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SOME LIGHT-TRAFFIC AND HEAVY-TRAFFIC RESULTS FOR THE *GI/G/n/* 0 QUEUE USING THE *GM Heuristic*

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1. INTRODUCTION

In this paper we study a heuristic approach called the *GM Heuristic* [1] for the numerical analysis of the general, multi-server queueing loss system, the $GI/G/n/0$ queue. The most important practical property of such systems is the loss probability, p_{loss}, which is the proportion of customers that are lost, or balk, in the long run. In many applications, the value of this probability determines the number and cost of resources needed to provide a given level of service. For a survey of methods available to calculate exact or approximate values of p_{loss}, see [1].

It was shown in Atkinson that the *GM Heuristic* gives values of p_{loss} that are sufficiently accurate for most practical purposes in normal traffic. The approximations also satisfy certain experimental bounds that have potentially useful applications in practice. Numerical examples illustrating the use of the *GM Heuristic* in conjunction with the experimental bounds and in conjunction with alternative heuristic methods were also given.

In this paper we concentrate on the calculation of loss probabilities in light traffic and heavy traffic. In Sec. 2 we introduce some background information. Section 3 summarises the steps of the *GM Heuristic* and Sec. 4 gives some illustrative numerical results for the loss probability in light traffic. This is followed in Sec. 5 by a proof that, when the $GI/G/n/0$ queue satisfies a specified set of sufficient conditions, the *GM Heuristic* is asymptotically exact in light traffic. Section 6 gives some numerical results in heavy traffic. This is followed in Sec. 7 by a proof that the *GM Heuristic*, under very general conditions, is asymptotically exact in heavy traffic as the number of servers n tends to infinity. Section 8 gives some brief concluding remarks.

2. BACKGROUND INFORMATION

2.1. Notation. The inter-arrival times are assumed to be positive, independent, identically distributed random variables with distribution function $A(t)$. Let the Laplace–Stieltjes transform (LST) of $A(t)$ be $\varphi(s)$ where

$$\varphi(s)=\int_0^\infty e^{-st}\,dA(t).$$

Its mean λ^{-1} is

$$\lambda^{-1}=\int_0^\infty t\,dA(t).$$

We also assume that the service times are non-negative, independent, identically distributed random variables with the distribution function $B(t)$ and the mean service time μ^{-1} is

$$\mu^{-1}=\int_0^\infty t\,dB(t).$$

The traffic intensity ρ is given by $\rho=\lambda/\mu$.

The coefficient of variation of a random variable, c, is defined as the ratio of the standard deviation to the mean. Let c_A^2 and c_B^2 be the squared coefficients of variation of the inter-arrival time and service time respectively.

Let $q_k(A/B/n,\rho)$ $(k=0,1,\ldots,n)$ be the steady-state probability that an arriving customer finds k customers ahead of itself in the $A/B/n/0$ queue having traffic intensity ρ. Let $\hat{q}_k(A/B/n,\rho)$ be the approximation to q_k that is obtained when applying the *GM Heuristic* to the same problem. Where the context is clear, we also use the terms p_{loss} and $\hat{p}_{\text{loss}}$ to represent the probabilities q_n and $\hat{q}_n$ respectively.

Using $\hat{p}_{\text{loss}}$ as an approximation for p_{loss}, the relative error (*RE*) is defined as follows:

$$RE = \frac{\hat{p}_{\text{loss}} - p_{\text{loss}}}{p_{\text{loss}}}.$$

2.2. The Coxian C_2 distribution. The two-phase Coxian C_2 distribution can be defined as a probability density whose Laplace–Stieltjes transform (LST) is rational with a denominator of degree 2 (see [2]). The C_2 density has a squared coefficient of variation c_X^2 that is at least 0.5. When $c_X^2 > 1$, the C_2 density is equivalent to the hyperexponential (H_2) density, which is a mixture of two distinct exponential densities, and when $c_X^2 = 1$, the C_2 density is equivalent to the exponential density. The set of C_2 densities for which $0.5 \le c_X^2 < 1$ includes the two-phase generalised Erlangian density (GE_2), which is the density of a sum of two independent exponential random variables. The unique C_2 density for which $c_X^2 = 0.5$ is the Erlang E_2 density.

In general, the C_2 density has three independent parameters but, in many applications, this is reduced to two. In the gamma normalisation, for example, the third moment is set equal to that of a gamma density with the same mean, $E(X)$, and the same squared coefficient of variation, c_X^2. The parameters of the density are then expressed in terms of $E(X)$ and c_X^2; see Appendix B of Tijms [7] for details. The gamma normalisation of the C_2 density was used for both the inter-arrival time and service-time distributions in the numerical work described below. The heuristic calculations for obtaining the $\hat{q}_k(C_2/C_2/n,\rho)$ values were carried out according to the steps set out in Sec. 3. The method used for calculating the corresponding exact values of $q_k(C_2/C_2/n,\rho)$ are described in [1].

3. THE *GM Heuristic*

Here we summarise the general steps of the *GM Heuristic* based on [1] with some slight changes in notation. In this heuristic we approximate the properties of the $GI/G/n/0$ queue using results from the corresponding $GI/M/n/0$ and $GI/G/1/0$ queues.

Based on our previous definition, $q_n(GI/M/n,\rho)$ is the steady-state loss probability for the $GI/M/n/0$ queue, and $q_1(GI/M/1,\rho)$ is the steady-state loss probability for the corresponding single-server queue.

Takács [6] gives an algorithm for finding the $q_k(GI/M/n,\rho)$ probabilities in terms of $\varphi(s), \lambda,$ and μ.

Step 1 of the *GM Heuristic* is to re-write Takács's algorithm so that the $q_k(GI/M/n,\rho)$ probabilities for the n-server queue are calculated in terms of the $q_1(GI/M/1,\rho)$ probability for the corresponding single-server $GI/M/1/0$ queue.

Following Takács [6, Chapter 4, Theorem 2], with some changes in notation, we have

$$q_k(GI/M/n,\rho) = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} B_r, \tag{1}$$

where B_r is the rth binomial moment of the $\{q_k\}$ and is given by

$$B_r = C_r \frac{\sum_{j=r}^{n} \binom{n}{j} \frac{1}{C_j}}{\sum_{j=0}^{n} \binom{n}{j} \frac{1}{C_j}},$$

with $C_0 = 1$ and

$$C_r = \prod_{i=1}^{r} \left(\frac{\varphi(i\mu)}{1-\varphi(i\mu)} \right), \quad r = 1,2,\ldots,n.$$

For the single-server system, using (1) we can easily show that:

$$q_1(GI/M/1,\rho)=\varphi(\mu).$$

Now consider a version of the problem in which the random variable X, representing the inter-arrival time, is re-scaled to give a new random variable Y such that $Y=iX$ where i is a positive constant. The traffic intensity would then be equal to ρ/i, and Y would have an LST equal to $\varphi(i\mu)$. Hence, we have

$$q_1(GI/M/1,\rho/i)=\varphi(i\mu). \tag{2}$$

The re-writing of Takács's algorithm involves using (2) to substitute for $\varphi(i\mu)$ in (1), thus expressing $q_k(GI/M/n,\rho)$ as a function of $q_1(GI/M/1,\rho/i)$ $(i=1,2,\ldots,n)$.

Step 2 of the heuristic uses the above re-written version of Takács's algorithm. We replace the $q_1(GI/M/1,\rho/i)$ probabilities by the corresponding $q_1(GI/G/1,\rho/i)$ probabilities. The result of using (1) will then be the approximation $\hat{q}_k(GI/G/n,\rho)$.

We also mention here that the *GM Heuristic* is extremely fast. The calculations were programmed in MATLAB on a P.C. with an AMD Athlon™ MP2200+, 1.80 GHz processor, operating under Windows XP (2002). For example, a typical problem with 100 servers and a given value of ρ took 0.004 s to solve.

4. NUMERICAL RESULTS FOR THE $C_2/C_2/n/0$ QUEUE IN LIGHT TRAFFIC USING THE *GM Heuristic*

The loss probability (p_{loss}) for the $GI/G/n/0$ queue in light traffic (i.e., as the traffic intensity $\rho\to 0$) has been widely studied because of its practical importance, for example in the analysis of reliability models and emergency communication networks. Kovalenko et al. in [5] give an extensive survey of relevant publications and analyse specific cases of insensitivity in such queues; see also [4] and [1]. The form of insensitivity considered was the property that the steady-state probabilities, including the loss probability, depend on the service-time distribution only through its mean value. Such results are useful, for example, in the approximate calculation of p_{loss} values in light traffic.

In the present work, a number of experiments were carried out to investigate the light-traffic properties of the *GM Heuristic*. Figure 1 shows p_{loss} values for systems in which $c_A^2=2, c_B^2=4$ or 20, and the number of servers $n=2$ or 8. The relative errors (R.E.), which decrease sharply as ρ decreases, suggest that the heuristic is asymptotically exact in light traffic for these cases. The high level of accuracy of the heuristic is also apparent from the graphs. It will be seen in the next section that the queueing systems in Fig. 1 satisfy a set of sufficient conditions for which the *GM Heuristic* is asymptotically exact in light traffic. We also detect, in the examples of Fig. 1, the approximate insensitivity of p_{loss} to the nature of the service-time distribution. This corresponds to Case 1 in [5]. Such insensitivity results hold, in the $C_2/C_2/0$ queue, for those cases in which $c_A^2>1$.

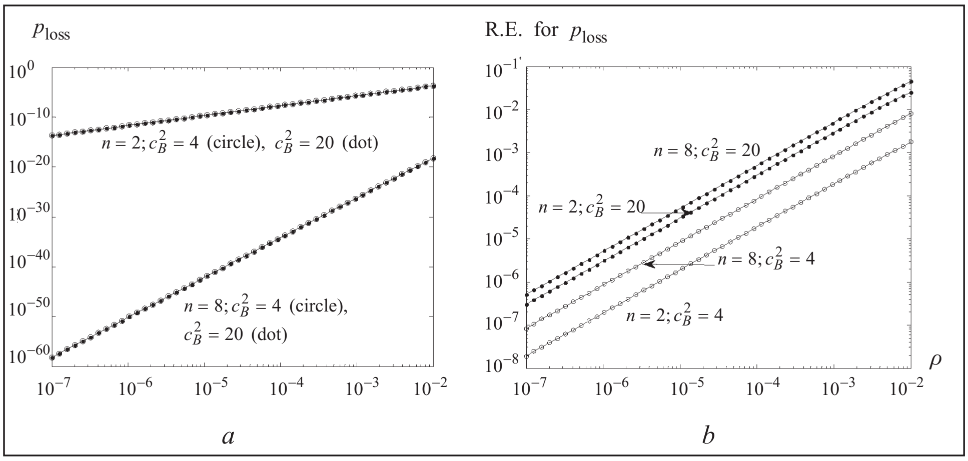

Fig. 1. (*a*): loss probabilities in light traffic when $c_A^2=4$ for the *GM Heuristic* (dots and circles) compared to exact values (lines); (*b*): the corresponding relative errors

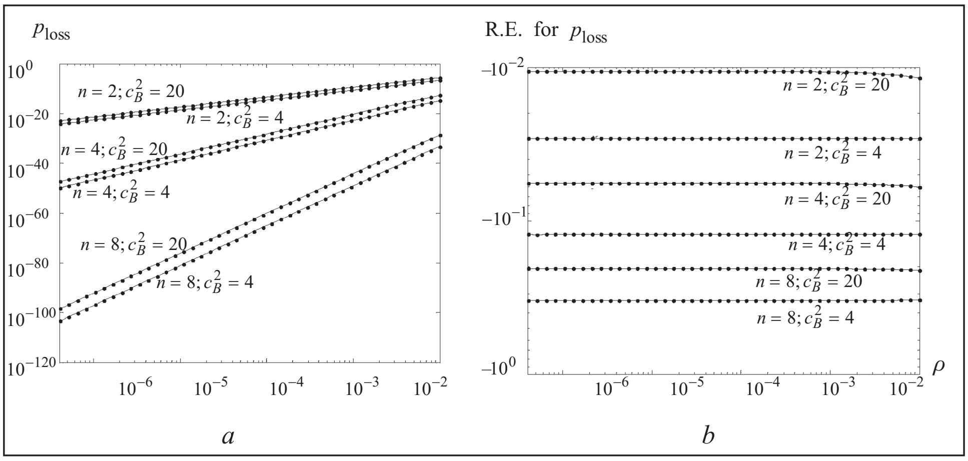

Fig. 2. (*a*): loss probabilities in light traffic when $c_A^2 = 0.5$ for the *GM Heuristic* (dots) compared to exact values (lines); (*b*): the corresponding relative errors

This feature can be compared in Fig. 2 with results for systems in which $c_A^2 = 0.5$ (the Erlang-2 distribution) where again $c_B^2 = 4$ or 20 and $n = 2$, 4, or 8. For these cases the relative error is now approximately constant in light traffic for a given queueing system. In keeping with this result, it will be seen in the next section that such queueing systems do not satisfy the set of sufficient conditions referred to above.

We also note that the relative errors for the p_{loss} values in Fig. 1 are positive while those in Fig. 2 are negative. These results agree with established empirical rules, for the signs of the errors in the $C_2 / C_2 / n / 0$ queue, which relate to the values of c_A^2 and c_B^2 and were given in [1].

5. SUFFICIENT CONDITIONS FOR THE *GM Heuristic* TO BE ASYMPTOTICALLY EXACT IN LIGHT TRAFFIC

In this section, we show that the *GM Heuristic* is asymptotically exact in light traffic for any $GI / G / n / 0$ queue in which the set of conditions, (i) to (v) below, are satisfied.

One of the special cases of Theorem 1 from [5] can be formulated as follows. Consider a family of queueing systems $A / B_\tau / n / 0$ where A is a fixed inter-arrival d.f., B_τ is a parameter τ-dependent service time d.f. and n is a fixed number of channels. Assume that each system is initially empty. Then let $J_{j\tau}$ be the indicator of a loss of customer j;

$$q_{n\tau} = \lim \mathbf{E}\{J_{j\tau}\} \text{ as } j \to \infty$$

if such a limit exists. Assume that the following conditions hold true:

(i) $\lambda^{-1} = \int_0^\infty x dA(x) < \infty$;

(ii) $1 + c_A^2 = \lambda^2 \int_0^\infty x^2 dA(x) < \infty$;

(iii) $A'(x) \to \lambda_0 > 0$ as $x \downarrow 0$;

(iv) $\mu^{-1} = \tau = \int_0^\infty x\, dB_\tau(x),\ \rho = \lambda\tau,\ \tau > 0$;

(v) $1 + c_B^2 = \tau^{-2} \int_0^\infty x^2 dB_\tau(x) \le 1 + \bar{c}_B^2 < \infty$,

where $\bar{c}_B^2$ is a constant.

Then $q_{n\tau} \sim \dfrac{(\lambda_0 \tau)^n}{n!}$ asymptotically as $\tau \to \infty$.

We note, with respect to the numerical examples in the previous section, that condition (iii) is satisfied by the inter-arrival time distribution for the systems in Fig. 1 and it is not satisfied by the inter-arrival time distribution for the systems in Fig. 2.

To indicate the dependence of q_τ upon the parameters involved, one can write that

$$q_{n\tau}(A/B_\tau/n,\rho) \sim \frac{(\lambda_0\tau)^n}{n!} \quad \text{as } \tau \to 0, \tag{3}$$

the conditions (i) to (v) being satisfied.

Note that $J_{j\tau}$ can be treated as $J_{j\tau}^{(n)}$, meaning that the jth arriving customer sees n busy channels. Similarly, let $J_{j\tau}^{(k)}$ be the indicator of the event {customer j sees k busy channels at the instant of its arrival}. Then let $q_{k\tau} = \lim \mathbf{P}\{J_{j\tau}^{(k)} = 1\}$ as $j \to \infty$ if such a limit exists.

A proof following the same steps as those for the cited statement above yields the following more general theorem.

Theorem 1. Under conditions (i) to (v),

$$q_{k\tau} \sim \frac{(\lambda_0\tau)^k}{k!} \quad \text{as } \tau \to 0 \tag{4}$$

for any k, $0 \le k \le n$, and any $n \ge 1$.

Informally, the main point in the proof of relation (4) is the fact that, due to condition (iii), the arrival process is nearly poissonian in a small interval of time.

It is worthwhile to note that the loss probability for a $GI/G/1/0$ loss system can be evaluated directly, avoiding the heavy machinery of the arguments used in [5]. Indeed,

$$q_{1\tau}(A/B_\tau/1,\rho) = \left(\int_0^\infty H_A(x)\,dB_\tau(x) - 1\right) \Big/ \int_0^\infty H_A(x)\,dB_\tau(x), \tag{5}$$

the $H_A(x)$ being the renewal function of the arrival process.

From condition (iii), given $\varepsilon > 0$, a value $\delta > 0$ can be chosen such that

$$(1-\varepsilon)\lambda_0 x < A(x) < (1+\varepsilon)\lambda_0 x \tag{6}$$

as $0 \le x \le \delta$. From a well-known inequality

$$H_A(x) \le 1 + A(x) H_A(x), \quad 0 \le x \le \delta,$$

and inequality (6), one obtains the double inequality

$$1 + (1-\varepsilon)\lambda_0 x < H_A(x) < 1 + (1+\varepsilon')\lambda_0 x$$

for $0 \le x \le \delta$, where each of the parameters δ, ε, and ε' can be chosen to be arbitrarily small.

Now we can bound the integral in (5). Indeed,

$$\int_0^\infty H_A(x)\,dB_\tau(x) \ge 1 + \int_0^\delta (H_A(x)-1)\,dB_\tau(x) \ge 1 + (1-\varepsilon)\lambda_0 \int_0^\delta x\,dB_\tau(x) \ge$$

$$\ge 1 + (1-\varepsilon)\lambda_0 \left(\tau - \frac{1}{\delta}\tau^2 (1+\bar{c}_B^2)\right). \tag{7}$$

By condition (iii) and a well-known inequality,

$$H_A(x) \le 1 + \lambda x + c_A^2,$$

one obtains

$$\int_0^\infty H_A(x) dB_\tau(x) \le 1 + \int_0^\infty (1+\varepsilon')\lambda_0 x dB_\tau(x) + \int_\delta^\infty (\lambda x + c_A^2 - 1) dB_\tau(x) \le$$

$$\le 1 + (1+\varepsilon')\lambda_0\tau + \tau^2(1+\bar{c}_B^2)\left(\frac{\lambda}{\delta} + \frac{1}{\delta^2}|c_A^2 - 1|\right). \tag{8}$$

As δ, ε, and ε' can be chosen to be sufficiently small independently of τ, inequalities (7) and (8) yield the relation

$$\int_0^\infty H_A(x)\, dB_\tau(x) = 1 + \lambda_0 \tau + o(\tau) \text{ as } \tau \to 0$$

which, due to (5), implies the relation

$$q_{1\tau}(A/B_\tau/1, \rho) \sim \lambda_0\, \tau \text{ as } \tau \to 0.$$

In particular, one concludes that

$$\varphi(\mu) = q_{1\tau}(A/M_\tau/1, \rho) \sim \lambda_0\, \tau \text{ as } \tau \to 0$$

where the service time d.f.

$$M_\tau(x) = 1 - e^{-x/\tau}, \quad x \ge 0.$$

From this,

$$q_1\left(A/M_\tau/1, \frac{\rho}{i}\right) \sim \frac{\lambda_0\, \tau}{i} \text{ as } \tau \to 0. \tag{9}$$

Inserting (9) into [6], one gets

$$\hat{q}_k(A/B_\tau/n, \rho) \sim \frac{(\lambda_0 \tau)^k}{k!}. \tag{10}$$

Hence, the relation for the relative error of our approximation

$$\frac{\hat{q}_k(A/B_\tau/n, \rho)}{q_k(A/B_\tau/n, \rho)} - 1 \to 0 \text{ as } \tau \to 0.$$

6. NUMERICAL RESULTS FOR THE $C_2/C_2/n/0$ QUEUE IN HEAVY TRAFFIC USING THE *GM Heuristic*

The $GI/G/n/0$ queue in heavy traffic (i.e., as the traffic intensity $\rho \to \infty$) has less direct practical importance than the light-traffic case discussed above. However, methods for the combination of light-traffic and heavy-traffic results, both numerical and analytic, have been a fertile area of study for the approximation of various queues in normal traffic. Examples include the interpolation approaches of Whitt [8] and Kimura [3].

A number of experiments were carried out to investigate the heavy-traffic properties of the *GM Heuristic*. Figure 3 shows values of $1 - p_{\text{loss}}$ for systems in which $c_A^2 = 2$ and $c_B^2 = 4$ or 20, while the number of servers $n = 2$ or 8. The absolute values of the relative errors of $1 - p_{\text{loss}}$ are seen to decrease sharply as ρ increases, suggesting that the heuristic is asymptotically exact in heavy traffic for these cases. Figure 4 shows comparable

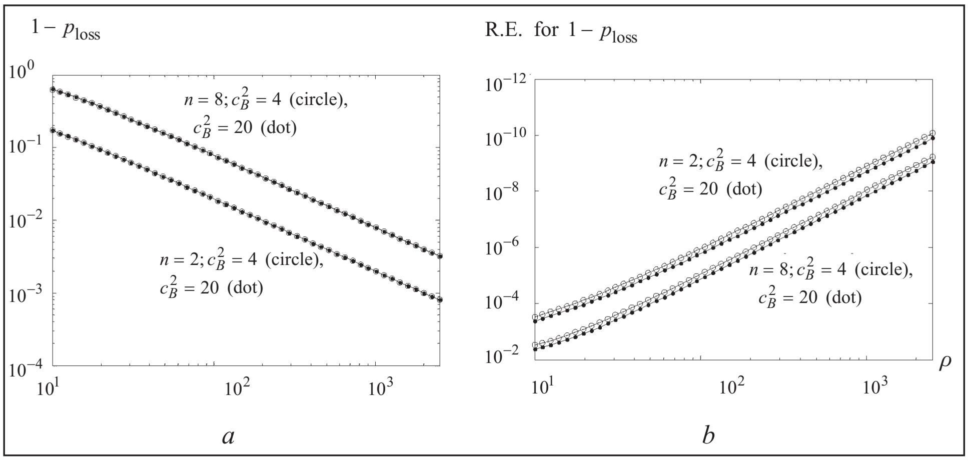

Fig. 3. (*a*): loss probabilities in heavy traffic when $c_A^2 = 2$ for the *GM Heuristic* (dots and circles) compared to exact values (lines); (*b*): the corresponding relative errors

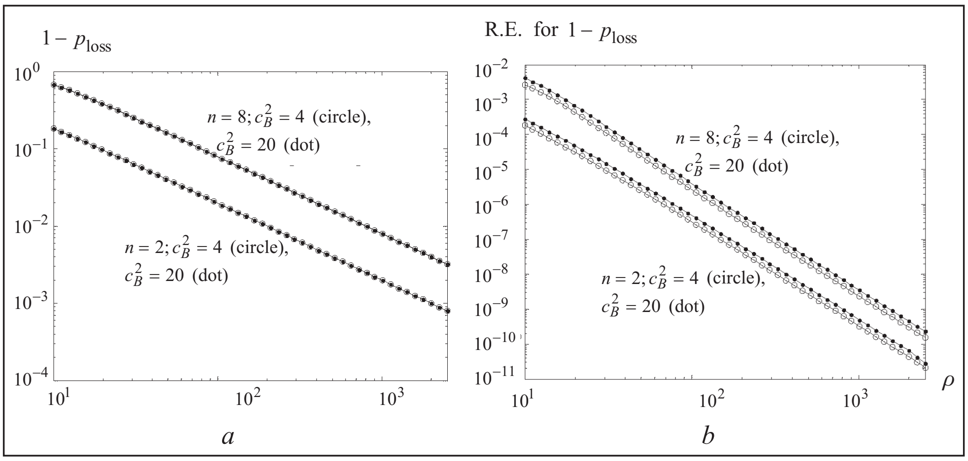

Fig. 4. (*a*): loss probabilities in heavy traffic when $c_A^2=0.5$ for the *GM Heuristic* (dots and circles) compared to exact values (lines); (*b*): the corresponding relative errors

results for the otherwise equivalent cases in which the value of c_A^2 is changed from 2 to 0.5. Again, the heuristic appears to be asymptotically exact in heavy traffic. This is in contrast to the previous results in light traffic, where there was a distinct difference between the nature of the results obtained for the two cases, $c_A^2=2$ and $c_A^2=0.5$.

It will be noted that the relative errors for the values of $1-p_{\text{loss}}$ in Fig. 3 are negative while those in Fig. 4 are positive. These results agree with established empirical rules, for the signs of the errors in the $C_2/C_2/n/0$ queue, which relate to the values of c_A^2 and c_B^2 and were given in [1].

The high level of accuracy of the heuristic is again apparent from the graphs. It will be seen in the next section that, for queueing systems such as those illustrated in Fig. 3 and 4, which satisfy certain general conditions, the *GM Heuristic* is asymptotically exact in heavy traffic.

7. A HEAVY-TRAFFIC ANALYSIS OF THE *GI/G/n/*0 QUEUE AND THE *GM Heuristic*

Without loss of generality, let $\lambda=1$. Also let $\rho=\dfrac{\lambda}{\mu}=\dfrac{1}{\mu}=nz$, where z is a constant and $z>1$;

$$\mu=\frac{1}{nz}\quad B(x)=B_0\left(\frac{1}{nz}x\right),$$

where $B_0(x)$ is a fixed d.f. with mean 1.

We assume the following analytical conditions:

(i) $\int\limits_0^\infty x^2\,dA(x)=\alpha_2<\infty$;

(ii) $A(x)$ and $B_0(x)$ are non-latticed d.f.

Let $q_n=q_n(GI,G,n,\rho)$. Also note that, for the $GI/M/n/0$ queue, $B_0(x)$ is given by $1-e^{-x}$, $x\ge 0$. The purpose of this section is to discover the behavior of q_n as $n\to\infty$.

(Elementary) statement 1. The inequality

$$q_n\ge 1-\frac{1}{z} \tag{11}$$

holds true.

Proof. If there had not been intervals between service times then the system would have serviced $n\mu T$ customers in a large time T; hence $q_n\ge 1-\dfrac{n\mu}{\lambda}=1-\dfrac{1}{z}$.

Statement 2. The inequality $\limsup_{n\to\infty} q_n \le 1-\frac{1}{z}$ holds true.

Proof. Any allocation of the input to specific groups of channels produces an increase in q_n. Thus, set $n = m + m + \ldots + m + m'$, where $m \le m' \le 2m$; this means that the channels are divided into groups of size m (and one of size m') and each customer is directed to an m-group with a probability m/n or to the m'-group with a probability m'/n. Denote by q'_m (respectively $q'_{m'}$) the loss probability for a group. The groups behave like independent loss systems.

Then obviously

$$q_n \le \max(q'_m, q'_{m'}). \tag{12}$$

Denote by $A_m(x)$ the inter-arrival time d.f. for an m-group. We have the following equation for its LST $\varphi_m(s)$:

$$\varphi_m(s) = \frac{m}{n}\sum_{k=1}^{\infty}\left(1-\frac{m}{n}\right)^{k-1}\varphi^k(s) = \frac{(m/n)\varphi(s)}{1-\left(1-\frac{m}{n}\right)\varphi(s)}. \tag{13}$$

Let $X^{(m)}$ be a generic r.v. with an LST given by (13); let Y be a generic service time. We will change the scaling and pass to variables $\frac{m}{n}X^{(m)}$ and $\frac{m}{n}Y$. By analysis of the RHS of (13) or directly by Rényi's theorem one obtains that

$$\mathbf{P}\left\{\frac{m}{n}X^{(m)} < x\right\} \to 1-e^{-x},\ \ x \ge 0 \text{ as } n\to\infty.$$

Further

$$\mathbf{P}\left\{\frac{m}{n}Y < x\right\} = B_0\left(\frac{x}{mz}\right).$$

In the limit, as $n\to\infty$, we have an $M/G/m/0$ system with a traffic intensity mz. (The details of a proper continuity theorem are omitted here.) We have

$$q'_m \sim \frac{(mz)^m}{m!}\Big/\sum_{k=0}^{m}\frac{(mz)^k}{k!} = \frac{1}{\sum_{i=0}^{m}\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{i-1}{m}\right)/z^i}. \tag{14}$$

If we make m large enough, the RHS of (14) can be made arbitrarily close to $1-\frac{1}{z}$. The same holds true for the m'-group.

Theorem 2. The inequality

$$q_n \to 1-\frac{1}{z} \quad \text{as } n\to\infty$$

holds true.

Proof. Apply (11) and (12).

Application of Takács's theory. As set out in Sec. 3, we have

$$\varphi(\mu) = \int_0^{\infty}\exp\{-x/(nz)\}dA(x) \sim 1-\frac{1}{nz}.$$

Thus

$$\frac{\varphi(\mu)}{1-\varphi(\mu)} \sim nz$$

and similarly

$$\frac{\varphi(i\mu)}{1-\varphi(i\mu)} \sim nz/i. \tag{15}$$

Hence

$$C_j \sim \prod_{i=1}^{j}(nz-i) \sim \frac{(nz)^j}{j!};$$

$$q_n(GI, M, n, \rho) = B_n = \frac{1}{\sum_{j=0}^{n} C_n^j \frac{1}{C_j}} \sim \frac{1}{\sum_{j=0}^{n} C_n^j \frac{j!}{(nz)^j}} \sim \frac{1}{1 + \frac{1}{z} + \frac{1}{z^2} + \ldots} \sim 1 - \frac{1}{z}.$$

Application of the *GM Heuristic.* The $\hat{q}(GI, G, n, \rho)$ values will be computed using the *GM Heuristic*, as set out in Sec. 3. In order to compute $q_1(GI, G, 1, \rho)$, consider the renewal function of the arrival process

$$H(t) = 1 + \sum_{k=1}^{\infty} A^{*k}(x),$$

where $A^{*k}(x)$ is a k-fold convolution of $A(x)$ with itself. We start with the formula

$$q_1(GI, G, 1, \rho) = 1 - \frac{1}{\gamma}, \quad (16)$$

where

$$\gamma = \int_0^{\infty} H(x) dB(x). \quad (17)$$

Since $\alpha_2 < \infty$, an inequality $|H(x) - x| \le c$ holds true with a constant c. Hence $\left|\gamma - \frac{1}{\mu}\right| = |\gamma - nz| \le c$, so that, by (16), (17),

$$q_1(GI, G, 1, \rho) \sim 1 - \frac{1}{nz}$$

and

$$\frac{q_1(GI, G, 1, \frac{\rho}{i})}{1 - q_1(GI, G, 1, \frac{\rho}{i})} \sim \frac{nz}{i}. \quad (18)$$

The RHSs of (15) and (18) are the same. We observe that

$$\hat{q}_n(GI, G, n, \rho) \sim 1 - \frac{1}{x} \sim q_n(GI, G, n, \rho).$$

Therefore the *GM Heuristic* is asymptotically exact in the domain $\{n \to \infty, \ \rho = nx, x > 1\}$.

8. CONCLUDING REMARKS

It has been shown that the *GM Heuristic* is asymptotically exact in light traffic for the $GI/G/n/0$ queue when the set of conditions (i) to (v) in Sec. 5 are satisfied. Under more general conditions, the heuristic is also asymptotically exact in heavy traffic as the number of servers n tends to infinity (for $\rho = nx, x > 1$). These results were illustrated by numerical calculations. Since the *GM Heuristic* is extremely fast, it is potentially useful for practical applications involving either light or heavy traffic.

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