Theory of Stochastic Processes
Vol. 14 (30), no. 1, 2008, pp. 49–59

UDC 519.21

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ON SOME CHARACTERISTICS OF THE CLAIM SURPLUS PROCESS

The distribution of characteristics, which describes the behaviour of the claim surplus process after the ruin, is investigated and compared with some previous results on distributions of risk functionals. Main attention is given to total durations of a sojourn time for risk processes in a risk (red) zone and in a survival (green) zone.

The characteristics of the behaviour of classic risk processes after a ruin are closely related to the overjump functionals of semicontinuous Poisson processes. The distributions of the overjump functionals for processes with stationary independent increments are studied by many authors (particularly in articles [1]–[3] and in monographs [4]–[6]). But a lot of results of the mentioned distributions (obtained in the boundary-value problems for processes) sometimes is ignored in the studies of applied problems. Some risk characteristics under consideration are studied in [7]–[11].

In [10], the following classic risk reserve process $U(t)$ is considered:

$$U(t) = u + ct - S(t),\ S(t) = \sum_{k\le\nu(t)} \xi_k,\ u \ge 0,$$

$$F(x) = \mathrm{P}\{\xi_k < x\},\ x \ge 0;\ m(z) = \mathrm{E}e^{z\xi_k},\ \forall k \ge 1;\ \overline{F}(x) = 1 - F(x).$$

Let $\nu(t)$ be a simple Poisson process with intensity $\lambda > 0$. The ruin time, the severity (penalty) of a ruin, and the red time for $u = 0$ (according to [10]) were denoted correspondingly

$$T = \tau^-(0),\ Y = U(T), \widetilde{T}_1 = \tau'(0) - T,\ \tau'(0) = \inf\{t > T, ct - S(t) > 0\}.$$

Instead of $U(t)$, we consider a claim surplus process $\zeta(t)$ (which is lower semicontinuous) with initial capital $u > 0$

$$\zeta(t) = S(t) - ct,\ \zeta^+(t) = \sup_{0\le t'\le t} \zeta(t') \underset{t\to\infty}{\longrightarrow} \zeta^+,$$

$$\tau^+(u) = \inf\{t > 0; \zeta(t) > u\}, \tau^+(0) \doteq T;$$

$$\gamma^+(u) = \zeta(\tau^+(u)) - u,\ u \ge 0,$$

$$\gamma_+(u) = u - \zeta(\tau^+(u) - 0),\ \gamma^+_u = \gamma^+(u) + \gamma_+(u);$$

$$\tau'(u) = \inf\{t > \tau^+(u), \zeta(t) < u\};$$

and, instead of $\widetilde{T}_1$, we introduce the red time $T'(u)$ $(u \ge 0,\ T'(0) \doteq \widetilde{T}_1)$

$$T'(u) = \begin{cases} \tau'(u) - \tau^+(u), & \tau^+(u) < \infty; \\ \infty, & \tau^+(u) = \infty. \end{cases}$$

2000 *AMS Mathematics Subject Classification.* Primary 60G50; Secondary 60K10.

Key words and phrases. Claim surplus process, risk reserve process; sojourn time in red zone and in survival zone; multivariate and marginal ruin functions.

This article was partly supported by Deutshe Forschungsgemeinschaft.

The risk level $y = u > 0$ divides the right half-plane on a risk "red" zone $\{y > u\}$ and on a survival "green" zone $\{y \le u\}$. The ruin time $\tau^+(u)$ defines the duration of the first survival "green" period, whereas $T'(u)$ does the duration of the first "red" period.

In what follows, we use the notation of a random variable $\theta_s > 0$:

$$\mathrm{P}\{\theta_s > t\} = e^{-st},\ s > 0,\ t > 0.$$

All risk characteristics for $U(t)$ and $\zeta(t)$ are considered under the condition of the positiveness of a security load $\delta > 0$,

$$\delta = \frac{c - \lambda\mu}{\lambda\mu} = \frac{c}{\lambda\mu} - 1\ (\mu = \mathrm{E}\xi_1,\ \mu_2 = \mathrm{E}\xi_1^2,\ m = \mathrm{E}\zeta(1)).$$

We need some relations from [3] and [13] (see Theorem 3 in [13] and Corollary 1 in [3] or in [13]) which are gathered in

Lemma. *If $m = \lambda\mu - c < 0$ $(\delta > 0)$, then, for $u \ge 0$,*

$$g_u(s) =: \mathrm{E}[e^{-sT'(u)}, T'(u) < \infty] = \frac{\lambda}{c}\int_0^\infty e^{-z\rho_-(s)}\overline{F}(u+z)dz +$$

$$+\frac{\lambda}{|m|}\int_0^\infty e^{-z\rho_-(s)}\int_0^u \overline{F}(u+z-y)dP_+(y)dz, \tag{1}$$

where we denote $P_+(u) = \mathrm{P}\{\zeta^+ < u\} = \delta(u) = \phi(u),\ \Psi(u) = \overline{\phi}(u)$,

$$\overline{F}(x) = 1 - F(x), \overline{\overline{F}}(x) = \int_x^\infty \overline{F}(y)dy, x > 0.$$

$$g_u(0) = \mathrm{P}\{T'(u) < \infty\} = \frac{\lambda}{c}\overline{\overline{F}}(u) + \frac{\lambda}{|m|}\int_{0+}^u \overline{\overline{F}}(u-y)dP_+(y), \tag{2}$$

where $f(s) = -\rho_-(s)$ is the root of Lundberg's equation

$$s = \lambda[m(f(s)) - 1] - cf(s),\ m(f) = \lambda\int_0^\infty e^{fx}dF(x).$$

For $u = 0$ and $\delta > 0$, $P_+(u) \underset{u\to 0}{\longrightarrow} = p_+ = \mathrm{P}\{\zeta^+ = 0\}$

$$g_u(s) \underset{u\to 0}{\longrightarrow} g(s) =: \mathrm{E}[e^{-sT'(0)}, T'(0) < \infty] = \frac{\lambda}{c}\int_0^\infty e^{-x\rho_-(s)}\overline{F}(x)dx. \tag{3}$$

The m.g.f. of $\{\tau^+(u), \gamma^+(u)\}$

$$q_+(s,u,z) =: \mathrm{E}[e^{-s\tau^+(u) - z\gamma^+(u)}, \tau^+(u) < \infty]$$

is defined for $u \ge 0$ by the relations

$$q_+(s,u,z) = \int_{0-}^u G_0(s, u-y, z)dP_+(s,y),$$

$$q_+(s,0,z) = \mathrm{E}[e^{-s\tau^+(0) - z\gamma^+(0)}, \tau^+(0) < \infty] = p_+(s)G_0(s,0,z), \tag{4}$$

where $p_+(s) = \mathrm{P}\{\zeta^+(\theta_s) = 0\}, P_+(s,y) = \mathrm{P}\{\zeta^+(\theta_s) < y\}, y > 0$; moreover,

$$G_0(s,u,z) = \frac{s^{-1}\lambda\rho_-(s)}{\rho_-(s) - z}\int_0^\infty [e^{-zy} - e^{-\rho_-(s)y}]dF(u+y),$$

$$G_0(0,u,z) = \frac{\lambda}{c}\int_0^\infty e^{-zy}\overline{F}(u+y)dy \quad (c\rho_-(s)p_+(s) = s). \tag{5}$$

From $(4) - (5)$ (as $s \to 0$), the m.g.f. of $\gamma^+(u)$ is defined as

$$g_+(u,z) = q_+(0,u,z) = \frac{\lambda}{c}\int_0^\infty e^{-zx}\overline{F}(u+x)dx +$$

$$+\frac{\lambda}{|m|}\int_0^\infty e^{-zx}\overline{F}(u-y+x)dxdP_+(y). \tag{6}$$

By comparing (1) *and* (6), *the corrected Dos Reis relation is established:*

$$g_u(s) = g_+(u, \rho_-(s)) = \mathrm{E}[e^{-\gamma_+(u)\rho_-(s)}, \tau^+(u) < \infty]. \tag{7}$$

The m.g.f. of $\tau^+(u)$,

$$q_+(s,u) = q_+(s,u,0) = \overline{P}_+(s,u) = \mathrm{P}\{\zeta^+(\theta_s) > u\},$$

is represented by the convolution (see (4) for $z = 0$*)*

$$q_+(s,u) = p_+(s)G_0(s,u) + \int_{0+}^{u} G_0(s, u-y)dP_+(s,y), \tag{8}$$

$$G_0(s,u) = G_0(s,u,0) = s^{-1}\lambda\rho_-(s)\int_0^\infty e^{-\rho_-(s)y}\overline{F}(u+y)dy.$$

By substitution G_0 *in* (8), $q_+(s,u)$ *is defined by the relation*

$$q_+(s,u) = \frac{\lambda}{c}\int_0^\infty e^{-\rho_-(s)y}\overline{F}(u+y)dy +$$

$$+\frac{\lambda}{s}\rho_-(s)\int_{0+}^{u}\int_0^\infty e^{-\rho_-(s)y}\overline{F}(u+y-z)dP_+(s,z)dy. \tag{9}$$

After the Laplace transformation with respect to u, (8) *is reduced to the relation*

$$\widetilde{q}_+(s,\nu) =: \int_0^\infty e^{-\nu u}q_+(s,u)du = \frac{\widetilde{G}_0(s,\nu)}{1+\nu\widetilde{G}_0(s,\nu)}, \tag{10}$$

$$\widetilde{G}_0(s,\nu) = \frac{\lambda\rho_-(s)s^{-1}}{\rho_-(s)-\nu}\int_0^\infty [e^{-\nu z} - e^{-\rho_-(s)z}]\overline{F}(z)dz.$$

It should be remarked that, for $s \to 0$,

$$\widetilde{q}_+(\nu) = \lim_{s\to 0}\widetilde{q}_+(s,\nu) = \frac{1}{\nu}(1 - \mathrm{E}e^{-\nu\zeta^+}) = \frac{\widetilde{C}_0(0,\nu)}{1+\nu\widetilde{C}_0(0,\nu)}, \tag{11}$$

$$\mathrm{E}e^{-\nu\zeta^+} = \frac{1}{1+\nu\widetilde{C}_0(0,\nu)},\ \widetilde{G}_0(0,\nu) = \frac{\lambda}{|m|\nu}\int_0^\infty [1-e^{-\nu z}]\overline{F}(z)dz.$$

The last relation is a rephrasing of the Pollaczec–Khinchin formula for $\mathrm{E}e^{-\nu\zeta^+}$. Comparing (3) and (9) for $u = 0$, we obtain the relation

$$q_+(s,0) = q_+(s) = g(s) = \frac{\lambda}{c}\int_0^\infty e^{-z\rho_-(s)}\overline{F}(z)dz, \tag{12}$$

which yields $\tau^+(0) \doteq T'(0)$.

Comparing (1) and (9) for $u > 0$, we can say that $\tau^+(u)$ and $T'(u)$ have a common "defect" of the distribution. But their m.g.f. are different. Hence, the following important assertion is true.

Corollary 1. *For* $u = 0$, *the first regeneration period of* $\zeta(t)$

$$\tau_1'(0) = \inf\{t > \tau_1^+(0) : \zeta(t) < 0\} \doteq \tau_1^+(0) + T_1'(0) \tag{13}$$

consists of two random variables with a common m.g.f. (12) *and*

$$\varphi(s) = \mathrm{E}[e^{-s\tau_1'(0)}, \tau_1'(0) < \infty] = \mathrm{E}[e^{-sT_1'(0)}, \zeta^+(\theta_s) > 0]. \tag{14}$$

For $u > 0$, *the first regeneration period*

$$\tau_1'(u) = \inf\{t > \tau_1^+(u) : \zeta(t) < u\} \doteq \tau_1^+(u) + T_1'(u) \tag{15}$$

consists of two random variables $\tau_1^+(u), T_1'(u)$ *which have a common defect of the distribution (see* (2)*), but their m.g.f.* $q_+(s,u)$ *and* $g_u(s)$ *are different (see* (1) *and* (9)*) and*

$$\varphi_u(s) = \mathrm{E}[e^{-s\tau_1'(u)}, \tau_1'(u) < \infty] = \mathrm{E}[e^{-sT_1'(u)}, \zeta^+(\theta_s) > u] < q_+(s,u). \tag{16}$$

The Laplace transform of $q_+(s,u)$ *is defined by* (10). *The regeneration periods* $\tau'_k(u)$ *for* $u \geq 0$

$$\tau'_k(u) \doteq \tau^+_k(0) + T'_k(0), \ \ k \geq 2, \tag{17}$$

consist of two random variables and

$$\mathrm{E}[e^{-s\tau'_k(u)}, \tau'_k(u) < \infty] = \mathrm{E}[e^{-sT'_k(0)}, \zeta^+(\theta_s) > 0] < q_+(s). \tag{18}$$

Now let us compare (1)–(3) with some results in [10], where, for $u = 0$, the m.g.f. of $Y \doteq \gamma^+(0)$ (according to the notation in [10]) is defined by the relation

$$M_Y(z,0) = \mathrm{E}[e^{zY}|u = 0] = \frac{1}{\mu z}[m(z) - 1]. \tag{19}$$

It should be remarked that $Y = U(\tau^-(0))$ depends on $\tau^-(0)$ ($\gamma^-(x)$ depends on $\tau^-(x)$ for $x < 0$). In the case where claims $\xi_k > 0$ are exponentially distributed,

$$\mathrm{E}[e^{z\gamma^-(x)/\tau^-(u)<\infty}] = \frac{b}{b-z} = m(z), \forall x \leq 0, m(z) - 1 = \frac{z}{b-z}.$$

In the general case, relation (19) and the following two relations for the red time $\widetilde{T}_1 \doteq T'_1(0)$

$$\mathrm{E}[e^{-s\widetilde{T}_1}|u = 0] = M_{\widetilde{T}_1}(s,0) = \frac{1}{\mu f(s)}[m(f(s)) - 1], \tag{20}$$

$$M_{\widetilde{T}}(s,0) = 1 + \theta - \frac{s}{s\mu f(s)}, \ \ \theta = \delta = \frac{c}{\lambda\mu} - 1 > 0,$$

(relations (20) in [10], p. 27, rows 14,16) do not agree with (6), (9), and (12). Really, relation (6) yields

$$g_+(0,z) = \mathrm{E}[e^{-z\gamma^+(0)}, \zeta^+ > 0] = \frac{\lambda}{c}\int_0^\infty e^{-zx}\overline{F}(x)dx, m = \mathrm{E}\zeta(1) < 0. \tag{21}$$

The last relation does not accord with (19), because, for $m < 0$,

$$M_Y(z,0) \underset{z\to 0}{\longrightarrow} 1, \ \text{but} \ \ \mathrm{E}[e^{-z\gamma^+(0)}, \zeta^+ > 0] \underset{z\to 0}{\longrightarrow} \frac{\lambda\mu}{c} = q_+ = \Psi(0) < 1,$$

$$\mathrm{E}[e^{-s\tau^+(0)}, \tau^+(0) < \infty] \underset{s\to 0}{\longrightarrow} \mathrm{P}\{\tau^+(0) < \infty\} = q_+ < 1.$$

Now we consider a renewal scheme generated by the sums

$$S_n = \sum_{k\leq n} \tau'_k(0), u = 0; \ S_n^{(u)} = \tau'_1(u) + \sum_{k=2}^n \tau'_k(0), u > 0. \tag{22}$$

For $u = 0$, we denote

$$N(t) = \max\{n : S_n \leq t\}, \ H(t) = \mathrm{E}N(t),$$

$$N(\theta_s) \text{ – the randomly stopped renewal process.}$$

Proposition 1. *The Laplace–Stieltjes transform of the renewal function* $H(t)$ *is defined by the relation (in terms* $\varphi(s)$ *from (14))*

$$\widetilde{h}(s) =: \int_0^\infty e^{-st}dH(t) = \frac{\varphi(s)}{1-\varphi(s)}. \tag{23}$$

The distribution and the m.g.f. of $N(\theta_s)$ *are defined by the relations*

$$p(k,s) =: \mathrm{P}\{N(\theta_s) = k\} = (1-\varphi(s))\varphi(s)^k, k \geq 0,$$

$$\pi(z,s) =: \mathrm{E}z^{N(\theta_s)} = \frac{1-\varphi(s)}{1-z\varphi(s)}, \ \ |z| \leq 1. \tag{24}$$

For $m < 0$, the number of returns of $\zeta(t)$ on $[0,\infty)$ into the survival zone $\{y \le u\}$ ($N = N(\infty)$) has a geometric distribution with the m.g.f.

$$\pi(z) =: \mathrm{E}z^N = \lim_{s\to 0}\pi(z,s) = \frac{1-q_+}{1-zq_+}, q_+ = \Psi(0), \tag{25}$$
$$p(k) = \lim_{s\to 0} p(k,s) = \mathrm{P}\{N = k\} = (1-q_+)q_+^k, k \ge 0.$$

To prove an analogous assertion for $u > 0$, we consider the second random walk in (22) (with the m.g.f. $\varphi_u(s)$ for $\tau_1'(u)$ see (16))

$$S_n^{(u)} = \tau_1'(u) + \sum_{k=2}^{n} \tau_k'(0).$$

We denote, for $u > 0$,

$$N_u(t) = \max\{n : S_n^{(u)} \le t\}$$
$$H_u(t) = \mathrm{E}N_u(t),\ \widetilde{h}_u(s) = \int_0^\infty e^{-st} dH_u(t).$$

Theorem 1. *The Laplace–Stieltjes transform $\widetilde{h}_u(s)$ is defined by the relation*

$$\widetilde{h}_u(s) = \frac{\varphi_u(s)}{1-\varphi(s)},\ \varphi_u(s) = \mathrm{E}[e^{-sT'(u)}, T'(u) < \infty], u \ge 0. \tag{26}$$

The distribution and the m.g.f. of $N_u(\theta_s)$ are defined by the relations

$$p_u(0,s) =: \mathrm{P}\{N_u(\theta_s) = 0\} = 1-\varphi_u(s),\ (k=0),$$
$$p_u(k,s) =: \mathrm{P}\{N_u(\theta_s) = k\} = (1-\varphi(s))\varphi^{k-1}(s)\varphi_u(s), k \ge 1; \tag{27}$$
$$\pi_u(z,s) =: \mathrm{E}z^{N_u(\theta_s)} = 1 - \varphi_u(s)\frac{1-z}{1-z\varphi(s)}. \tag{28}$$

If $m < 0$, then the m.g.f. and the distribution of $N_u = N_u(\infty)$, being the total number of negative surpluses, are defined by the relations

$$\pi_u(z) =: \mathrm{E}z^{N_u} = 1 - \varphi_u(0)\frac{1-z}{1-zq_+},$$
$$\varphi_u(0) = \overline{P}_+(u) = \mathrm{P}\{\zeta^+ > u\}, \tag{29}$$
$$p_u(0) = \mathrm{P}\{N_u = 0\} = 1-\varphi_u(0),\ k = 0,$$
$$p_u(k) = \mathrm{P}\{N_u = k\} = (1-q_+)\varphi_u(0)q_+^{(k-1)},\ k \ge 1. \tag{30}$$

The first two moments of $N_u(\theta_s)$ are defined as

$$\mathrm{E}N_u(\theta_s) = \frac{\varphi_u(s)}{1-\varphi(s)} = \widetilde{h}_u(s),\ \mathrm{E}N_u^2(\theta_s) = \widetilde{h}_u(s)\frac{1+\varphi(s)}{1-\varphi(s)}, \tag{31}$$
$$DN_u(\theta_s) = VN_u(\theta_s) = \widetilde{h}_u(s)\frac{1+\varphi(s)-\varphi_u(s)}{1-\varphi(s)},$$

and those of N_u look as

$$\begin{aligned} \mathrm{E}N_u &= \frac{1}{1-q_+}\overline{P}_+(u), \\ DN_u &= \frac{1}{1-q_+}\mathrm{E}N_u[1+q_+-\overline{P}_+(u)]. \end{aligned} \tag{32}$$

For $u = 0$, relations (31) and (32) yield

$$\begin{aligned} \mathrm{E}N(\theta_s) &= \widetilde{h}(s),\ DN(\theta_s) = VN(\theta_s) = \frac{\widetilde{h}(s)}{1-\varphi(s)}, \\ \mathrm{E}N &= \widetilde{h}(0),\ DN = \frac{\widetilde{h}(0)}{1-q_+},\ \widetilde{h}(0) = \frac{q_+}{1-q_+}. \end{aligned} \tag{33}$$

Proof. Relation (26) is established after the Laplace–Stieltjes transformation of the renewal function

$$H(t) = \sum_{n\geq 1} \mathrm{P}\{S_n \leq t\}.$$

The distribution of $N_u(\theta_s)$ (27) follows from the relation

$$p_{k,u}(t) = \mathrm{P}\{N_u(t) = k\} = \mathrm{P}\{S_k^{(u)} \leq t\} - \mathrm{P}\{S_{k+1}^{(u)} \leq t\}$$

after the Laplace–Carson transformation (with the use of the integration by parts)

$$p_u(k,s) = s\int_0^\infty e^{-st} p_{k,u}(t)dt = -\int_0^\infty p_{k,u}(t)de^{-st}.$$

On the basis of (27), the m.g.f. of $N_u(\theta_s)$ is defined in (28). From (27)–(28) after the limit passage ($s \to 0$), relations (29)–(30) are established.

To calculate the moments in (31)–(33), the derivatives of the corresponding m.g.f. are used (particularly, $\mathrm{E}N_u(\theta_s) = \frac{\partial}{\partial z}\pi_u(z,s)|_{z=1}$)

$$\mathrm{E}N_u^2(\theta_s) = \frac{\partial^2}{\partial z^2}\pi_u(z,s)|_{z=1} + \mathrm{E}N_u(\theta_s);$$

$$\frac{\partial}{\partial z}\pi_u(z,s) = \frac{(1-\varphi(s))\varphi_u(s)}{(1-z\varphi(s))^2} \underset{z\to 1}{\longrightarrow} \frac{\varphi_u(s)}{1-\varphi(s)}, \tag{34}$$

$$\frac{\partial^2}{\partial z^2}\pi_u(z,s) = \frac{2(1-\varphi(s))\varphi_u(s)\varphi(s)}{(1-z\varphi(s))^3} \underset{z\to 1}{\longrightarrow} \frac{2\varphi_u(s)\varphi(s)}{(1-\varphi(s))^2}.$$

Let us separate the random walk (22) into two partial walks

$$\begin{aligned} \theta_n^{(u)} &\doteq \tau_1^+(u) + \sum_{k=2}^n \tau_k^+(0) \ \left(\theta_n = \sum_{k=1}^n \tau_k^+(0), u=0\right), \\ \sigma_n^{(u)} &\doteq T_1'(u) + \sum_{k=2}^n T_k'(0) \ \left(\sigma_n = \sum_{k=1}^n T_k'(0), u=0\right). \end{aligned} \tag{35}$$

For a claim surplus process $\zeta(t)$ with risk level $u > 0$, we denote

$\theta_{N_u} = \theta_{N_u}^{(u)}$ – the total duration of survival periods,

$\sigma_{N_u} = \sigma_{N_u}^{(u)}$ – the total duration of red periods,

$$\theta_N = \theta_{N_u}\big|_{u=0}, \ \sigma_N = \sigma_{N_u}\big|_{u=0}.$$

The following assertion is true.

Theorem 2. *For $u = 0$ and $m < 0$, the m.g.f. θ_N and σ_N are identical:*

$$\mathrm{E}[e^{-s\theta_N}, \theta_N < \infty] = \mathrm{E}[e^{-s\sigma_N}, \sigma_N < \infty] =$$

$$= \frac{1-q_+}{1-q_+ g(s)} \quad (q_+(s) = g(s) \textit{ see in (12)}). \tag{36}$$

For $u > 0$ and $m < 0$, the m.g.f. of θ_{N_u} is defined in terms of $q_+(s,u)$ and $\varphi_u(0) = \Psi(u)$ as (see (9) and (16))

$$\begin{aligned} &\mathrm{E}[e^{-s\theta_{N_u}}, \theta_{N_u} < \infty] = \\ &= 1 - \Psi(u) + \frac{(1-q_+)\Psi(u)}{1-q_+ q_+(s)} q_+(s,u). \end{aligned} \tag{37}$$

For $u > 0$ and $m < 0$, the m.g.f. of σ_{N_u} and S_{N_u} is defined in terms of $g_u(s)$ and $\varphi_u(0)$ as (see (1) and (16))

$$\begin{gathered}\mathrm{E}[e^{-s\sigma_{N_u}}, \sigma_{N_u} < \infty] = \\ = 1 - \Psi(u) + \frac{p_+ \Psi(u)}{1 - q_+ q_+(s)} g_u(s), (g_u(s) \textit{see (7)}) \\ \mathrm{E}[e^{-sS_{N_u}}, S_{N_u} < \infty] = \\ = 1 - \Psi(u) + \frac{p_+ \Psi(u)}{1 - q_+ \varphi(s)} \varphi_u(s), \varphi_u(s) \textit{see in (16).}\end{gathered} \tag{38}$$

For $s \to 0$,

$$\begin{gathered}\mathrm{P}\{\theta_N < \infty\} = \mathrm{P}\{\sigma_N < \infty\} = \mathrm{P}\{S_N < \infty\} = \frac{1}{1 + \Psi(0)}, \\ \mathrm{P}\{\theta_{N_u} < \infty\} = \mathrm{P}\{\sigma_{N_u} < \infty\} = \mathrm{P}\{S_{N_u} < \infty\} = \\ = 1 - \Psi(u) + \frac{\Psi^2(u)}{1 + \Psi(0)} \underset{u \to 0}{\longrightarrow} \frac{1}{1 + \Psi(0)}, \\ \frac{1}{2} < \mathrm{P}\{\theta_N < \infty\} = \mathrm{P}\{\sigma_N < \infty\} < 1.\end{gathered} \tag{39}$$

Proof. Relations (36) are established by averaging the m.g.f. of θ_n and σ_n in (35) over the geometric distribution (25). The proof of (37) and (38) follows from averaging the m.g.f. of $\theta_n^{(u)}$, $\sigma_n^{(u)}$, and $S_n^{(u)}$ in (35) over distribution (30) for N_u.

To analyze the last relation in extreme cases, we denote

$$A_* = \{\omega : \zeta(t) \to -\infty\}$$

and remark that $\delta = \frac{c - \lambda\mu}{\lambda\mu} = \frac{1}{q_+} - 1 = \frac{p_+}{q_+}$.

1) Let $q_+ = 1 - \epsilon$ (ϵ is small), $\delta = \frac{\epsilon}{1-\epsilon}$ $(|m| = O(\epsilon))$. Then we have

$$\mathrm{P}\{\sigma_N < \infty\} = \mathrm{P}\{\theta_N < \infty\} = \frac{1}{2 - \epsilon} \approx \frac{1}{2} \text{ for } \omega \in A_*,$$

$$\mathrm{P}\{\sigma_N = \infty\} = \mathrm{P}\{\theta_N = \infty\} = \frac{1 - \epsilon}{2 - \epsilon} \approx \frac{1}{2} \text{ for } \omega \in \overline{A}_*.$$

2) Let $q_+ = \Psi(0) = \epsilon$, $\delta = \frac{1-\epsilon}{\epsilon}$ $(|m| = O(\frac{1}{\epsilon}))$. Then we have

$$\mathrm{P}\{\sigma_N < \infty\} = \mathrm{P}\{\theta_N < \infty\} = \frac{1}{1 + \epsilon} \approx 1 \text{ for } \omega \in A_*,$$

$$\mathrm{P}\{\sigma_N = \infty\} = \mathrm{P}\{\theta_N = \infty\} = \frac{\epsilon}{1 + \epsilon} \approx 0 \text{ for } \omega \in \overline{A}_*.$$

3) Let $p_+ = q_+ = \frac{1}{2}$. Then we have

$$\mathrm{P}\{\sigma_N < \infty\} = \mathrm{P}\{\theta_N < \infty\} = \frac{2}{3} \text{ for } \omega \in A_*,$$

$$\mathrm{P}\{\sigma_N = \infty\} = \mathrm{P}\{\theta_N = \infty\} = \frac{1}{3} \text{ for } \omega \in \overline{A}_*.$$

Namely the last case $\delta = 1$ is regarded as the extreme one in the risk theory, because, as in the case of a "fair play" in the classic ruin problem, $p_+ = q_+ = \frac{1}{2}$.

Let us consider the multivariate ruin function which is determined by the joint m.g.f. for

$$\begin{gathered}\{\tau^+(u), \gamma_1(u), \gamma_2(u), \gamma_3(u)\}; \gamma_1(u) = \gamma^+(u), \gamma_2(u) = \gamma_+(u), \gamma_3(u) = \gamma_u^+ : \\ V(s, u, u_1, u_2, u_3) = \mathrm{E}\left[e^{-s\tau^+(u) - \sum_{k=1}^3 u_k \gamma_k(u)}, \tau^+(u) < \infty\right] = \\ = \mathrm{E}[e^{-\sum_{k=1}^3 u_k \gamma_k(u)}, \zeta^+(\theta_s) > u].\end{gathered}$$

The results obtained in [3], [12] yield the following theorem for a semicontinuous process $\zeta(t)$.

Theorem 3. *The joint m.g.f. of* $\{\tau^+(u), \gamma_1(u), \gamma_2(u), \gamma_3(u)\}$ *and* $\{\tau^+(u), \gamma_k(u)\}_{k=\overline{1,3}}$ *is determined in terms of the convolutions*

$$G(s,x,u_1,u_2,u_3) = s^{-1}\int_{-\infty}^{0} A_{x-y}(u_1,u_2,u_3)dP_-(s,y),$$
$$P_\pm(s,y) = \mathrm{P}\left\{\zeta^\pm(\theta_s) < y\right\}, \pm y > 0, P_-(s,y) = q_-(s)e^{y\rho_-(s)},$$
$$A_x(u_1,u_2,u_3) = \lambda\int_x^\infty e^{(u_1-u_2)x-(u_1+u_3)z}dF(z), x > 0,$$

by the relations

$$\begin{gathered} V(s,u,u_1,u_2,u_3) = \int_0^u G(s,u-y,u_1,u_2,u_3)dP_+(s,y); \\ V_k(s,u,u_k) = \int_0^u G_k(s,u-y,u_k)dP_+(s,z), \\ G_k(s,u,u_k) = G(s,u,u_1,u_2,u_3)|_{u_r=0,r\neq k}\ (1\le k\le 3). \end{gathered} \tag{40}$$

For the triple $\{\tau^+(u), \gamma_1(u), \gamma_2(u)\}$*, the m.g.f. is determined by the relation*

$$V(s,u,u_1,u_2) =: V(s,u,u_1,u_2,0) = \int_0^u G(s,u-z,u_1,u_2)dP_+(s,z), \tag{41}$$
$$G(s,u,u_1,u_2) =: G(s,u,u_1,u_2,0) =$$
$$\frac{s^{-1}\lambda\rho_-(s)e^{-u_2u}}{\rho_-(s)-u_1+u_2}\int_0^\infty\left[e^{-u_1y} - e^{-(\rho_-(s)+u_2)y}\right]dF(u+y).$$

It should be remarked that $G(s,u,u_1,u_2)$ and $G_k(s,u,u_k)$ are invertible with respect to u_k, particularly, $G_k(s,u,u_k)$ have inversions

$$g_1(s,u,x) = s^{-1}\lambda\rho_-(s)\int_x^\infty e^{\rho_-(s)(x-y)}dF(u+y),$$
$$g_2(s,u,y) = s^{-1}\lambda\rho_-(s)e^{\rho_-(s)(u-y)}\overline{F}(y)I\{y>u\},$$
$$g_3(s,u,z) = s^{-1}\lambda F'(z)[1-e^{\rho_-(s)(u-z)}]I\{z>u\}.$$

On the basis of Theorem 3, the following assertion (see Corollary 1 in [3]) is proved.

Corollary 2. *For the claim surplus process* $\zeta(t)$*, the densities of multivariate ruin functions (a prelimit for* $s>0$ *and the limit for* $s=0$*) are determined for* $y>0$ $(y\neq u)$ *by the relations*

$$\left\{\begin{array}{l} s\dfrac{\partial^2}{\partial x\partial y}\mathrm{P}\left\{\zeta^+(\theta_s)>u, \gamma^+(u)<x, \gamma_+(u)<y\right\} = \\ \quad = \left\{\begin{array}{l}\lambda\rho_-(s)F'(x+y)\int_0^u e^{\rho_-(s)(u-y-z)}dP_+(s,z), y>u, \\ \lambda\rho_-(s)F'(x+y)\int_{u-y}^u e^{\rho_-(s)(u-z-y)}dP_+(s,z), 0<y<u.\end{array}\right. \\ \dfrac{\partial^2}{\partial x\partial y}\mathrm{P}\left\{\zeta^+>u, \gamma^+(u)<x, \gamma_+(u)<y\right\} = \\ \quad = \left\{\begin{array}{l}\lambda|m|^{-1}F'(x+y)\mathrm{P}\{\zeta^+<u\}, y>u, m=\lambda\mu-c<0, \\ \lambda|m|^{-1}F'(x+y)\mathrm{P}\left\{u-y<\zeta^+<u\right\}, 0<y<u.\end{array}\right.\end{array}\right. \tag{42}$$

After inversion (40) with respect to u_k $(k=\overline{1,3})$*, the marginal densities of ruin functions*

$$p_k(s,u,x_k) = \frac{\partial}{\partial x_k}\mathrm{P}\{\zeta^+(\theta_s)>u, \gamma_k(u)<x_k\}, k=\overline{1,3},$$

are determined by the relations

$(k=1)$ *for* $x>0$

$$\begin{cases} p_1(s,u,x)=\lambda c^{-1}\displaystyle\int_x^\infty e^{\rho_-(s)(x-y)}dF(u+y)+ \\ +s^{-1}\lambda\rho_-(s)\displaystyle\int_0^u\int_x^\infty e^{\rho_-(s)(x-y)}dF(u+y-z)dP_+(s,z); \\ p_1(0,u,x)=\frac{\partial}{\partial x}\mathrm{P}\left\{\zeta^+>u,\gamma^+(u)<x\right\}= \\ \frac{\lambda}{c}\overline{F}(u+x)+\frac{\lambda}{|m|}\int_0^u\overline{F}(u+x-z)d\mathrm{P}\left\{\zeta^+<z\right\},m<0; \end{cases} \tag{43}$$

$(k=2)$ *for* $y>0$ $(y\neq u)$

$$\begin{cases} p_2(s,u,y)=\begin{cases} s^{-1}\lambda\rho_-(s)\overline{F}(y)\int_{-0}^u e^{\rho_-(s)(u-y-z)}dP_+(s,z),y>u, \\ s^{-1}\lambda\rho_-(s)\overline{F}(y)\int_{u-y}^u e^{\rho_-(s)(u-y-z)}dP_+(s,z),0<y<u. \end{cases} \\ p_2(0,u,y)=\begin{cases} \lambda|m|^{-1}\overline{F}(y)\mathrm{P}\{\zeta^+<u\},y>u,m<0, \\ \frac{\lambda}{|m|}\overline{F}(y)\mathrm{P}\left\{u-y<\zeta^+<u\right\},0<y<u; \end{cases} \end{cases} \tag{44}$$

$(k=3)$ *for* $z>0$ $(z\neq 0)$

$$\begin{cases} p_3(s,u,z)=p_+(s)g_3(s,u,z)+\int_{0+}^u g_3(s,u-y,z)dP_+(s,y), \\ p_3(0,u,z)=\frac{\lambda}{|m|}F'(z)\int_0^u(z-u+y)d\mathrm{P}\{\zeta^+<y\}I\{z>u\}+ \\ \frac{\lambda F'(z)}{|m|}\int\limits_0^z(z+v)d\mathrm{P}\{\zeta^+<v-u\}I\{0<z<u\}. \end{cases} \tag{45}$$

For $u=0$, *the marginal ruin functions are determined by the relations*

$$\begin{aligned} \mathrm{P}\{\zeta^+(\theta_s)>0,\gamma^+(0)>x\}&=\frac{\lambda}{c}\int_x^\infty e^{\rho_-(s)(x-y)}\overline{F}(y)dy\underset{s\to0}{\longrightarrow}\frac{\lambda}{c}\overline{\overline{F}}(x), \\ \mathrm{P}\{\zeta^+(\theta_s)>0,\gamma_+(0)>x\}&=\frac{\lambda}{c}\int_x^\infty e^{-\rho_-(s)(y)}\overline{F}(y)dy\underset{s\to0}{\longrightarrow}\frac{\lambda}{c}\overline{\overline{F}}(x), \\ \mathrm{P}\{\zeta^+(\theta_s)>0,\gamma_0^+>x\}&=\frac{\lambda}{c\rho_-(s)}\int_x^\infty(1-e^{-\rho_-(s)y}dF(y)\underset{s\to0}{\longrightarrow}\frac{\lambda}{c}\int_x^\infty ydF(y). \end{aligned} \tag{46}$$

This implies that, only for $s\to0,u=0$,

$$p_1(0,0,x)=p_2(0,0,x)=\frac{\lambda}{c}\overline{F}(x),\ \ x>0. \tag{47}$$

But, for $s>0$ *and* $u=0$, *the densities of* $\gamma_1(0)=\gamma^+(0)$ *and* $\gamma_2(0)=\gamma_+(0)$ *on* $\{\zeta^+(\theta_s)>0\}$ *are different:*

$$\begin{aligned} p_1(s,0,x)&=\frac{\lambda}{c}\int_0^\infty e^{-\rho_-(s)y}dF(x+y),x>0; \\ p_2(s,0,x)&=\frac{\lambda}{c}e^{-\rho_-(s)x}\overline{F}(x)\neq p_1(s,0,x); \\ p_3(s,0,x)&=\frac{\lambda}{c\rho_-(s)}(1-e^{-x\rho_-(s)})F'(x)\underset{s\to0}{\longrightarrow}\frac{\lambda}{c}xF'(x),\ \ x>0. \end{aligned} \tag{48}$$

Some results on the distributions of overjump functionals for semicontinuous processes were stated in our doctoral-degree thesis ([12], particularly, relations (5), (6), (40), (41), and (46).

The more complete results on the overjump functionals for the processes with stationary independent increments are stated in [3] and [12], on the basis of which the multivariate ruin function and other risk characteristics are studied in [13] and in our monograph [14].

Example (see [13], [14]). Let $\zeta(t)=\sum_{k\le\nu(t)}\xi_k-ct$ be the claim surplus process with exponentially distributed claims $\xi_k>0$, $c>0$. We have

$$\overline{F}(x)=\mathrm{P}\{\xi_k>x\}=e^{-bx}, x>0, \mu=\mathrm{E}\xi_k=b^{-1}.$$

The m.g.f. of $\zeta(t)$, $\zeta(\theta_s)$ are defined by the cumulant $k(r)$

$$\mathrm{E}e^{r\zeta(\theta_s)}=\frac{s}{s-k(r)},\ Re\, r=0,$$

$$k(r)=\frac{\lambda r-cr(b-r)}{b-r}, m=\mathrm{E}\zeta(1)=\frac{\lambda-cb}{b}<0. \tag{49}$$

Lundberg's equation

$$s-k(r)=0\sim cr^2+(s-|m|b)r+sb=0 \tag{50}$$

has two roots $r_1(s)=-\rho_-(s)$, $r_2(s)=\rho_+(s)>0$, which define the m.g.f. of $\zeta^{\pm}(\theta_s)$:

$$\mathrm{E}e^{-z\zeta^+(\theta_s)}=\frac{p_+(s)(b+z)}{\rho_+(s)+z}, \rho_+(s)=bp_+(s),$$

$$\mathrm{E}e^{-z\zeta^-(\theta_s)}=\frac{p_-(s)}{\rho_-(s)-z}, c\rho_-(s)p_+(s)=s. \tag{51}$$

Hence,

$$P_-(s,x)=\mathrm{P}\{\zeta^-(\theta_s)<x\}=e^{\rho_-(s)x}, x\le 0;$$
$$P_+(s,x)=q_+(s)e^{-\rho_+(s)x}, x>0, p_+(s)=\mathrm{P}\{\zeta^+(\theta_s)=0\}>0. \tag{52}$$

If $m<0$ and $s\to 0$, then $\rho_-(s)\to 0$, $\rho_+(s)\to\frac{b|m|}{c}$,

$$\Psi(u)=\mathrm{P}\{\zeta^+>u\}=q_+e^{-\rho_+u}, p_+(s)\to p_+=\frac{|m|}{c}, \rho_-'(0)=\frac{1}{|m|}.$$

The m.g.f. for $\tau^+(u),\gamma^+(u)$, $T'(u)$, and $\tau'(u)$ are simply calculated as

$$\begin{cases} q_+(s,u)=\mathrm{E}[e^{-s\tau^+(u)},\tau^+(u)<\infty]=q_+(s)e^{-\rho_+(s)u}.\\ g_+(z,u)=\mathrm{E}[e^{-z\gamma_+(u)},\tau^+(u)<\infty]=\Psi(u)\frac{b}{b+z},\\ g_u(s)=\mathrm{E}[e^{-sT'(u)},\tau^+(u)<\infty]=\Psi(u)\frac{b}{b+\rho_-(s)},\\ \varphi_u(s)=\mathrm{E}[e^{-s\tau'(u)},\tau^+(u)<\infty]=q_+(s,u)\frac{b}{b+\rho_-(s)},\end{cases} \tag{53}$$
$$q_+(s,u)\underset{s\to 0}{\longrightarrow}\Psi(u)=q_+e^{-\rho_+u}.$$

So the conditional m.g.f. of $\gamma_1^+(u)$ and $T_1'(u)$,

$$\begin{aligned}\mathrm{E}[e^{-z\gamma_1^+(u)}/\tau^+(u)<\infty]=\mathrm{E}e^{-z\tilde{\gamma}_1^+(u)}&=\frac{b}{b+z},\\ \mathrm{E}[e^{-sT_1'(u)}\tau^+(u)<\infty]=\mathrm{E}e^{-s\tilde{T}_1'(u)}&=\frac{b}{b+\rho_-(s)},\end{aligned} \tag{54}$$

are independent of u. For $u=0$,

$$g_+(z,0)=\Psi(0)\frac{b}{b+z}, g(s)=\Psi(0)\frac{b}{b+\rho_-(s)}$$

$$\varphi(s)=q_+(s)\frac{b}{b+\rho_-(s)}\underset{s\to 0}{\longrightarrow}\Psi(0)=q_+. \tag{55}$$

After the substitution of (53) in (30), we obtain

$$p_u(0)=1-\Psi(u), \Psi(u)=q_+e^{-\rho_+u},$$
$$p_u(k)=(1-q_+)\Psi(u)q_+^{k-1}, k\ge 1,$$
$$\pi_u(z)=1-q_+e^{-\rho_+u}\frac{1-z}{1-zq_+}. \tag{56}$$

By substituting (53) and (54) in (37) and (38), it is easy to calculate the m.g.f. of $\tilde{\sigma}_{N_u} = \sum_{k\le N_u} \tilde{T}_k(u)$, σ_{N_u}, and S_{N_u}:

$$\mathrm{E}e^{-s\tilde{\sigma}_{N_u}} = \mathrm{E}e^{-s\sum_{k\le N_u}\tilde{T}_k(u)} = 1 - \frac{\rho_-(s)\Psi(u)}{\rho_-(s)+\rho_+}, \tag{57}$$

$$\mathrm{E}[e^{-s\sigma_{N_u}}, \sigma_{N_u} < \infty] = 1 - \Psi(u) + \frac{p_+\Psi^2(u)}{1-q_+q_+(s)}\frac{b}{b+\rho_-(s)}, \tag{58}$$

$$\mathrm{E}[e^{-sS_{N_u}}, S_{N_u} < \infty] = 1 - \Psi(u) + \frac{p_+\Psi(u)}{1-q_+q_+(s)}\frac{bq_+(s)}{b+\rho_-(s)}e^{-\rho_+(s)u},$$

$$\mathrm{P}\{\sigma_{N_n} < \infty\} = \mathrm{P}\{S_{N_u} < \infty\} = 1 - \Psi(u) + \frac{\Psi(u)^2}{1+q_+}, \Psi(u) = e^{-\rho_+ u}.$$

From the Vieta formulas, we get

$$\rho_-(s)\rho_+(s) = \frac{bs}{c}, \quad \rho_+(s) - \rho_-(s) = \frac{|m|b-s}{c}, \ \rho_+ = \frac{|m|b}{c}.$$

Then, according to §2.4 in [14], the total sojourn time of $\zeta(t)$ in $\{y > u\}$ is defined by the integral functional

$$Q_u(\infty) = \int_0^\infty I\{\zeta(t) > u\}dt.$$

Its m.g.f. is defined by the relation (see (2.84) in [14])

$$D_u^+(0,\mu) = \mathrm{E}e^{-\mu Q_u(\infty)} = 1 - \frac{\rho_+(s)-\rho_+}{\rho_+(s)}e^{-\rho_+ u}$$

which is similar to (57) ($\tilde{\sigma}_{N_u} \doteq Q_u(\infty)$).

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