left\{\lambda c^{-1/2}(b^{-1}-c-2)\right\}.
\end{aligned} \tag{3}
$$

3. Main results and proofs

Let L be a line as shown in Fig. 2 and given by the formula

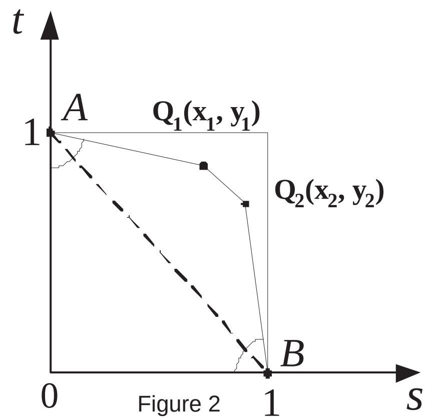

Figure 2

$$
L = \begin{cases} t = 1 - \frac{s(1-y_1)}{x_1}, & s \in [0, x_1] \\ t = -\frac{s(y_1-y_2)}{x_2-x_1} + \frac{x_2y_1 - x_1y_2}{x_2-x_1}, & s \in (x_1, x_2] \\ t = -\frac{sy_2}{1-x_2} + \frac{y_2}{1-x_2}, & s \in (x_2, 1]. \end{cases} \tag{4}
$$

Theorem 2. *Let $X(s,t)$ be a standard Chentsov random field on a unit square. Let the polygonal line L have two points of break $Q_1(x_1,y_1)$ and $Q_2(x_2,y_2)$ and be given by formula (4). Let the coordinates of Q_1 and Q_2 satisfy the conditions*

1) $y_2 < y_1$;
2) $\frac{x_2}{y_2} > \frac{x_1}{y_1}$.

Then

$$
P_2 = P\left\{\sup_{(s,t)\in L} X(s,t) < \lambda\right\}
$$

$$
\times \int_{-\infty}^{\frac{\lambda}{y_1}} \int_{-\infty}^{\frac{\lambda}{y_2}} \frac{1}{2\pi\sqrt{\frac{x_1}{y_1}\left(\frac{x_2}{y_2}-\frac{x_1}{y_1}\right)}} \exp\left\{-\frac{u_1^2}{\frac{2x_1}{y_1}}\right\} \exp\left\{-\frac{(u_2-u_1)^2}{2\left(\frac{x_2}{y_2}-\frac{x_1}{y_1}\right)}\right\}
$$

$$
\times \left(1 - \exp\left\{-\frac{2\lambda y_1}{x_1}\left(\frac{\lambda}{y_1}-u_1\right)\right\}\right)\left(1 - \exp\left\{-2\lambda\left(\frac{\lambda}{y_2}-u_2\right)\right\}\right)
$$

$$\times\left(1-\exp\left\{-\frac{2(\lambda-u_1y_1)(\lambda-u_2y_2)}{(x_2y_1-x_1y_2)}\right\}\right)\,du_1du_2 \tag{5}$$

Corollary 1. *Passing to the limit as $Q_1 \longrightarrow Q_2$ and using (5), we obtain a result which agrees with Park's result for a polygonal line with a single point of break (Theorem 1).*

Let us denote that $x_0 = 0$, $x_{n+1} = 1$, $y_0 = 1$, $y_{n+1} = 0$. Let L be a line given by the formula

$$L = \{(s,t) : t = v(s), s \in [0,1]\}\,. \tag{6}$$

For which $(x_1, y_1), \dots, (x_n, y_n)$ are the points of break where

$$v(s) = \sum_{i=1}^{n+1}\left(-\frac{s(y_{i-1}-y_i)}{x_i-x_{i-1}} + \frac{x_iy_{i-1}-x_{i-1}y_i}{x_i-x_{i-1}}\right) I_{(x_{i-1};x_i]}(s).$$

Let us denote $\Delta_0 = 0, \Delta_i = \frac{x_i}{y_i}, i = \overline{1,n}, \Delta_{n+1} = \infty$.

The following theorem is a generalization of Theorem 2.

Theorem 3. *Let $X(s,t)$ be a standard Chentsov random field on a unit square. Let $u_0 = u_{n+1} = 0$. Let the polygonal line L have n points of break and be given by formula (6). Let the coordinates of these points satisfy the conditions*

1) $y_1 > \dots > y_n$;
2) $\frac{x_1}{y_1} < \dots < \frac{x_n}{y_n}$.

Then

$$P_n = P\left\{\sup_{(s;t)\in L} X(s;t) < \lambda\right\} = \int_{-\infty}^{\frac{\lambda}{y_1}} \dots \int_{-\infty}^{\frac{\lambda}{y_n}} \prod_{i=1}^{n} \varphi_{0,\Delta_i-\Delta_{i-1}}(u_i - u_{i-1})$$

$$\times\prod_{i=1}^{n+1}\left(1-\exp\left\{-\frac{2\left(\frac{\lambda}{y_{i-1}}-u_{i-1}\right)\left(\frac{\lambda}{y_i}-u_i\right)}{(\Delta_i-\Delta_{i-1})}\right\}\right)du_1\dots du_n$$

where $\varphi_{0,\Delta}(u)$ is the density of the Gaussian random variable with variance Δ.

Proof. Let the restriction of $X(s,t)$ over L be denoted by $w_1(s)$. Then

$$w_1(s) = X(s, v(s))$$

Let us find the derivation of $v(s)$.

$v'(s) = \sum_{i=1}^{n+1} -\frac{(y_{i-1}-y_i)}{x_i-x_{i-1}} I_{(x_{i-1};x_i]}(s) < 0$ because of conditions over coordinates of points. This means that $v(s)$ is monotone decreasing function.

Using the covariance property of $X(s,t)$, we can write

$$\mathrm{cov}(w_1(s_1), w_1(s_2)) = \mathrm{cov}\left(X(s_1, v(s_1)), X(s_2, v(s_2))\right) = s_1v(s_2), 0 < s_1 \leqslant s_2 \leqslant 1$$

$a(s) = \frac{s}{v(s)}$ is continuous monotone increasing function. We can write $a(s)$ in an explicit form:

$$a(s) = \sum_{i=1}^{n+1}\frac{s}{-\frac{s(y_{i-1}-y_i)}{x_i-x_{i-1}} + \frac{x_iy_{i-1}-x_{i-1}y_i}{x_i-x_{i-1}}} I_{(x_{i-1},x_i]}(s)$$

It is enough to prove a continuity $a(s)$ in the points $x_i, i = \overline{1,n}$:

$$a(x_i) = \frac{x_i}{-\frac{x_i(y_{i-1}-y_i)}{x_i-x_{i-1}} + \frac{x_iy_{i-1}-x_{i-1}y_i}{x_i-x_{i-1}}} = \frac{x_i}{y_i}.$$

$$a(x_i+) = \frac{x_i}{-\frac{x_i(y_i-y_{i+1})}{x_{i+1}-x_i} + \frac{x_{i+1}y_i-x_iy_{i+1}}{x_{i+1}-x_i}} = \frac{x_i}{\frac{y_i(x_{i+1}-x_i)}{x_{i+1}-x_i}} = \frac{x_i}{y_i}.$$

That is $a(x_i) = a(x_{i+})$ continuous in the point x_i. That is $a(s)$ continuous in $(0;1)$. s is a monotone increasing function and $v(s)$ is a monotone decreasing function. That is why $a(s)$ is a monotone increasing function. For $a(s)$ the inverse will be the function:

$$a^{-1}(s) = \sum_{i=1}^{n+1} \frac{s(x_iy_{i-1} - x_{i-1}y_i)}{s(y_{i-1} - y_i) + x_i - x_{i-1}} I_{[\Delta_{i-1},\Delta_i)}.$$

It is necessary to notice that $v(0) = -\frac{x_0(y_0-y_1)}{x_1-x_0} + \frac{x_1y_0-x_0y_1}{x_1-x_0} = y_0 = 1$ and $v(1) = -\frac{x_{n+1}(y_n-y_{n+1})}{x_{n+1}-x_n} + \frac{x_{n+1}y_n-x_ny_{n+1}}{x_{n+1}-x_n} = y_{n+1} = 0$. That is why $a(0) = 0$ and $\lim\limits_{t\to 1} a(t) = \infty$.

$$\frac{1}{v(a^{-1}(s))} = \sum_{i=1}^{n+1} \left(\frac{s(y_{i-1} - y_i) + x_i - x_{i-1}}{x_iy_{i-1} - x_{i-1}y_i} \right) I_{[\Delta_{i-1},\Delta_i)}(s)$$

The functions $a(s)$ and $v(\cdot)$ satisfy the conditions of Doob's transformation theorem. Thus,

$$w^*(s) = \sum_{i=1}^{n+1} \left(\frac{s(y_{i-1} - y_i) + x_i - x_{i-1}}{x_iy_{i-1} - x_{i-1}y_i} \right)$$
$$\times w_1 \left(\frac{s(x_iy_{i-1} - x_{i-1}y_i)}{s(y_{i-1} - y_i) + x_i - x_{i-1}} \right) I_{[\Delta_{i-1},\Delta_i)}(s)$$

and $w(t)$ are stochastically equivalent processes.

$$P_n(\lambda) = P\left\{ \sup_{(s;t)\in L} X(s;t) < \lambda \right\} = P\left\{ \sup_{s\in[0,1]} X(s;v(s)) < \lambda \right\}$$
$$= P\left\{ \sup_{s\in[0,1]} w_1(s) < \lambda \right\} = P\left\{ \sup_{s\in[0,\infty)} w_1\left(a^{-1}(s)\right) < \lambda \right\}$$
$$= P\left(\bigcap_{s\geqslant 0} \left\{ w_1\left(a^{-1}(s)\right) < \lambda \right\} \right)$$
$$= P\left(\bigcap_{s>0} \left\{ w_1\left(a^{-1}(s)\right) < \lambda \right\} \bigcap \left\{ X\left(a^{-1}(0), v\left(a^{-1}(0)\right)\right) < \lambda \right\} \right)$$
$$= P\left(\bigcap_{s>0} \left\{ w_1\left(a^{-1}(s)\right) < \lambda \right\} \bigcap \Omega \right)$$

Because $X\left(a^{-1}(0), v\left(a^{-1}(0)\right)\right) = X(0,1) = 0$ and that is why

$$\left\{ X\left(a^{-1}(0), v\left(a^{-1}(0)\right)\right) < \lambda \right\} = \Omega$$

$$P_n = P\left(\bigcap_{s>0} \left\{ w_1\left(a^{-1}(s)\right) < \lambda \right\} \right) = P\left\{ \bigcap_{s>0} \frac{w_1\left(a^{-1}(s)\right)}{v\left(a^{-1}(s)\right)} - \frac{\lambda}{v\left(a^{-1}(s)\right)} < 0 \right\} =$$
$$= P\left\{ \sup_{s\in(0,\infty)} \frac{w_1\left(a^{-1}(s)\right)}{v\left(a^{-1}(s)\right)} - \frac{\lambda}{v\left(a^{-1}(s)\right)} < 0 \right\} = P\left\{ \sup_{s\in(0,\infty)} w(t) - \frac{\lambda}{v\left(a^{-1}(s)\right)} < 0 \right\} =$$
$$= P\left\{ w(t) < \frac{\lambda(x_i - x_{i-1})}{x_iy_{i-1} - x_{i-1}y_i} + \frac{\lambda t(y_{i-1} - y_i)}{x_iy_{i-1} - x_{i-1}y_i}; t \in (\Delta_{i-1};\Delta_i], i = \overline{1, n+1} \right\}$$

$$= \int_{-\infty}^{\frac{\lambda}{y_1}} \dots \int_{-\infty}^{\frac{\lambda}{y_n}} \frac{1}{(2\pi)^{n/2}} P\left\{ w(s) < \lambda + \frac{(1-y_1)s\lambda}{x_1}, s \in (0;\Delta_1] \,|w(\Delta_1) = u_1 \right\} \times$$

$$\times \prod_{i=2}^{n} P\Bigg\{ w(t) < \frac{\lambda(x_i - x_{i-1})}{x_i y_{i-1} - x_{i-1} y_i} + \frac{\lambda t(y_{i-1} - y_i)}{x_i y_{i-1} - x_{i-1} y_i}; t \in (\Delta_{i-1};\Delta_i] \;\Big|\; w(\Delta_{i-1}) = u_{i-1}, w(\Delta_i) = u_i \Bigg\}$$

$$\times P\left\{ w(t) < \frac{\lambda(1-x_n)}{y_n} + \lambda t,\ t > \Delta_n | w(\Delta_n) = u_n \right\} \times \prod_{i=1}^{n} \frac{\varphi_{0,\Delta_i - \Delta_{i-1}}(u_i - u_{i-1})}{\sqrt{\Delta_i - \Delta_{i-1}}} du_i.$$

Then, by using Lemma 2 and Lemma 3, we get

$$P_n = P\left\{ \sup_{(s;t)\in L} X(s;t) \leqslant \lambda \right\}$$

$$= \int_{-\infty}^{\frac{\lambda}{y_1}} \dots \int_{-\infty}^{\frac{\lambda}{y_n}} \prod_{i=1}^{n+1} \left(1 - \exp\left\{ -\frac{2\left(\frac{\lambda}{y_{i-1}} - u_{i-1}\right)\left(\frac{\lambda}{y_i} - u_i\right)}{(\Delta_i - \Delta_{i-1})} \right\} \right)$$

$$\times \prod_{i=1}^{n} \varphi_{0,\Delta_i - \Delta_{i-1}}(u_i - u_{i-1}) du_1 \dots du_n$$

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