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SPECTRAL ANALYSIS OF MULTIVARIATE STATIONARY RANDOM FUNCTIONS ON SOME MASSIVE GROUPS

The spectral representations for wide sense stationary multivariate random functions and for their covariance functions on two classes of additive vector groups are obtained under some assumptions about continuity of such functions. The first class is nuclear topological groups and the second class is additive group of real vector space equipped with the finite topology.

1. Introduction

The spectral theory of stationary random functions of second order on abelian locally compact groups is well developed. This theory is based on spectral representations of such random functions and their covariance functions in one-dimensional and multivariate cases [1]-[3].

Now actual problems are connected with obtaining similar results for weakly stationary random functions on abelian massive groups, i.e. groups which are not generally locally compact.

We consider two classes of abelian massive groups in this paper. The first class is nuclear topological groups, which theory was developed in [4]-[6]. Second class is related with additive groups of general real vector spaces and corresponding results of harmonic analysis on such groups [7].

The objects of investigation in this paper are spectral representations for generalized wide sense stationary random functions in Hilbert space H on nuclear groups and real vector spaces and for their covariance functions. Under some assumptions of continuity for stationary random functions these spectral representations are obtained in the form of Fourier transforms of orthogonally scattered random measures in H and operator positive finite measures in H respectively.

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2. Wide sense stationary random functions in Hilbert space H on abelian groups

Let $L_2(\Omega)$ be Hilbert space of all complex-valued random variables of second order, which defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and H be a complex Hilbert space. Then the set $L(H, L_2(\Omega))$ of all linear continuous mappings of the space H into $L_2(\Omega)$ may be considered as the set of generalized random elements of second order in H, which defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Such generalized random elements may be realized as usual random elements in some quasinuclear extension H_- of the space H (see [3], [8]).

Denote by $B(H)$ Banach algebra of all linear bounded operators in H and denote by $B_+(H)$ the convex cone of all nonnegative Hermitian operators in $B(H)$.

The expectation $m = \mathbb{E}\Xi \in H$ and covariance operator $[\Xi, \Psi] \in B(H)$ for generalized random elements $\Xi, \Psi \in L(H, L_2(\Omega))$ are uniquely determinated by the relations

$$(x|m) = \mathbb{E}(\Xi x), ([\Xi, \Psi]x|y) = \mathbb{E}(\Xi x)\overline{(\Psi y)}, x, y \in H,$$

where $(\cdot|\cdot)$ is an inner product in H. Note that $[\Xi, \Psi]$ is sesquilinear form on $L(H, L_2(\Omega))$ and $[\Xi, \Xi] \in B_+(H)$.

If $dimH = d < \infty$ and $\{e_j\}_{j=1}^d$ is orthonormal basis in H, then random element $\Xi \in L(H, L_2(\Omega))$ may be identified with random vector $\xi = \{\xi_j\}_{j=1}^d$, with $\xi_j = \Xi e_j$ and expectation $\mathbb{E}\Xi$ - with vector $\mathbb{E}\xi = \{\mathbb{E}\xi_j\}_{j=1}^d$ and covariance operator $[\Xi, \Psi], \Psi \in L(H, L_2(\Omega))$ - with matrix $\{\mathbb{E}\xi_j\overline{\varphi}_k\}_{j,k=1}^d$, where $\varphi_k = \Psi e_k, k = 1, .., d$.

Definition 2.1 The (generalized) random function of second order $\Xi_t, t \in T$, in a Hilbert space H defined on set T is the family of generalized random elements $\Xi_t \in L(H, L_2(\Omega))$ indexed by $t \in T$.

The mean function m_t and covariance function $Q(s,t)$ of $\Xi_t, t \in T$ are defined by the equalities

$m_t = \mathbb{E}\Xi_t, Q(t,s) = [\Xi_t, \Xi_s], t, s \in T$.

Note that Q is positive definite operator kernel on T, i.e. for all integers $n \in \mathbb{N}$, elements $t_k \in T$ and vectors $x_k \in H, k = 1, ..n$

$$\sum_{k=1}^{n}\sum_{j=1}^{n}(Q(t_k, t_j)x_k|x_j) \geq 0.$$

Conversely, every positive definite operator kernel Q on T with values in $B(H)$ is a covariance function of some (in particulary gaussian) generalized random function of second order $\Xi_t, t \in T$ (see [3]).

Let G be some Abelian group with operation which is written as addition.

Definition 2.2 A random function of second order $\Xi_g, g \in G$ on group G in H is called a wide-sense (or weakly) stationary if its mean function is constant $m_g = \mathbb{E}\Xi_g = m, g \in G$ and its covariance function $Q(g, s)$ depends only on difference $g - s$:

$$Q(g, s) = R(g - s), g, s \in G. \qquad (1)$$

The function $R(g)$ in (1) is called a covariance function of stationary function $\Xi_g, g \in G$.

Note that definition of stationarity for random function $\Xi_g, g \in G$ is equivalent to assumption of shift invariance its two moment function m_g and $Q(g, s)$.

3. Spectral representations for stationary random functions in H on nuclear group

Let V be a real vector space and A, B be non-empty balanced subsets of V such that A is absorbed by B. For vector subspace Y of V we set

$$d(A, B; Y) = inf\{t > 0 : A \subseteq tB + Y\}.$$

The Kolmogorov diameter d_k of A with respect to $B, k \in \mathbb{N}$ is defined by the equality

$$d_k(A, B) = inf\ \{d(A, B; X) : X\},$$

where X ranges through the vector subspaces of V with $dim(X) < k$.

Definition 3.1 A locally convex vector group is a real vector space V, equipped with a Hausdorff group topology such that the filter of zero-neighborhoods possesses a base of symmetric convex sets.

A locally convex vector group V is called nuclear if for every balanced zero-neighborhood W there exists a balanced zero-neighborhood W_0 such that W_0 is absorbed by W and for all $k \in \mathbb{N}$ $d_k(W_0, W) \leq k^{-1}$.

Note that that every locally convex space is a locally convex vector group.

Definition 3.2 An Abelian topological group is called nuclear if it is isomorphic to a Hausdorff quotient group of subgroup of some nuclear vector group.

Note that the class of nuclear groups is a variety of topological Hausdorff groups, i.e., is closed under the formation of cartesian products, subgroups, Hausdorff quotients, and passage to isomorphic topological groups; it comprises the classes of locally compact Abelian groups and nuclear locally convex spaces [6]. A topological vector space is a nuclear group if and only if it is a nuclear locally convex space [6]. The further information of nuclear group is contained in [4]-[6].

Let G be a nuclear group with operation, which is written as addition, and with dual group $\widehat{G}$, equipped by admissible topology τ, and for $g \in G$ $\langle \chi, g \rangle$ denotes the value of character $\chi \in \widehat{G}$ on g.

The ultra-weak operator topology on $B(H)$ is the initial topology on $B(H)$ with respect to family of linear functionals from $B(H)$ into $\mathbb{C}$ of the form: $A \mapsto tr(BA), A \in B(H)$, where B ranges through the set of trace class operators in $B(H)$.

Theorem 3.3 1. *Let $\{\Xi_g, g \in G\}$ be a generalized wide sense stationary random function on nuclear group G in H, which covariance function $R(g)$ is continuous in ultra-weak topology on $B(H)$. Then exists uniquely defined random $L(H, L_2(\Omega))$-valued strongly regular measure Φ on $(\widehat{G}, \tau)$ such that Ξ_g is the Fourier transform of Φ:*

$$\Xi_g = \int_{\widehat{G}} \langle \chi, g \rangle \Phi(d\chi), g \in G. \tag{2}$$

2. *The covariance function $R(g)$ of the stationary random function Ξ_g admits the spectral representation in form of the Fourier transform of $B_+(H)$-valued finite strongly regular operator measure F on $(\widehat{G}, \tau)$:*

$$R(g) = \int_{\widehat{G}} \langle \chi, g \rangle F(d\chi), g \in G. \tag{3}$$

The random spectral measure Φ of stationary random function Ξ_g is connected with its spectral measure F by the following relation:

$$[\Phi(\Delta_1), \Phi(\Delta_2)] = F(\Delta_1 \cap \Delta_2) \tag{4}$$

Proof. Denote by $K(\Xi)$ the subspace of $L_2(\Omega)$ generated by family of random variables $\{\Xi_g x : g \in G, x \in H\}$. Define on set of elements in $K(\Xi)$ of the form $\sum_{j=1}^{n} \Xi_{g_j} x_j$, which is dense in $K(\Xi)$, the shift operators $\widetilde{U_s}, s \in G$ by equalities

$$\widetilde{U_s}(\sum_{j=1}^{n} \Xi_{g_j} x_j) = \sum_{j=1}^{n} \Xi_{g_j+s} x_j$$

It is easy to see that these operators are isometric (because Ξ_g is stationary function) and may extended by unique way to unitary operators $U_s, s \in G$ on $K(\Xi)$. It is follows from the definition that the family $\{U_s, s \in G\}$ forms unitary representation of group G in $K(\Xi)$. These unitary representation is connected with stationary function Ξ_g and its covariance function $R(g)$ by the equalities

$$\Xi_g = U_g \Xi_0, R_g = \Xi_0^* U_g \Xi_0, g \in G, \tag{5}$$

where 0 is neutral element in G and $\Xi_0^*: L_2(\Omega) \to H$ is adjoint operator for Ξ_0, and also is ultra-weak continuous.

Then from the theorem 13.3 in [6] and theorem 15.4 in [7] it follows that unitary representation $U_g, g \in G$ is the Fourier transform of uniquely defined strongly regular Radon spectral measure P on $(\widehat{G}, \tau)$ with values in $B(K(\Xi))$. Now spectral representations (2),(3) and property of orthogonality (4) are consequences of equalities (5) with $\Phi(\Delta) = P(\Delta)\Xi_0, F(\Delta) = \Xi_0^* P(\Delta)\Xi_o$.

Remark 3.4. *If group G is k-space, i.e., if it is the direct limit of its compact subsets, than the assumption of ultra-weak continuity of covariance function $R(g)$ in theorem 3.3 is equivalent to condition of weak continuity of $R(G)$ in $B(H)$. In particular, this is true for locally compact or metrizable groups G.*

The result of Remark 3.4 follows from corollary 1.9 and Remark 15.5 from [7].

4. Spectral representations for stationary random function in H on vector space

For real vector space, equipped with its finest locally convex topology, denote by V^* its topological dual space, equipped with weak -*-topology.

Let $\Sigma(V^*)$ the coarsest σ-algebra on V^*, which makes the point evaluations $\lambda \mapsto \lambda(\nu), \lambda \in V^*$ measurable for all vector $\nu \in V$.

Let us consider generalized wide sense stationary random function Ξ_ν, $\nu \in V$ in Hilbert space H with covariance function $R(\nu - u) = [\Xi_\nu, \Xi_u], \nu, u \in V$.

Theorem 4.1. *Suppose that $B(H)$-valued covariance function $R(\nu), \nu \in V$ of stationary random function $\Xi_\nu, \nu \in V$ in H is ultra-weakly continuous on each finite-dimensional subspace Y of space V. Then it exists such $B_+(H)$-valued operator measure F on $(V^*, \Sigma(V^*))$ that $R(g)$ admits the spectral representation in the form of the Fourier transform of measure F:*

$$R(\nu) = \int_{V^*} exp\{i\lambda(\nu)\} F(d\lambda), \nu \in V, \tag{6}$$

Itself stationary random function $\Xi_\nu, \nu \in V$ admits the spectral representation

$$\Xi_\nu = \int_{V^*} exp\{i\lambda(\nu)\} \Phi(d\lambda), \nu \in V, \tag{7}$$

where Φ is random $L(H, L_2(\Omega))$ -valued measure on $(V^*, \Sigma(V^*))$ with the following orthogonal property

$$[\Phi(\Delta_1), \Phi(\Delta_2)] = F(\Delta_1 \cap \Delta_2). \tag{8}$$

Proof. One possible way to prove this theorem - it is using the method, which is similar to the approach of proving of theorem 3.4 above. This method is based on spectral expansion of unitary representation of additive group V,which connected with stationary random function $\Xi_\nu, \nu \in V$. We choose here another short way.

First of all we use the fact that $R(\nu)$ is positive definite $B(H)$-valued function on V. Then through the generalized Bochner theorem for vector space (Theorem 15.6 from [7]) the spectral representation (6) is valid and we have the equality

$$R(\nu - u) = \int_{V^*} exp\{i\lambda(\nu)\}\overline{exp\{i\lambda(u)\}}F(d\lambda), \nu, u \in V. \tag{9}$$

Now by mean of application to expansion (9)Theorem 3 from (9) about integral spectral representations of generalized random function of second order in vector space we have representation (7) with property (8).

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