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WEAK CONVERGENCE OF FIRST-RARE-EVENT TIMES FOR SEMI-MARKOV PROCESSES

Necessary and sufficient conditions for weak convergence of first-rare-event times for semi-Markov processes with finite set of states in series of schemes are obtained.

1. Introduction

Limit theorems for random functionals of similar first-rare-event times known under such names as first hitting times, first passage times, first record times, etc. were studied by many authors. Revue of the literature related to the subject can be found in Silvestrov (2004) and in the recent papers by Silvestrov and Drozdenko (2005, 2006a, 2006b).

The main features for the most previous results are that they give sufficient conditions of convergence for such functionals. As a rule, those conditions involve assumptions, which imply convergence of distributions for sums of i.i.d random variables distributed as sojourn times for the semi-Markov process (for every state) to some infinitely divisible laws plus some ergodicity condition for the embedded Markov chain plus condition of vanishing probabilities of occurring rare event during one transition step for the semi-Markov process.

Our results are related to the model of semi-Markov processes with a finite set of states. In the papers by Silvestrov and Drozdenko (2005, 2006a, 2006b) necessary and sufficient conditions of first-rare-event times for semi-Markov processes were obtained for the non-triangular-array case of stable type asymptotics for sojourn times distributions.

In the present paper we generalize results of those papers to a general triangular array model.

Instead of using traditional approach based on conditions for "individual" distributions of sojourn times, we use more general and weaker conditions imposed on distributions of sojourn times averaged by the stationary

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distribution of the limit embedded Markov chain. Moreover, we show that these conditions are not only sufficient but also necessary conditions for the weak convergence for first-rare-event times, and describe the class of all possible not-concentrated in zero limit laws. The results presented in the paper give some kind of a “final solution” for limit theorems for first-rare-event times for semi-Markov process with a finite set of states in triangular array mode.

In addition to the references given in Silvestrov and Drozdenko (2005, 2006a, 2006b), we would like to mention some recent publications relevant to our research: Anisimov (2005), Avrachenkov and Haviv (2003), Dayar (2005), Di Crescenzo and Nastro (2004), Fuh (2004), Harrison and Knottenbelt (2002), Hunter (2005), Janssen and Manca (2006), Koroliuk and Limnios (2005), Limnios, Ouhbi, and Sadek (2005), Nguyen, Vuong, and Tran (2005), Solan and Vielle (2003), Symeonaki and Stamou (2006), Szewczak (2005).

The paper is organized in the following way. In Section 2, we formulate and prove our main Theorem 1, which describes the class of all possible limit distributions for first-rare-event times for semi-Markov processes and give necessary and sufficient conditions of weak convergence to distributions from this class. Several lemmas describing asymptotical solidarity cyclic properties for sum-processes defined on Markov chains are used in the proof of Theorem 1. These lemmas and their proofs are collected in Section 3.

2. Main results

Let $\left(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)}\right)$, $n = 0, 1, \ldots$ be, for every $\varepsilon > 0$, a Markov renewal process, i.e. a homogenous Markov chain with phase space $Z = X \times [0, +\infty) \times Y$ (here $X = \{1, 2, \ldots, m\}$, and Y is some measurable space with σ–algebra of measurable sets B_Y) and transition probabilities,

$$\begin{aligned} &\mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j, \varkappa_{n+1}^{(\varepsilon)} \le t, \zeta_{n+1}^{(\varepsilon)} \in A/\eta_n^{(\varepsilon)} = i, \varkappa_n^{(\varepsilon)} = s, \zeta_n^{(\varepsilon)} = y\right\} \\ &\quad = \mathsf{P}\left\{\eta_{n+1}^{(\varepsilon)} = j, \varkappa_{n+1}^{(\varepsilon)} \le t, \zeta_{n+1}^{(\varepsilon)} \in A/\eta_n^{(\varepsilon)} = i\right\} \\ &\quad = Q_{ij}^{(\varepsilon)}(t, A),\ i, j \in X,\ s, t \ge 0,\ y \in Y, A \in B_Y. \end{aligned} \tag{1}$$

The characteristic property, which specifies Markov renewal processes in the class of general multivariate Markov chains $\left(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)}\right)$, is (as shown in (1)) that transition probabilities do depend only of the current position of the first component $\eta_n^{(\varepsilon)}$.

As is known, the first component $\eta_n^{(\varepsilon)}$ of the Markov renewal process is also a homogenous Markov chain with the phase space X and transition probabilities $p_{ij}^{(\varepsilon)} = Q_{ij}^{(\varepsilon)}(+\infty, Y)$, $i, j \in X$.

Also, the first two components of Markov renewal process (namely $\eta_n^{(\varepsilon)}$ and $\varkappa_n^{(\varepsilon)}$) can be associated with the semi-Markov process $\eta^{(\varepsilon)}(t)$, $t \geq 0$ defined as,

$$\eta^{(\varepsilon)}(t) = \eta_n^{(\varepsilon)} \quad \text{for} \quad \tau_n^{(\varepsilon)} \leq t < \tau_{n+1}^{(\varepsilon)}, \quad n = 0, 1, \ldots,$$

where $\tau_0^{(\varepsilon)} = 0$ and $\tau_n^{(\varepsilon)} = \varkappa_1^{(\varepsilon)} + \ldots + \varkappa_n^{(\varepsilon)}$, $n \geq 1$.

Random variables $\varkappa_n^{(\varepsilon)}$ represent inter–jump times for the process $\eta^{(\varepsilon)}(t)$. As far as random variables $\zeta_n^{(\varepsilon)}$ are concerned, they are so-called, "flag variables" and are used to record "rare" events.

Let D_ε, $\varepsilon > 0$ be a family of measurable "small" in some sense subsets of Y. Then events $\left\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}$ can be considered as "rare".

Let us introduce random variables

$$\nu_\varepsilon = \min\left(n \geq 1 : \zeta_n^{(\varepsilon)} \in D_\varepsilon\right),$$

and

$$\xi_\varepsilon = \sum_{n=1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}.$$

A random variable ν_ε counts the number of transitions of the embedded Markov chain $\eta_n^{(\varepsilon)}$ up to the first appearance of the "rare" event, while a random variable ξ_ε can be interpreted as the first-rare-event time for the semi-Markov process $\eta^{(\varepsilon)}(t)$.

Let us consider the distribution function of the first-rare-event time ξ_ε, under fixed initial state of the embedded Markov chain $\eta_n^{(\varepsilon)}$,

$$F_i^{(\varepsilon)}(u) = \mathsf{P}_i\{\xi_\varepsilon \leq u\}, \; u \geq 0.$$

Here and henceforth, P_i and E_i denote, respectively, conditional probability and expectation calculated under condition that $\eta_0 = i$.

We give necessary and sufficient conditions for weak convergence of distribution functions $F_i^{(\varepsilon)}(uu_\varepsilon)$, where $u_\varepsilon > 0$, $u_\varepsilon \to \infty$ as $\varepsilon \to 0$ is a non-random normalising function, and describe the class of possible limit distributions.

The problem is solved under the four general model assumptions.

The first assumption **A** guaranties that the last summand in the random sum ξ_ε is negligible under any normalization u_ε, i.e. $\varkappa_{\nu_\varepsilon}^{(\varepsilon)} \big/ u_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$:

A: $\lim\limits_{t\to\infty} \overline{\lim\limits_{\varepsilon\to 0}} \, \mathsf{P}_i\left\{\varkappa_1^{(\varepsilon)} > t / \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} = 0, \; i \in X.$

Let us introduce the probabilities of occurrence of rare event during one transition step of the semi-Markov process $\eta^{(\varepsilon)}(t)$,

$$p_{i\varepsilon} = \mathsf{P}_i\left\{\zeta_1^{(\varepsilon)} \in D_\varepsilon\right\}, \; i \in X.$$

The second assumption **B**, imposed on probabilities $p_{i\varepsilon}$, specifies interpretation of the event $\left\{\zeta_n^{(\varepsilon)} \in D_\varepsilon\right\}$ as "rare" and guarantees the possibility for such event to occur:

B: $0 < \max_{1\le i\le m} p_{i\varepsilon} \to 0$ as $\varepsilon \to 0$.

The third assumption **C** is a condition of convergence of transition matrix of embedded perturbed Markov chain $\eta_n^{(\varepsilon)}$ to transition matrix of embedded limit Markov chain $\eta_n^{(0)}$:

C: $p_{ij}^{(\varepsilon)} \to p_{ij}^{(0)}$ as $\varepsilon \to 0$, $i, j \in X$.

The forth assumption **D** is a standard ergodicity condition for the limit embedded Markov chain $\eta_n^{(0)}$:

D: Markov chain $\eta_n^{(0)}$ with matrix of transition probabilities $\left\|p_{ij}^{(0)}\right\|$ is ergodic with stationary distribution $\pi_i^{(0)}$, $i \in X$.

Let us define a probability which is the result of averaging of the probabilities of occurrence of rare event in one transition step by the stationary distribution of the embedded limit Markov chain $\eta_n^{(0)}$,

$$p_\varepsilon = \sum_{i=1}^{m} \pi_i^{(0)} p_{i\varepsilon}.$$

Let us also introduce the distribution functions of a sojourn times $\varkappa_1^{(\varepsilon)}$ for the semi-Markov processes $\eta^{(\varepsilon)}(t)$,

$$G_i^{(\varepsilon)}(t) = \mathsf{P}_i\left\{\varkappa_1^{(\varepsilon)} \le t\right\},\ t \ge 0,\ i \in X,$$

and the distribution function, which is a result of averaging of distribution functions of sojourn times by the stationary distribution of the embedded Markov chain $\eta_n^{(0)}$,

$$G^{(\varepsilon)}(t) = \sum_{i=1}^{m} \pi_i^{(0)} G_i^{(\varepsilon)}(t),\ t \ge 0.$$

Now we are in position to formulate the necessary and sufficient conditions for weak convergence of distribution functions of first-rare-event times ξ_ε. Mentioned conditions have the following form:

E: $p_\varepsilon^{-1}\left(1 - G^{(\varepsilon)}(uu_\varepsilon)\right) \to h(u)$ as $\varepsilon \to 0$ for all $u > 0$, which are points of continuity of the limit function $h(u)$.

F: $p_\varepsilon^{-1}\int_0^{uu_\varepsilon} s\, G^{(\varepsilon)}(ds) \to f(u)$ as $\varepsilon \to 0$ for some $u > 0$ which is a point of continuity of $h(u)$.

The limits here satisfy a number of conditions:

($\mathbf{a}_1$) $h(u)$ is a non-negative, non-increasing, and right-continuous function for $u > 0$ and $h(\infty) = 0$;

($\mathbf{a}_2$) The measure $\mathrm{H}(A)$ on σ-algebra $\mathcal{H}^+$, the Borel σ-algebra of subsets of $(0, \infty)$, defined by the relation $\mathrm{H}((u_1, u_2]) = h(u_1) - h(u_2)$, $0 < u_1 \le u_2 < \infty$, satisfies the condition $\int_0^\infty \frac{s}{1+s}\mathrm{H}(ds) < \infty$;

($\mathbf{a}_3$) Under **E**, condition **F** can only hold simultaneously for all continuity points of $h(u)$ and $f(u_1) = f(u_2) - \int_{u_1}^{u_2} s\mathrm{H}(ds)$ for any such points $0 < u_1 < u_2 < \infty$;

($\mathbf{a}_4$) $f(u)$ is a non-negative function.

We use the symbol $\Rightarrow$ to show weak convergence of distribution functions (pointwise convergence in points of continuity of the limit distribution function).

Conditions **E** and **F** are necessary and sufficient conditions for the weak convergence,

$$\vartheta^{(\varepsilon)}(t) = \sum_{k=1}^{[tp_\varepsilon^{-1}]} \frac{\vartheta_k^{(\varepsilon)}}{u_\varepsilon}, t \ge 0 \Rightarrow \vartheta(t), t \ge 0 \text{ as } \varepsilon \to 0, \tag{2}$$

where $\vartheta_k^{(\varepsilon)}$ are i.i.d. random variables with joint distribution $G^{(\varepsilon)}(t)$ and the cumulant $a(s)$ of the limit process $\vartheta(t)$ (i.e. $Ee^{-s\vartheta(t)} = e^{-a(s)t}$), according to *Lévy-Khintchine representation formula*, has the following form

$$a(s) = as - \int_0^\infty (e^{-sx} - 1)\mathrm{H}(dx), \tag{3}$$

where the constant

$$a = f(u) - \int_0^u s\mathrm{H}(ds)$$

does not depend on the choice of the point u in condition **F**.

The main result of the paper is the following theorem.

Theorem. *Let conditions* **A**, **B**, **C**, *and* **D** *hold. Then*:

(i): *The class of all possible non-concentrated in zero limit distribution functions (in the sense of weak convergence) for the distribution functions of first-rare-event times $F_i^{(\varepsilon)}(uu_\varepsilon)$ coincides with the class of distribution functions $F(u)$ with Laplace transforms $\phi(s) = \frac{1}{1+a(s)}$.*

(ii): *Conditions* **E** *and* **F** *are necessary and sufficient for the following relation of weak convergence to hold (for some or every $i \in X$, respectively, in the statements of necessity and sufficiency),*

$$F_i^{(\varepsilon)}(uu_\varepsilon) \Rightarrow F(u) \text{ as } \varepsilon \to 0, \tag{4}$$

where $F(u)$ is the distribution function with Laplace transform $\frac{1}{1+a(s)}$.

Remark 1. $F(u)$ is the distribution function of a random variable $\xi(\rho)$, where ($\mathbf{b}_1$) $\xi(t)$, $t \geq 0$ is a non-negative homogeneous stable process with independent increments and the Laplace transform $\mathsf{E}e^{-s\xi(t)} = e^{-a(s)t}$, $s, t \geq 0$, ($\mathbf{b}_2$) ρ is an exponentially distributed random variable with parameter 1, ($\mathbf{b}_3$) the random variable ρ and the process $\xi(t)$, $t \geq 0$ are independent.

Proof. We split the proof of Theorem 1 into several steps.

As the first step, we obtain an appropriate representation for the first-rare-event time ξ_ε in the form of geometric type random sum of random variables connected with cyclic returns of the semi-Markov process $\eta^{(\varepsilon)}(t)$ to a fixed state $i \in X$.

Let $\tau_i^{(\varepsilon)}(n)$ be the number of transitions after which the embedded Markov chain $\eta_n^{(\varepsilon)}$ reaches a state $i \in X$ for the n-th time,

$$\tau_i^{(\varepsilon)}(n) = \min\left\{k > \tau_i^{(\varepsilon)}(n-1) : \eta_k^{(\varepsilon)} = i\right\},\ n = 1, 2, \ldots,$$

where $\tau_i^{(\varepsilon)}(0) = 0$. For simplicity, we will write $\tau_i^{(\varepsilon)}(1)$ as $\tau_i^{(\varepsilon)}$.

Let $\beta_i^{(\varepsilon)}(n)$ be the duration of the n-th i-cycle between the moments of $(n-1)$-th and n-th return of the semi-Markov process $\eta^{(\varepsilon)}(t)$ to the state i,

$$\beta_i^{(\varepsilon)}(n) = \sum_{k=\tau_i^{(\varepsilon)}(n-1)+1}^{\tau_i^{(\varepsilon)}(n)} \varkappa_k^{(\varepsilon)},\ n = 1, 2, \ldots.$$

For simplicity, we will also write $\beta_i^{(\varepsilon)}(1)$ as $\beta_i^{(\varepsilon)}$. The moments of return of the semi-Markov process $\eta^{(\varepsilon)}(t)$ to a fixed state $i \in X$ are regenerative moments for this process. Due to this property, $\beta_i^{(\varepsilon)}(n), n = 1, 2\ldots$ are i.i.d. random variables for $n \geq 2$. As far as the random variable $\beta_i^{(\varepsilon)}(1)$ is concerned, it has the same distribution as $\beta_i^{(\varepsilon)}(2)$ if the initial distribution of the embedded Markov chain $\eta_n^{(\varepsilon)}$ is concentrated in state i. Otherwise, the distribution of $\beta_i^{(\varepsilon)}(1)$ can differ from the distribution of $\beta_i^{(\varepsilon)}(2)$.

Let us also introduce the random variable $\nu_{i\varepsilon}$ which counts the number of cycles ended before the moment ν_ε,

$$\nu_{i\varepsilon} = \max\left\{n : \tau_i^{(\varepsilon)}(n) \leq \nu_\varepsilon\right\}.$$

Finally, let $\tilde{\beta}_{i\varepsilon}$ be the duration of the residual sub-cycle, between the moment of the last return of the semi-Markov process $\eta^{(\varepsilon)}(t)$ to the state i before the first-rare-event time ξ_ε, and the time ξ_ε,

$$\tilde{\beta}_{i\varepsilon} = \sum_{n=\tau_i^{(\varepsilon)}(\nu_{i\varepsilon})+1}^{\nu_\varepsilon} \varkappa_n^{(\varepsilon)}.$$

Now, the following representation, in the form of random sum, can be written down for the first-rare-event time ξ_ε,

$$\xi_\varepsilon = \sum_{n=1}^{\nu_{i\varepsilon}} \beta_i^{(\varepsilon)}(n) + \tilde{\beta}_{i\varepsilon}. \tag{5}$$

It should be noted that the random index $\nu_{i\varepsilon}$ and summands $\beta_i^{(\varepsilon)}(n), n = 1, 2, \ldots$, and $\tilde{\beta}_{i\varepsilon}$ are not independent random variables. However, they are conditionally independent with respect to the indicator random variables $\chi_{i\varepsilon}(n) = \chi\left(\tau_i^{(\varepsilon)}(n-1) < \nu_\varepsilon \le \tau_i^{(\varepsilon)}(n)\right), n = 1, 2, \ldots$. It will be seen in the best way when we shall rewrite the representation formula (5) in terms of Laplace transforms.

Let us introduce Laplace transforms of the first-rare-event time,

$$\Phi_{i\varepsilon}(s) = \mathsf{E}_i \exp\{-s\xi_\varepsilon\}, \ s \ge 0, \ i \in X.$$

Let us denote $q_{i\varepsilon}$ the probability of occurrence the rare event during the first i-cycle,

$$q_{i\varepsilon} = \mathsf{P}_i\left\{\nu_\varepsilon \le \tau_i^{(\varepsilon)}\right\}, \ i \in X.$$

Let us also introduce the conditional Laplace transforms of the duration of the first i-cycle $\beta_i^{(\varepsilon)}$ under condition $\nu_\varepsilon > \tau_i^{(\varepsilon)}$ of non-occurrence of the rare event in the first i-cycle,

$$\bar{\psi}_{i\varepsilon}(s) = \mathsf{E}_i\left\{\exp\left\{-s\beta_i^{(\varepsilon)}\right\} / \nu_\varepsilon > \tau_i^{(\varepsilon)}\right\}, \ s \ge 0,$$

and the conditional Laplace transform of the duration of residual sub-cycle $\beta_{i\varepsilon}$ under condition that $\nu_\varepsilon \le \tau_i^{(\varepsilon)}$ of occurrence of the rare event in the first i-cycle,

$$\widetilde{\psi}_{i\varepsilon}(s) = \mathsf{E}_i\left\{\exp\left\{-s\tilde{\beta}_{i\varepsilon}\right\} / \nu_\varepsilon \le \tau_i^{(\varepsilon)}\right\}, \ s \ge 0.$$

The Markov renewal process $\left(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)}\right)$ regenerates at moments of return to every state i, and ν_ε is a Markov moment for this process. Due to these properties the representation formula (5) takes, in terms of Laplace transforms, the following form,

$$\begin{aligned}\Phi_{i\varepsilon}(s) &= \mathsf{E}_i \exp\{-s\xi_\varepsilon\} \\ &= \sum_{n=0}^{\infty}(1-q_{i\varepsilon})^n q_{i\varepsilon}\bar{\psi}_{i\varepsilon}(s)^n\widetilde{\psi}_{i\varepsilon}(s) \\ &= \frac{q_{i\varepsilon}\widetilde{\psi}_{i\varepsilon}(s)}{1-(1-q_{i\varepsilon})\bar{\psi}_{i\varepsilon}(s)} \\ &= \frac{\widetilde{\psi}_{i\varepsilon}(s)}{1+(1-q_{i\varepsilon})\frac{\left(1-\bar{\psi}_{i\varepsilon}(s)\right)}{q_{i\varepsilon}}}, \ s \geq 0. \end{aligned} \tag{6}$$

As the second step, we prove that the weak convergence for the first-rare-event times is invariant with respect to the choice of initial distribution of the embedded Markov chain $\eta_n^{(\varepsilon)}$.

At this stage we are interested in solidarity statements concerned the relation of weak convergence,

$$F_i^{(\varepsilon)}(uu_\varepsilon) \Rightarrow F(u) \text{ as } \varepsilon \to 0, \tag{7}$$

where ($\mathbf{c}_1$) $F(u)$ is a distribution function concentrated on non-negative half-line but not concentrated in zero, and ($\mathbf{c}_2$) u_ε is a positive normalizing function such that $u_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We shall prove that, under conditions **A**, **B**, **C**, and **D**, (**d**) the assumption that relation (7) holds for some $i \in X$ implies that this relation holds for every $i \in X$ and, in this case, (**e**) the limit distribution function $F(u)$ is the same for all $i \in X$.

In terms of Laplace transforms relation (7) is equivalent to the relation,

$$\Phi_{i\varepsilon}(s/u_\varepsilon) \to \Phi(s) \text{ as } \varepsilon \to 0, \ s \geq 0, \tag{8}$$

where (**f**) $\Phi(s)$ is a Laplace transform of some non-negative random variable, (**g**) $\Phi(s) < 1$ for $s > 0$ (this is equivalent to the requirement that the corresponding limit distribution function is not concentrated in zero).

Thus, in order to prove the solidarity statement formulated above, we should prove that, under conditions **A**, **B**, **C**, and **D**, (**h**) the assumption that relation (8) holds for some $i \in X$ implies that this relation holds for every $i \in X$ and, in this case, (**i**) the limit Laplace transform $\Phi(s)$ is the same for all $i \in X$.

In what follows, we the use several lemmas describing asymptotical solidarity cyclic properties for functional defined on trajectories of Markov renewal processes $\left(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)}\right)$.

It will be proved in Lemma 3 that conditions **B**, **C**, and **D** imply the following asymptotic relation, for every $i \in X$,

$$q_{i\varepsilon} \sim \frac{p_\varepsilon}{\pi_i^{(0)}} \text{ as } \varepsilon \to 0. \tag{9}$$

Here and henceforth relation $a(\varepsilon) \sim b(\varepsilon)$ as $\varepsilon \to 0$ means that $a(\varepsilon)/b(\varepsilon) \to 1$ as $\varepsilon \to 0$.

It follows from (9) that, for every $i \in X$,

$$q_{i\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{10}$$

It will be shown in Lemma 4, with the use of (9), that conditions **A**, **B**, **C**, and **D** implies the following asymptotic relation, for every $i \in X$,

$$\widetilde{\psi}_{i\varepsilon}(s/u_\varepsilon) \to 1 \text{ as } \varepsilon \to 0, \ s \geq 0. \tag{11}$$

Relation (11) implies that, under conditions **A**, **B**, **C**, and **D** for every $i \in X$,

$$\Phi_{i\varepsilon}(s/u_\varepsilon) \sim \frac{1}{1 + (1 - q_{i\varepsilon})\frac{\left(1 - \bar{\psi}_{i\varepsilon}(s/u_\varepsilon)\right)}{q_{i\varepsilon}}} \text{ as } \varepsilon \to 0, \ s \geq 0. \tag{12}$$

It follows from relations and (10) and (12) that, under conditions **A**, **B**, **C**, and **D** relation (8) holds, for given $i \in X$, if and only if,

$$\frac{1 - \bar{\psi}_{i\varepsilon}(s/u_\varepsilon)}{q_{i\varepsilon}} \to \varsigma(s) \text{ as } \varepsilon \to 0, \ s \geq 0, \tag{13}$$

where $\varsigma(s)$ is a function such that (**j**) $\frac{1}{1+\varsigma(s)}$ is a Laplace transform of some non-negative random variable, and (**k**) $\varsigma(s) > 0$ for $s > 0$.

Obviously, that the limit functions in relations (8) and (13) are connected by the following relation,

$$\Phi(s) = \frac{1}{1 + \varsigma(s)}, \ s \geq 0. \tag{14}$$

To simplify the following asymptotic analysis, we shall now try to replace the conditional Laplace transform $\bar{\psi}_{i\varepsilon}(s)$ in the relation (13) by the unconditional Laplace transform of the duration of the first i-cycle $\beta_i^{(\varepsilon)}$,

$$\psi_i^{(\varepsilon)}(s) = \mathsf{E}_i \exp\left\{-s\beta_i^{(\varepsilon)}\right\}, \ s \geq 0.$$

The Laplace transform $\psi_i^{(\varepsilon)}(s)$ can obviously be represented in the following form,

$$\psi_i^{(\varepsilon)}(s) = (1 - q_{i\varepsilon})\bar{\psi}_{i\varepsilon}(s) + q_{i\varepsilon}\widehat{\psi}_{i\varepsilon}(s), \ s \geq 0, \tag{15}$$

where $\widehat{\psi}_{i\varepsilon}(s)$ is the conditional Laplace transform of the duration of the first i-cycle $\beta_i^{(\varepsilon)}$ under condition $\nu_\varepsilon \leq \tau_i^{(\varepsilon)}$ of occurrence of the rare event in the first i-cycle,

$$\widehat{\psi}_{i\varepsilon}(s) = \mathsf{E}_i \left\{ \exp\left\{ -s\beta_i^{(\varepsilon)} \right\} / \nu_\varepsilon \leq \tau_i^{(\varepsilon)} \right\}, \ s \geq 0.$$

Relation (15) can be re-written in the following form,

$$\frac{1 - \psi_i^{(\varepsilon)}(s/u_\varepsilon)}{q_{i\varepsilon}} = (1 - q_{i\varepsilon}) \frac{1 - \bar{\psi}_{i\varepsilon}(s/u_\varepsilon)}{q_{i\varepsilon}} + q_{i\varepsilon} \frac{1 - \widehat{\psi}_{i\varepsilon}(s/u_\varepsilon)}{q_{i\varepsilon}}, \ s \geq 0. \quad (16)$$

It will be shown in Lemma 3 that conditions **A**, **B**, **C**, and **D** imply that, for every $i \in X$,

$$\widehat{\psi}_{i\varepsilon}(s/u_\varepsilon) \to 1 \text{ as } \varepsilon \to 0, \ s \geq 0. \quad (17)$$

It follows from relation (17) that, under conditions **A**, **B**, **C**, and **D**, relation (13) holds, for given $i \in X$, if and only if,

$$\frac{1 - \psi_i^{(\varepsilon)}(s/u_\varepsilon)}{q_{i\varepsilon}} \to \varsigma(s) \text{ as } \varepsilon \to 0, \ s \geq 0, \quad (18)$$

where $\varsigma(s)$ is a function such that (**j**) $\frac{1}{1+\varsigma(s)}$ is a Laplace transform of some non-negative random variable, and (**k**) $\varsigma(s) > 0$ for $s > 0$.

It will be shown in Lemma 4 that, under conditions **B**, **C**, and **D**, (**l**) the assumption that relation (18) holds for some $i \in X$ implies that this relation holds for every $i \in X$ and, in this case, (**m**) the limit function $\varsigma(s)$ is the same for all $i \in X$, (**n**) $\varsigma(s)$ is a cumulant of an infinitely divisible law concentrated on non-negative half-line and not concentrated in zero.

Note that, in this case, ($\mathbf{o}_1$) the function $\frac{1}{1+\varsigma(s)}$ is a Laplace transform of the random variable $\xi(\rho)$, where ($\mathbf{o}_2$) $\xi(t)$, $t \geq 0$ is a non-negative homogeneous process with independent increments and the Laplace transform $\mathsf{E}e^{-s\xi(t)} = e^{-\varsigma(s)t}$, ($\mathbf{o}_3$) ρ is exponentially distributed random variable, with parameter 1, ($\mathbf{o}_4$) the random variable ρ is independent of the process $\xi(t)$, $t \geq 0$, and ($\mathbf{o}_5$) $\varsigma(s) > 0$ for $s > 0$. These properties are consistent with requirements (**j**) and (**k**).

Let introduce the Laplace transforms for the sojourn times $\varkappa_1^{(\varepsilon)}$,

$$\varphi_i^{(\varepsilon)}(s) = \mathsf{E}_i \exp\left\{ -s\varkappa_1^{(\varepsilon)} \right\} = \int_0^\infty e^{-st} G_i^{(\varepsilon)}(dt), \ s \geq 0,$$

and the corresponding Laplace transform averaged by the stationary distribution of the embedded Markov chain $\eta_n^{(\varepsilon)}$,

$$\varphi^{(\varepsilon)}(s) = \sum_{i=1}^m \pi_i^{(0)} \varphi_i^{(\varepsilon)}(s) = \int_0^\infty e^{-st} G^{(\varepsilon)}(dt), \ s \geq 0.$$

Finally, it will be shown in Lemma 5 that, under conditions **A**, **B**, **C**, and **D**, relation (18) holds, for given $i \in X$, if and only if,

$$\frac{1 - \varphi^{(\varepsilon)}(s/u_\varepsilon)}{p_\varepsilon} \to \varsigma(s) \text{ as } \varepsilon \to 0, \ s \geq 0, \tag{19}$$

where (**p**) $\varsigma(s)$ is a cumulant of an infinitely divisible law concentrated on non-negative half-line and not concentrated in zero.

Relation (19) is the final point in series the solidarity statements concerned the distributions of first-rare-event times and based on conditions **A**, **B**, **C**, and **D**.

The last step in the proof is the standard one. As was mentioned above **E** and **F** are equivalent to asymptotic relation (2). In terms of Laplace transform (2) is equivalent (for every $t > 0$) to the following relations

$$\begin{aligned} \mathsf{E} \exp\left\{-s\vartheta^{(\varepsilon)}(t)\right\} &= \left(\varphi^{(\varepsilon)}(s/u_\varepsilon)\right)^{[t/p_\varepsilon]} \\ &\sim \exp\left\{-(1 - \varphi^{(\varepsilon)}(s/u_\varepsilon))t/p_\varepsilon\right\} \\ &\to \exp\{-a(s)t\} \text{ as } \varepsilon \to 0. \end{aligned} \tag{20}$$

It follows from (20) that relation (2) is equivalent to (19) and in this case

$$\varsigma(s) = a(s), \ s \geq 0. \tag{21}$$

This completes the proof of Theorem 1. □

3. Cyclic conditions of convergence

In this section we prove Lemmas 1-7 used in the proof of Theorem 1. These lemmas present a series of so-called cyclic solidarity conditions of convergence connected with the first-rare-event times and, as we think, have their own value.

Conditions **C** and **D** obviously imply that Markov chain $\eta_n^{(\varepsilon)}$ is also ergodic for all ε small enough. Let denote by $\pi^{(\varepsilon)}$ stationary distribution of Markov chain $\eta_n^{(\varepsilon)}$. As is known, stationary distributions are unique solution of the system

$$\begin{cases} \pi_i^{(\varepsilon)} = \sum_{k=1}^m \pi_k^{(\varepsilon)} p_{ki}^{(\varepsilon)}, \quad i = \overline{1, m}, \\ \sum_{k=1}^m \pi_k^{(\varepsilon)} = 1. \end{cases} \tag{22}$$

Lemma 1. *Conditions* **C**, **D** *imply that*

$$\pi_i^{(\varepsilon)} \to \pi_i^{(0)} \ \textit{as } \varepsilon \to 0, \quad i \in X. \tag{23}$$

Proof. For every $L \in (0,1)$ exists n such that

$$\max_{i\in X} \mathsf{P}_i \left\{ \tau_j^{(0)} \geq n \right\} < L. \tag{24}$$

By condition **C**, for any $i, j \in X$, and $n \geq 1$,

$$\begin{aligned} \mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq n \right\} &= \sum_{i_0=i,\cdots,i_n\neq j} \left(\prod_{k=1}^{n} p_{i_{k-1},i_k}^{(\varepsilon)} \right) \\ &\to \sum_{i_0=i,\cdots,i_n\neq j} \left(\prod_{k=1}^{n} p_{i_{k-1},i_k}^{(0)} \right) \\ &= \mathsf{P}_i \left\{ \tau_j^{(0)} \geq n \right\} \text{ as } \varepsilon \to 0. \end{aligned} \tag{25}$$

Relation (25) means that random variables $\tau_j^{(\varepsilon)}$ converge weakly to $\tau_j^{(0)}$ as $\varepsilon \to 0$ for any $j \in X$. Relations (24) and (25) imply that exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$, $j \in X$

$$\max_{i\in X} \mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq n \right\} < L, \tag{26}$$

Using (26) we get for $r = 1, 2, \cdots$ and $\varepsilon \leq \varepsilon_0$ and $i, j \in X$

$$\begin{aligned} &\mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq rn \right\} = \\ &\sum_k \mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq r(n-1), \eta_{r(n-1)} = k \right\} \mathsf{P}_k \left\{ \tau_j^{(\varepsilon)} \geq n \right\} \leq L^r. \end{aligned} \tag{27}$$

Finally, for any $x > 0$, $\varepsilon \leq \varepsilon_0$ and $i, j \in X$, using (27), we get

$$\begin{aligned} \max_{i\in X} \mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq x \right\} &\leq \max_{i\in X} \mathsf{P}_i \left\{ \tau_j^{(\varepsilon)} \geq \left[\frac{x}{n} \right] n \right\} \\ &\leq L^{x/n}. \end{aligned} \tag{28}$$

Relation (28) implies, in an obvious way, that, for any $m \geq 1$ and $i, j \in X$

$$\sup_{\varepsilon\leq\varepsilon_0} \mathsf{E}_j [\tau_i^{(\varepsilon)}]^m < \infty. \tag{29}$$

It follows from (25) and (29), for $i, j \in X$,

$$\mathsf{E}_i [\tau_j^{(\varepsilon)}]^m \to \mathsf{E}_i [\tau_j^{(0)}]^m \text{ as } \varepsilon \to 0. \tag{30}$$

As is known,

$$\pi_i^{(\varepsilon)} = \left[\mathsf{E}_j \tau_j^{(\varepsilon)} \right]^{-1}, \; j \in X. \tag{31}$$

Relations (30) and (31) imply asymptotic relation given in Lemma 1. □

Let us define

$$\bar{p}_\varepsilon = \sum_{i=1}^{m} \pi_i^{(\varepsilon)} p_{i\varepsilon}.$$

Lemma 2. *Conditions* **B**, **C**, *and* **D** *imply that*

$$\bar{p}_\varepsilon \sim p_\varepsilon \text{ as } \varepsilon \to 0.$$

Proof. Using Lemma 1, we get

$$\begin{aligned}
\left| \frac{\bar{p}_\varepsilon - p_\varepsilon}{p_\varepsilon} \right| &= \frac{\left| \sum_{i=1}^m \pi_i^{(\varepsilon)} p_{\varepsilon i} - \sum_{i=1}^m \pi_i^{(0)} p_{\varepsilon i} \right|}{\sum_{i=1}^m \pi_i^{(0)} p_{\varepsilon i}} \\
&\leq \sum_{i=1}^{m} \left| \pi_i^{(\varepsilon)} - \pi_i^{(0)} \right| \cdot \frac{p_{\varepsilon i}}{\sum_{j=1}^m \pi_j^{(0)} p_{\varepsilon j}} \\
&\leq \sum_{i=1}^{m} \frac{|\pi_i^{(\varepsilon)} - \pi_i^{(0)}|}{\pi_i^{(0)}} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

□

It follows from this relation and condition **B** that normalizing function p_ε can be replaced by $\bar{p}_\varepsilon$ in conditions **E**, **F** and in Theorem 1.

The next lemma describes asymptotic behavior of the probability of occurrence the rare event during one i-cycle.

Lemma 3. *Let conditions* **B**, **C**, *and* **D** *hold. Then, for every* $i \in X$,

$$q_{i\varepsilon} \sim \frac{p_\varepsilon}{\pi_i^{(0)}} \text{ as } \varepsilon \to 0. \tag{32}$$

Proof. Let us define the probabilities of occurrence of the rare event before the first hitting of the embedded Markov chain to the state i under condition that the initial state of this Markov chain $\eta_0^{(\varepsilon)} = j$,

$$q_{ji\varepsilon} = \mathsf{P}_j \left\{ \nu_\varepsilon \leq \tau_i^{(\varepsilon)} \right\}, \ i, j \in X.$$

By the definition,

$$q_{ii\varepsilon} = q_{i\varepsilon}, \ i \in X. \tag{33}$$

The probabilities $q_{ji\varepsilon}, j \in X$ satisfy, for every $i \in X$, the following system of linear equations,

$$\begin{cases} q_{ji\varepsilon} = p_{j\varepsilon} + \sum\limits_{k \neq i} \bar{p}_{jk}^{(\varepsilon)} q_{ki\varepsilon} \\ j \in X, \end{cases} \tag{34}$$

where

$$\begin{aligned}\bar{p}_{jk}^{(\varepsilon)} &= \mathsf{P}_j\left\{\eta_1^{(\varepsilon)} = k, \zeta_1^{(\varepsilon)} \notin D_\varepsilon\right\}, \\ &= p_{jk}^{(\varepsilon)} - \mathsf{P}_j\left\{\eta_1^{(\varepsilon)} = k, \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\}, \ j,k \in X. \end{aligned} \tag{35}$$

System (34) can be rewritten, for every $i \in X$, in the following matrix form,

$$\mathbf{q}_{i\varepsilon} = \mathbf{p}_\varepsilon + {}_i\mathbf{P}^{(\varepsilon)}\mathbf{q}_{i\varepsilon}, \tag{36}$$

where

$$\mathbf{q}_{i\varepsilon} = \begin{bmatrix} q_{1i\varepsilon} \\ \vdots \\ q_{mi\varepsilon} \end{bmatrix}, \quad \mathbf{p}_\varepsilon = \begin{bmatrix} p_{1\varepsilon} \\ \vdots \\ p_{m\varepsilon} \end{bmatrix},$$

and

$${}_i\mathbf{P}^{(\varepsilon)} = \begin{bmatrix} \bar{p}_{11}^{(\varepsilon)} & \dots & \bar{p}_{1(i-1)}^{(\varepsilon)} & 0 & \bar{p}_{1(i+1)}^{(\varepsilon)} & \dots & \bar{p}_{1m}^{(\varepsilon)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{p}_{m1}^{(\varepsilon)} & \dots & \bar{p}_{m(i-1)}^{(\varepsilon)} & 0 & \bar{p}_{m(i+1)}^{(\varepsilon)} & \dots & \bar{p}_{mm}^{(\varepsilon)} \end{bmatrix}.$$

Let us show that the matrix $\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}$ has the inverse matrix for all ε small enough, and, therefore, the solution of the system (36) has the following form, for every $i \in X$,

$$\mathbf{q}_{i\varepsilon} = \left[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}\right]^{-1}\mathbf{p}_\varepsilon. \tag{37}$$

Let us also introduce the matrix,

$${}_i\mathbf{P}^{(0)} = \begin{bmatrix} p_{11}^{(0)} & \dots & p_{1(i-1)}^{(0)} & 0 & p_{1(i+1)}^{(0)} & \dots & p_{1m}^{(0)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p_{m1}^{(0)} & \dots & p_{m(i-1)}^{(0)} & 0 & p_{m(i+1)}^{(0)} & \dots & p_{mm}^{(0)} \end{bmatrix}.$$

By conditions **B** and **C**,

$${}_i\mathbf{P}^{(\varepsilon)} \to {}_i\mathbf{P}^{(0)} \text{ as } \varepsilon \to 0. \tag{38}$$

Let us introduce random variable $\delta_{ik}^{(\varepsilon)}$ which is the number of visits of the embedded Markov chain $\eta_n^{(\varepsilon)}$ to the state k up to the first visit to the sate i,

$$\delta_{ik}^{(\varepsilon)} = \sum_{n=1}^{\tau_i^{(\varepsilon)}} \chi\left(\eta_{n-1}^{(\varepsilon)} = k\right), \ i,k \in X.$$

As is known, due to the ergodicity of the Markov chain $\eta_n^{(0)}$, $\mathsf{E}_j\delta_{ik}^{(0)} < \infty$ for all $j,i,k \in X$. Moreover, for every $i \in X$, there exists the inverse matrix,

$$\left[\mathbf{I} - {}_i\mathbf{P}^{(0)}\right]^{-1} = \left\| \ \mathsf{E}_j\delta_{ik}^{(0)} \ \right\|. \tag{39}$$

This means that (**a**) $\det\big(\mathbf{I} - {}_i\mathbf{P}^{(0)}\big) \neq 0$. Thus by relation (38) (**b**) $\det\big(\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}\big) \neq 0$ for all ε small enough. Since the elements of the inverse matrix $\big[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}\big]^{-1}$ are continuous rational functions of the elements of the matrix $\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}$ with non-zero denominator $\det(\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)})$, we get

$$\big[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}\big]^{-1} \to \big[\mathbf{I} - {}_i\mathbf{P}^{(0)}\big]^{-1} \text{ as } \varepsilon \to 0. \tag{40}$$

Let us also introduce random variable $\delta_{ik\varepsilon}$ which is the number of visits of the embedded Markov chain $\eta_n^{(\varepsilon)}$ to the state k before the first visit to the sate i or the occurrence of the first-rare-event,

$$\delta_{ik\varepsilon} = \sum_{n=1}^{\tau_i^{(\varepsilon)} \wedge \nu_\varepsilon} \chi\left(\eta_{n-1}^{(\varepsilon)} = k\right), \ i, k \in X.$$

The matrix

$${}_i\mathbf{P}^{(\varepsilon)n} = \left\| \, \mathsf{P}_j \left\{ \eta_n^{(\varepsilon)} = k, \nu_\varepsilon \wedge \tau_i^{(\varepsilon)} > n \right\} \right\|, \quad n \geq 1$$

and, therefore,

$$\big[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}\big]^{-1} = \mathbf{I} + {}_i\mathbf{P}^{(\varepsilon)} + \big({}_i\mathbf{P}^{(\varepsilon)}\big)^2 + \cdots = \| \, \mathsf{E}_j \delta_{ik\varepsilon} \, \| \tag{41}$$

Using relations (33) and (41) we get the following formula,

$$q_{i\varepsilon} = \sum_{k=1}^{m} \mathsf{E}_i \delta_{ik\varepsilon} \, p_{k\varepsilon}, \tag{42}$$

and

$$\mathsf{E}_i \delta_{ik\varepsilon} \to \mathsf{E}_i \delta_{ik}^{(0)} \text{ as } \varepsilon \to 0. \tag{43}$$

As is known, the following formula holds, since the Markov chain $\eta_n^{(0)}$ is ergodic,

$$\mathsf{E}_i \delta_{ik}^{(0)} = \frac{\pi_k^{(0)}}{\pi_i^{(0)}}, \ i, k \in X. \tag{44}$$

Using formulas (42) and (44) we get,

$$\begin{aligned}
\frac{\left| q_{i\varepsilon} - \frac{p_\varepsilon}{\pi_i^{(0)}} \right|}{\frac{p_\varepsilon}{\pi_i^{(0)}}} &\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi_k^{(0)}}{\pi_i^{(0)}} \right| \cdot \frac{\pi_i^{(0)} p_{k\varepsilon}}{\sum_{j=1}^m \pi_j^{(0)} p_{j\varepsilon}} \\
&\leq \sum_{k=1}^{m} \left| \mathsf{E}_i \delta_{ik\varepsilon} - \frac{\pi_k^{(0)}}{\pi_i^{(0)}} \right| \cdot \frac{\pi_i^{(0)}}{\pi_k^{(0)}} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{45}$$

Relation (45) implies asymptotic relation (32). The proof is complete. □

Lemma 4. *Let conditions* **A**, **B**, **C**, **D** *hold. Then, for any normalization function* $0 < u_\varepsilon \to \infty$ *as* $\varepsilon \to 0$, *and for* $i \in X$,

$$\widetilde{\psi}_{i\varepsilon}(s/u_\varepsilon) \to 1 \text{ as } \varepsilon \to 0,\ s \geq 0. \tag{46}$$

Proof. Let us introduce the Laplace transforms,

$$\widetilde{\psi}_{ji\varepsilon}(s) = \mathsf{E}_j \exp\left\{-s\tilde{\beta}_{i\varepsilon}\right\} \chi\left(\nu_\varepsilon \leq \tau_i^{(\varepsilon)}\right),\ s \geq 0,\ i,j \in X.$$

Obviously,

$$\widetilde{\psi}_{i\varepsilon}(s) = \frac{\widetilde{\psi}_{ii\varepsilon}(s)}{q_{i\varepsilon}},\ s \geq 0,\ i \in X. \tag{47}$$

Let us also introduce the Laplace transforms,

$$\bar{p}_{jk}^{(\varepsilon)}(s) = \mathsf{E}_j e^{-s\varkappa_1^{(\varepsilon)}} \chi\left(\zeta_1^{(\varepsilon)} \notin D_\varepsilon, \eta_1^{(\varepsilon)} = k\right),\ s \geq 0,\ j,k \in X,$$

and

$$p_j^{(\varepsilon)}(s) = \mathsf{E}_j e^{-s\varkappa_1^{(\varepsilon)}} \chi\left(\zeta_1^{(\varepsilon)} \in D_\varepsilon\right) = \widehat{\varphi}_{j\varepsilon}(s) p_{j\varepsilon},\ s \geq 0,\ j \in X,$$

where

$$\widehat{\varphi}_{j\varepsilon}(s) = \mathsf{E}_j \left\{e^{-s\varkappa_1^{(\varepsilon)}} / \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\},\ s \geq 0,\ j \in X.$$

Functions $\widetilde{\psi}_{ji\varepsilon}(s/u_\varepsilon), j \in X$ satisfy, for every $s \geq 0$ and $i \in X$, the following system of linear equations,

$$\begin{cases} \widetilde{\psi}_{ji\varepsilon}(s/u_\varepsilon) = p_j^{(\varepsilon)}(s/u_\varepsilon) + \sum\limits_{k \neq i} \bar{p}_{jk}^{(\varepsilon)}(s) \widetilde{\psi}_{ki\varepsilon}(s/u_\varepsilon), \\ j \in X. \end{cases} \tag{48}$$

System (48) can be rewritten in the following equivalent matrix form

$$\widetilde{\mathbf{\Psi}}_i^{(\varepsilon)}(s/u_\varepsilon) = \mathbf{p}^{(\varepsilon)}(s/u_\varepsilon) + {}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon)\ \widetilde{\mathbf{\Psi}}_i^{(\varepsilon)}(s/u_\varepsilon), \tag{49}$$

where

$$\widetilde{\mathbf{\Psi}}_i^{(\varepsilon)}(s) = \begin{bmatrix} \widetilde{\psi}_{1i\varepsilon}(s) \\ \vdots \\ \widetilde{\psi}_{mi\varepsilon}(s) \end{bmatrix}, \quad \mathbf{p}^{(\varepsilon)}(s) = \begin{bmatrix} p_1^{(\varepsilon)}(s) \\ \vdots \\ p_m^{(\varepsilon)}(s) \end{bmatrix},$$

and

$${}_i\mathbf{P}^{(\varepsilon)}(s) = \begin{bmatrix} \bar{p}_{11}^{(\varepsilon)}(s) & \dots & \bar{p}_{1(i-1)}^{(\varepsilon)}(s) & 0 & \bar{p}_{1(i+1)}^{(\varepsilon)}(s) & \dots & \bar{p}_{1m}^{(\varepsilon)}(s) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{p}_{m1}^{(\varepsilon)}(s) & \dots & \bar{p}_{m(i-1)}^{(\varepsilon)} & 0 & \bar{p}_{m(i+1)}^{(\varepsilon)}(s) & \dots & \bar{p}_{mm}^{(\varepsilon)}(s) \end{bmatrix}.$$

Let us show that, for every $s \geq 0$ and $i \in X$, matrix $\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon)$ has the inverse matrix for all ε small enough, and, therefore, the solution to the system (49) has the following form,

$$\widetilde{\mathbf{\Psi}}_i^{(\varepsilon)}(s/u_\varepsilon) = \left[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon)\right]^{-1} \mathbf{p}^{(\varepsilon)}(s/u_\varepsilon). \tag{50}$$

Conditions **A** and **B** implies, in an obvious way, that, for every $s \geq 0$ and $j, k \in X$,

$$\begin{aligned} \bar{p}_{jk}^{(\varepsilon)}(s/u_\varepsilon) &= \mathsf{E}_j \exp\left\{-s\varkappa_1^{(\varepsilon)}/u_\varepsilon\right\} \chi\left(\zeta_1^{(\varepsilon)} \notin D_\varepsilon, \eta_1^{(\varepsilon)} = k\right) \\ &\quad -\mathsf{E}_j \chi\left(\eta_1^{(\varepsilon)} = k\right) \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{51}$$

Since

$$\mathsf{E}_j \chi\left(\eta_1^{(\varepsilon)} = k\right) = p_{jk}^{(\varepsilon)},$$

we conclude that

$$\bar{p}_{jk}^{(\varepsilon)}(s/u_\varepsilon) \to p_{jk}^{(0)} \text{ as } \varepsilon \to 0. \tag{52}$$

Thus (**c**) ${}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon) \to {}_i\mathbf{P}^{(0)}$ as $\varepsilon \to 0$, for every $s \geq 0$ and $i \in X$. It was shown in the proof of Lemma 3 that, under condition **D**, the inverse matrix $[\mathbf{I} - {}_i\mathbf{P}^{(0)}]^{-1}$ exists. Thus, (**c**) implies that (**d**) there exists, for every $s \geq 0$ and $i \in X$, the inverse matrix $[\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon)]^{-1}$ for all ε small enough. Moreover, for every $s \geq 0$ and $i \in X$,

$$\begin{aligned} [\mathbf{I} - {}_i\mathbf{P}^{(\varepsilon)}(s/u_\varepsilon)]^{-1} &= \|\Delta_{jik}^{(\varepsilon)}(s)\| \\ &\to [\mathbf{I} - {}_i\mathbf{P}^{(0)}]^{-1} = \|\mathsf{E}_j \delta_{ik}^{(0)}\| \text{ as } \varepsilon \to 0. \end{aligned} \tag{53}$$

Taking in account formulas (47), (50) and the definition of $p_j^{(\varepsilon)}(s)$, we get, for every $s \geq 0$ and $i \in X$,

$$\widetilde{\psi}_{ii\varepsilon}(s/u_\varepsilon) = \sum_{k=1}^{m} \Delta_{iik}^{(\varepsilon)}(s) \widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon) p_k(\varepsilon). \tag{54}$$

Condition **A** implies that, for every $s \geq 0$ and $k \in X$,

$$\widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon) \to 1 \text{ as } \varepsilon \to 0. \tag{55}$$

Indeed, using condition **A**, we get, for any $v > 0$,

$$\begin{aligned} 0 \leq \overline{\lim}_{\varepsilon\to 0}(1 - \widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon)) &\leq 1 - \exp\{-sv\} \\ &\quad + \overline{\lim}_{\varepsilon\to 0} \mathsf{P}_k \left\{\varkappa_1^{(\varepsilon)} > vu_\varepsilon / \zeta_1^{(\varepsilon)} \in D_\varepsilon\right\} \\ &= 1 - \exp\{-sv\} \to 0 \text{ as } v \to 0. \end{aligned} \tag{56}$$

Relations (53) and (55) imply that, for every $s \geq 0$ and $i, k \in X$,

$$\Delta^{(\varepsilon)}_{iik}(s)\widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon) \to \mathsf{E}_i\delta^{(0)}_{ik} = \frac{\pi^{(0)}_k}{\pi^{(0)}_i} \text{ as } \varepsilon \to 0. \tag{57}$$

Using relation (57) we get, for every $s \geq 0$ and $i, k \in X$,

$$\begin{aligned} &\frac{\left|\widetilde{\psi}_{ii\varepsilon}(s/u_\varepsilon) - \frac{p_\varepsilon}{\pi^{(0)}_i}\right|}{\frac{p_\varepsilon}{\pi^{(0)}_i}} \\ \leq\ & \sum_{k=1}^{m} \left|\Delta^{(\varepsilon)}_{iik}(s)\widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon) - \frac{\pi^{(0)}_k}{\pi^{(0)}_i}\right| \frac{\pi^{(0)}_i p_k(0)}{\sum_{j=1}^m \pi^{(0)}_j p_{j\varepsilon}} \\ \leq\ & \sum_{k=1}^{m} \left|\Delta^{(\varepsilon)}_{iik}(s)\widehat{\varphi}_{k\varepsilon}(s/u_\varepsilon) - \frac{\pi^{(0)}_k}{\pi^{(0)}_i}\right| \frac{\pi^{(0)}_i}{\pi^{(0)}_k} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{58}$$

Relation (58) means that, for every $s \geq 0$ and $i \in X$,

$$\widetilde{\psi}_{ii\varepsilon}(s/u_\varepsilon) \sim \frac{p_\varepsilon}{\pi^{(0)}_i} \text{ as } \varepsilon \to 0. \tag{59}$$

Finally, relation (32) given in Lemma 3, formula (47), and relation (59), we get, for every $s \geq 0$ and $i \in X$,

$$\widetilde{\psi}_{i\varepsilon}(s/u_\varepsilon) = \frac{\widetilde{\psi}_{ii\varepsilon}(s/u_\varepsilon)}{q_{i\varepsilon}} \to 1 \text{ as } \varepsilon \to 0. \tag{60}$$

The proof is complete. □

Lemma 5. *Let conditions* **A**, **B**, **C** *and* **D** *hold. Then for any normalization function* $0 < u_\varepsilon \to \infty$ *as* $\varepsilon \to 0$, *and for* $i \in X$,

$$\widehat{\psi}_{i\varepsilon}(s/u_\varepsilon) \to 1 \ \text{as } \varepsilon \to 0,\ s \geq 0. \tag{61}$$

Proof. The following representation can be written, for every $i \in X$,

$$\begin{aligned} \widehat{\psi}_{i\varepsilon}(s) &= q^{-1}_{i\varepsilon}\mathsf{E}_i \exp\left\{-s\beta^{(\varepsilon)}_i\right\}\chi\left(\nu_\varepsilon \leq \tau^{(\varepsilon)}_i\right) \\ &= \sum_{k=1}^{m} q^{-1}_{i\varepsilon}\mathsf{E}_i \exp\{-s(\sum_{n=1}^{\nu_\varepsilon}\varkappa^{(\varepsilon)}_n + \\ &\quad \sum_{n=\nu_\varepsilon+1}^{\tau^{(\varepsilon)}_i}\varkappa^{(\varepsilon)}_n)\}\chi\left(\nu_\varepsilon \leq \tau^{(\varepsilon)}_i, \eta^{(\varepsilon)}_{\nu_\varepsilon} = k\right) \\ &= \sum_{k=1}^{m} q^{-1}_{i\varepsilon}\mathsf{E}_i \exp\{-s\xi_\varepsilon\}\chi\left(\nu_\varepsilon \leq \tau^{(\varepsilon)}_i, \eta^{(\varepsilon)}_{\nu_\varepsilon} = k\right)\psi^{(\varepsilon)}_k(s). \end{aligned}$$

By condition **A**, $\psi_k^{(\varepsilon)}(s/u_\varepsilon) \to 1$ as $\varepsilon \to 0$ for every $s \geq 0$ and $k \in X$. Thus, for every $s \geq 0$ and $i \in X$,

$$\begin{aligned}\widehat{\psi}_{i\varepsilon}(s/u_\varepsilon) &\sim \sum_{k=1}^{m} q_{i\varepsilon}^{-1} \mathsf{E}_i \exp\{-s\xi_\varepsilon/u_\varepsilon\} \chi\left(\nu_\varepsilon \leq \tau_i^{(\varepsilon)}, \eta_{\nu_\varepsilon}^{(\varepsilon)} = k\right) \\ &= q_{i\varepsilon}^{-1} \mathsf{E}_i \exp\{-s\xi_\varepsilon/u_\varepsilon\} \chi\left(\nu_\varepsilon \leq \tau_i^{(\varepsilon)}\right) \\ &= \widetilde{\psi}_{i\varepsilon}(s/u_\varepsilon) \to 1 \text{ as } \varepsilon \to 0. \end{aligned} \tag{62}$$

The proof is complete. □

In what follows we assume that $\eta_0^{(\varepsilon)} = j$ and shall mark the corresponding processes based on the Markov renewal process $\left(\eta_n^{(\varepsilon)}, \varkappa_n^{(\varepsilon)}, \zeta_n^{(\varepsilon)}\right)$ by the index j in order to distinguish the cases with different initial states $\eta_0^{(\varepsilon)}$.

Let us introduce, for every $i, j \in X$, the following "cyclic" stochastic process,

$$\xi_{ji\varepsilon}(t) = \sum_{n=1}^{[tq_{i\varepsilon}^{-1}]+1} \frac{\beta_i^{(\varepsilon)}(n)}{u_\varepsilon}, \; t \geq 0. \tag{63}$$

Note that $\xi_{ji\varepsilon}(t)$ is a step sum-process with independent increments. Indeed, by the definition, random variables $\beta_i^{(\varepsilon)}(n), n = 1, 2, \dots$ are independent and,

$$\mathsf{E}\exp\left\{-s\beta_i^{(\varepsilon)}(n)\right\} = \begin{cases} \psi_{ji}^{(\varepsilon)}(s) & \text{for } n = 1, \\ \psi_{ii}^{(\varepsilon)}(s) & \text{for } n \geq 2, \end{cases} \tag{64}$$

where

$$\psi_{ji}^{(\varepsilon)}(s) = \mathsf{E}_j \exp\left\{-s\beta_i^{(\varepsilon)}\right\}, \; s \geq 0, \; i, j \in X.$$

We are interested to prove some solidarity statements concerned two asymptotic relations.

The first one is the following relation of weak convergence,

$$\xi_{ji\varepsilon}(t), \; t \geq 0 \Rightarrow \xi(t), \; t \geq 0 \text{ as } \varepsilon \to 0, \tag{65}$$

where (**e**) $\xi(t), t \geq 0$ is a non-zero, non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

The second one is the following asymptotic relation,

$$\frac{1 - \psi_i^{(\varepsilon)}(s/u_\varepsilon)}{q_{i\varepsilon}} \to \varsigma(s) \text{ as } \varepsilon \to 0, s \geq 0, \tag{66}$$

where (**f**) $\varsigma(s) > 0$ for $s > 0$.

The following lemma presents the variant of so-called solidarity proposition concerned weak convergence for cyclic step sum-processes $\xi_{ji\varepsilon}(t)$.

Lemma 6. *Let conditions* **B**, **C**, **D** *hold and* $\eta_0^{(\varepsilon)} = j$. *Then*: ($\boldsymbol{\alpha}$) *the assumption that the relation of weak convergence* (65) *holds for some* $i, j \in X$ *implies that this relation holds for every* $i, j \in X$; ($\boldsymbol{\beta}$) *the limit process* $\xi(t), t \geq 0$ *in* (65) *is the same for any* $i, j \in X$; ($\boldsymbol{\gamma}$) $\xi(t), t \geq 0$ *is a non-zero and non-decreasing homogenous process with independent increments;* ($\boldsymbol{\delta}$) *relation* (65) *holds for given* $i, j \in X$ *if and only if relation* (66) *holds for the same* i; ($\boldsymbol{\epsilon}$) *the limit function* $\varsigma(s)$ *in* (66) *is the same for any* $i \in X$; ($\boldsymbol{\zeta}$) $\varsigma(s)$ *is a cumulant of the process* $\xi(t), t \geq 0$, *i.e.* $\mathsf{E}e^{-s\xi(t)} = e^{-\varsigma(s)t},\ s, t \geq 0$; ($\boldsymbol{\eta}$) *conditions* **E** *and* **F** (*with replacement of function* p_ε *by* $q_{i\varepsilon}$ *in these conditions*), *imposed on the distribution of random variable* $\beta_i^{(\varepsilon)}$, *are necessary and sufficient for relation* (66) *to hold;* ($\boldsymbol{\theta}$) *cumulant* $\varsigma(s) = a(s)$ *in this case.*

Proof. Let us first prove that (**g**) the assumption that (65) holds for given $i, j \in X$ implies that this relation holds for the same i and every $j \in X$, moreover the limit process $\xi(t), t \geq 0$ does not depend on j.

Indeed, the pre-limit process $\xi_{ji\varepsilon}(t)$ can be represented in the form of the following sum,

$$\xi_{ji\varepsilon}(t) = \beta_i^{(\varepsilon)}/u_\varepsilon + \xi'_{i\varepsilon}(t),\ t \geq 0, \tag{67}$$

where

$$\xi'_{i\varepsilon}(t) = \sum_{n=2}^{[tq_{i\varepsilon}^{-1}]+1} \beta_i^{(\varepsilon)}(n)/u_\varepsilon,\ t \geq 0.$$

The random variable $\beta_i^{(\varepsilon)}/u_\varepsilon$ and the process $\xi'_{i\varepsilon}(t), t \geq 0$ are independent. The distribution of random variable $\beta_i^{(\varepsilon)}/u_\varepsilon$ depends on j while the finite-dimensional distributions of process $\xi'_{i\varepsilon}(t), t \geq 0$ do not depend on j. Conditions **B**–**F** readily imply $\beta_i^{(\varepsilon)}/u_\varepsilon \stackrel{\mathsf{P}}{\longrightarrow} 0$ as $\varepsilon \to 0$, for every $j \in X$, or, equivalently, ($\mathbf{f}_1$) the random variables $\xi_{ji\varepsilon}(t) - \xi'_{i\varepsilon}(t) \stackrel{\mathsf{P}}{\longrightarrow} 0$ as $\varepsilon \to 0$, for every $t > 0$ and $j \in X$. Thus, the assumption that (65) holds for given $i, j \in X$ implies weak convergence of the process $\xi'_{ji\varepsilon}(t), t \geq 0$ to the same limit process. This convergence, due to ($\mathbf{g}_1$), implies that ($\mathbf{g}_2$) the process $\xi_{ji\varepsilon}(t), t \geq 0$ converge weakly to the same limit process, for every $j \in X$, moreover, the finite-dimensional distributions of the limit process do not depend on j since it is so for the pre-limit process $\xi'_{i\varepsilon}(t), t \geq 0$.

Let us now prove that (**h**) the assumption that (65) holds for given $i, j \in X$ implies that this relation holds for the same j and every $i \in X$, moreover the limit process $\xi(t), t \geq 0$ does not depend on i.

Note that two partial solidarity propositions (**g**) and (**h**), formulated above, imply the solidarity statements ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$) formulated in Lemma 4.

To prove the proposition (**h**), let us introduce, for $j \in X$, the following step sum-processes based on sojourn times for semi-Markov process $\eta^{(\varepsilon)}(t)$,

$$\xi_{j\varepsilon}(t) = \sum_{n=1}^{[tp_\varepsilon^{-1}]} \frac{\varkappa_n^{(\varepsilon)}}{u_\varepsilon}, \ t \geq 0, \tag{68}$$

Let us also introduce, for $i, j \in X$, the processes $\mu_{ji\varepsilon}(t)$ which counts the number of transitions for the semi-Markov process $\eta^{(\varepsilon)}(t)$ that occurs in $[tq_{i\varepsilon}^{-1}] + 1$ cycles,

$$\mu_{ji\varepsilon}(t) = p_\varepsilon \tau_i^{(\varepsilon)} \left([tq_{i\varepsilon}^{-1}] + 1\right), \ t \geq 0.$$

The process $\xi_{ji\varepsilon}(t)$ can be represented, for every $i, j \in X$, in the form of superposition of the processes introduced above,

$$\xi_{ji\varepsilon}(t) = \xi_{j\varepsilon}(\mu_{ji\varepsilon}(t)), \ t \geq 0. \tag{69}$$

Let us now consider the following relation of weak convergence for the processes $\xi_{j\varepsilon}(t)$,

$$\xi_{j\varepsilon}(t), t \geq 0 \ \Rightarrow \ \xi(t), t \geq 0 \text{ as } \varepsilon \to 0, \tag{70}$$

where (**e**) $\xi(t), t \geq 0$ is non-zero, non-decreasing, and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us now prove that ($\mathbf{h}_1$) relation (65) holds, for given $i, j \in X$, if and only if the relation (70) holds, for the same j, moreover the limit process $\xi(t), t \geq 0$ can be taken the same in both relations.

Note that ($\mathbf{h}_1$) implies (**h**). Indeed, due to "iff" character, the relation (70) for given $j \in X$ implies that (65) should hold for the same j and every $i \in X$, and with the same limit process. Moreover, the limit process in (70) does not depend on i since the pre-limit process $\xi_{j\varepsilon}(t), t \geq 0$ does not depend on i.

We display the proof of ($\mathbf{h}_1$) for one-dimensional distributions. The proof for multi-dimensional distributions is similar.

Let us first prove that ($\mathbf{h}_2$) the weak convergence of random variables $\xi_{j\varepsilon}(t)$ in (70), assumed to hold for every $t > 0$ and given $j \in X$, implies the weak convergence of random variables $\xi_{ji\varepsilon}(t)$ in (65) for every $t > 0$, the same j and every $i \in X$, moreover the limit random variable $\xi(t)$ can be taken the same in both relations.

The process $\mu_{ji\varepsilon}(t)$ can be represented, for every $i, j \in X$, in the form of sum-process with independent increments,

$$\mu_{ji\varepsilon}(t) = p_\varepsilon \left([q_{i\varepsilon}^{-1}] + 1\right) \sum_{n=1}^{[tq_{i\varepsilon}^{-1}]+1} \frac{\alpha_i^{(\varepsilon)}(n)}{[q_{i\varepsilon}^{-1}] + 1}, t \geq 0, \tag{71}$$

where $\alpha_i^{(\varepsilon)}(n) = \tau_i^{(\varepsilon)}(n) - \tau_i^{(\varepsilon)}(n-1), n = 1, 2, \ldots$. Indeed, the random variables $\alpha_i^{(\varepsilon)}(n), n \geq 1$ are independent and,

$$\mathsf{E}\exp\left\{-s\alpha_i^{(\varepsilon)}(n)\right\} = \begin{cases} \vartheta_{ji}^{(\varepsilon)}(s) & \text{for } n = 1, \\ \vartheta_{ii}^{(\varepsilon)}(s) & \text{for } n \geq 2, \end{cases} \tag{72}$$

where

$$\vartheta_{ji}^{(\varepsilon)}(s) = \mathsf{E}_j \exp\left\{-s\alpha_i^{(\varepsilon)}(1)\right\}, \ s \geq 0, \ i, j \in X.$$

As was pointed out above conditions **C** and **D** Markov chain $\eta_n^{(\varepsilon)}$ is ergodic for all ε small enough and

$$\mathsf{E}_i\alpha_i^{(\varepsilon)}(1) = 1/\pi_i^{(\varepsilon)} \to \mathsf{E}_i\alpha_i^{(0)}(1) = 1/\pi_i^{(0)} \text{ as } \varepsilon \to 0.$$

Moreover, under conditions **B** and **C** exists limit

$$\overline{\lim_{\varepsilon\to 0}}\mathsf{E}_i(\alpha_i^{(\varepsilon)})^2 < \infty.$$

Thus, using the standard weak law of large numbers for i.i.d. random variables $\alpha_i^{(\varepsilon)}(n)$ with bounded variance, the asymptotic relation (32) given in Lemma 3, and representation (96), we get, for every $t > 0$ and $i, j \in X$,

$$\mu_{ji\varepsilon}(t) \xrightarrow{\mathsf{P}} \pi_i^{(0)} t\mathsf{E}_i\alpha_i^{(0)}(1) = t \text{ as } \varepsilon \ \to 0. \tag{73}$$

Let us choose an arbitrary $t > 0$ and a sequence $0 < c_n < t, n = 1, 2, \ldots$ such that $c_n \to 0$ as $n \to \infty$.

By the definition, the processes $\xi_{ji\varepsilon}(t)$, $\xi_{j\varepsilon}(t)$, and $\mu_{ji\varepsilon}(t)$ are non-negative and non-decreasing. Taking into account this fact and the representation (69), we get, for every $t > 0$, $i, j \in X$, any real-valued x, and $n \geq 1$,

$$\begin{aligned} \mathsf{P}\{\xi_{ji\varepsilon}(t) > x\} &= \mathsf{P}\{\xi_{ji\varepsilon}(t) > x, \mu_{ji\varepsilon}(t) \leq t + c_n\} \\ &\quad + \mathsf{P}\{\xi_{ji\varepsilon}(t) > x, \mu_{ji\varepsilon}(t) > t + c_n\} \\ &\leq \mathsf{P}\{\xi_{j\varepsilon}(t + c_n) > x\} \\ &\quad + \mathsf{P}\{\mu_{ji\varepsilon}(t) > t + c_n\}. \end{aligned} \tag{74}$$

Let U_t be the set of continuity points the distribution functions of the limit random variables $\xi(t)$ and $\xi(t \pm c_n), n = 1, 2, \ldots$ in (70). This set is the real line R except at most a countable set of points.

Using the estimate (74), relation (73), and the assumptions that relation (70) holds for one-dimensional distributions, for every $t > 0$ and given $j \in X$, and that the limit process $\xi(t)$ in (70) is stochastically continuous, we get, for every $t > 0$, the same j, and every $i \in X$,

$$\begin{aligned} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\xi_{ji\varepsilon}(t) > x\} &\leq \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\xi_{j\varepsilon}(t + c_n) > x\} \\ &\quad + \mathsf{P}\{\mu_{ji\varepsilon}(t) > t + c_n\}) \\ &= \lim_{n\to\infty} \mathsf{P}\{\xi(t + c_n) > x\} \\ &= \mathsf{P}\{\xi(t) > x\}, \ x \in U_t, \end{aligned} \tag{75}$$

or, equivalently,

$$\underline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\xi_{ji\varepsilon}(t) \le x\} \ge \mathsf{P}\{\xi(t) \le x\}, \ x \in U_t. \tag{76}$$

We can also employ the following estimate, for every $t > 0$, $i, j \in X$, any real x, and $n \ge 1$,

$$\mathsf{P}\{\xi_{ji\varepsilon}(t) \le x\} \le \mathsf{P}\{\xi_{j\varepsilon}(t - c_n) \le x\} + \mathsf{P}\{\mu_{ji\varepsilon}(t) \le t - c_n\}. \tag{77}$$

Then, using the estimate (77), relation (73), and the assumptions that relation (70) holds for one-dimensional distributions, for every $t > 0$ and given $j \in X$, and that the limit process $\xi(t)$ in (70) is stochastically continuous, we get, for every $t > 0$, the same j, and every $i \in X$,

$$\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\xi_{ji\varepsilon}(t) \le x\} \le \mathsf{P}\{\xi(t) \le x\}, \ x \in U_t. \tag{78}$$

Relations (76) and (78) implies that $\mathsf{P}\{\xi_{ji\varepsilon}(t) \le x\} \to \mathsf{P}\{\xi(t) \le x\}$ as $\varepsilon \to 0$, $x \in U_t$, Since the set U_t is dense in R, this relation implies that, for every $t > 0$, given j (for which relation (70) is assumed to hold) and every $i \in X$,

$$\xi_{ji\varepsilon}(t) \Rightarrow \xi(t) \text{ as } \varepsilon \to 0. \tag{79}$$

Let us now prove that ($\mathbf{h}_3$) the weak convergence of random variables $\xi_{ji\varepsilon}(t)$ in (65), assumed to hold for every $t > 0$ and given $i, j \in X$, implies the weak convergence of random variables $\xi_{j\varepsilon}(t)$ in (65) for every $t > 0$ and the same j, moreover the limit random variable $\xi(t)$ can be taken the same in both relations.

Let us choose an arbitrary $t > 0$ and a sequence $0 < d_n < t, n = 1, 2, \dots$ such that $d_n \to 0$ as $n \to \infty$.

Using again that the processes $\xi_{ji\varepsilon}(t)$, $\xi_{j\varepsilon}(t)$, and $\mu_{ji\varepsilon}(t)$ are non-negative and non-decreasing, and the representation (69), we get, for every $t > 0$, given $i, j \in X$, any real-valued x, and $n \ge 1$,

$$\begin{aligned} \mathsf{P}\{\xi_{j\varepsilon}(t) > x\} &= \mathsf{P}\{\xi_{j\varepsilon}(t) > x, \mu_{ji\varepsilon}(t + d_n) > t\} \\ &\quad + \mathsf{P}\{\xi_{j\varepsilon}(t) > x, \mu_{ji\varepsilon}(t + d_n) \le t\} \\ &\le \mathsf{P}\{\xi_{ji\varepsilon}(t + d_n) > x\} \\ &\quad + \mathsf{P}\{\mu_{ji\varepsilon}(t + d_n) \le t\}. \end{aligned} \tag{80}$$

Let V_t be the set of continuity points for the distribution functions of the limit random variables $\xi(t)$ and $\xi(t \pm d_n), n = 1, 2, \dots$ in (65). This set is the real line R except at most a countable set of points.

Using the estimate (80), relation (73), and the assumptions that relation (65) holds for one-dimensional distributions, for every $t > 0$ and given

$i, j \in X$, and that the limit process $\xi(t)$ in (65) is stochastically continuous, we get, for every $t > 0$ and the same j,

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\, \mathsf{P}\{\xi_{j\varepsilon}(t) > x\} &\leq \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\xi_{ji\varepsilon}(t + d_n) > x\} \\ &\quad + \mathsf{P}\{\mu_{ji\varepsilon}(t + d_n) \leq t\}) \\ &= \lim_{n\to\infty} \mathsf{P}\{\xi(t + d_n) > x\} \\ &= \mathsf{P}\{\xi(t) > x\},\ x \in V_t, \end{aligned} \tag{81}$$

or, equivalently,

$$\underline{\lim_{\varepsilon\to 0}}\, \mathsf{P}\{\xi_{j\varepsilon}(t) \leq x\} \geq \mathsf{P}\{\xi(t) \leq x\},\ x \in V_t. \tag{82}$$

We can also employ the following estimate, for every $t > 0$, $i, j \in X$, any real-valued x and $n \geq 1$,

$$\mathsf{P}\{\xi_{j\varepsilon}(t) \leq x\} \leq \mathsf{P}\{\xi_{ji\varepsilon}(t - d_n) \leq x\} + \mathsf{P}\{\mu_{ji\varepsilon}(t - d_n) \leq t\}. \tag{83}$$

Then, using the estimate (83), relation (73), and the assumptions that relation (65) holds for one-dimensional distributions, for every $t > 0$ and for given $i, j \in X$, and that the limit process $\xi(t)$ in (65) is stochastically continuous, we get, for every $t > 0$ and the same j,

$$\overline{\lim_{\varepsilon\to 0}}\, \mathsf{P}\{\xi_{j\varepsilon}(t) \leq x\} \leq \mathsf{P}\{\xi(t) \leq x\},\ x \in V_t. \tag{84}$$

Relations (82) and (84) implies that $\mathsf{P}\{\xi_{j\varepsilon}(t) \leq x\} \to \mathsf{P}\{\xi(t) \leq x\}$ as $\varepsilon \to 0, x \in V_t$. Since the set V_t is dense in R, this relation implies that, for every $t > 0$ and given j (for which relation (65) is assumed to hold),

$$\xi_{j\varepsilon}(t) \Rightarrow \xi(t) \text{ as } \varepsilon \to 0. \tag{85}$$

The proof of statements ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$) formulated in Lemma 6 is complete.

As was mention above $\xi_{ji\varepsilon}(t) - \xi'_{i\varepsilon}(t) \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$, for every $t \geq 0$, and, therefore, the weak convergence for the processes $\xi_{ji\varepsilon}(t), t \geq 0$ and $\xi'_{i\varepsilon}(t), t \geq 0$ is equivalent.

The statement ($\boldsymbol{\gamma}$) follows directly from the definition of the sum-process $\xi'_{i\varepsilon}(t), t \geq 0$ since the random variables $\beta_i^{(\varepsilon)}(n), n \geq 2$ are independent and identically distributed and $\xi'_{i\varepsilon}(t), t \geq 0$ is the homogeneous step sum-process with independent increments. As is known, the class of possible limit processes (in the sense of weak convergence) for such step sum-process coincides with the class of stochastically continuous homogeneous processes with independent increments.

Moreover, as is known, the weak convergence of finite-dimensional distributions follows in this case from the weak convergence of one-dimensional distributions. The statements ($\boldsymbol{\delta}$) and ($\boldsymbol{\epsilon}$) follows, in an obvious way, from the following formula,

$$\mathsf{E} \exp\{-s\xi'_{i\varepsilon}(t)\} = \psi_i^{(\varepsilon)}(s/u_\varepsilon)^{[tq_{i\varepsilon}^{-1}]}, \ s, t \geq 0, \ i \in X. \tag{86}$$

Indeed, (86) implies that, for given $t > 0$ and $i \in X$, the random variables $\xi'_{i\varepsilon}(t)$ converge weakly to some non-zero limit random variable if and only if relation (66) holds and, in this case,

$$\begin{aligned} \mathsf{E} \exp\{-s\xi'_{i\varepsilon}(t)\} &= \psi_i^{(\varepsilon)}(s/u_\varepsilon)^{[tq_{i\varepsilon}^{-1}]} \\ &\sim \exp\left\{-\left(1 - \psi_i^{(\varepsilon)}(s/u_\varepsilon)\right) tq_{i\varepsilon}^{-1}\right\} \\ &\to \exp\{-\varsigma(s)t\} \text{ as } \varepsilon \to 0, \ s \geq 0, \end{aligned} \tag{87}$$

where $\varsigma(s) > 0$ for $s > 0$.

Since, according the remarks above, the random variable $\xi(t)$ has, for every $t > 0$, an infinitely divisible distribution, and $\varsigma(s)t$ is the cumulant of this random variable. This proves the statement ($\boldsymbol{\zeta}$).

Last statements ($\boldsymbol{\eta}$) and ($\boldsymbol{\theta}$) of Lemma 6 are given in Lemma 7 and Remark 3. □

Remark 2. The proof presented above shows that the only property of the quantities $q_{i\varepsilon}$ and p_ε, used in the proof of Lemma 4, is (**i**) $0 < q_{i\varepsilon}/\pi_i^{(\varepsilon)} \sim p_\varepsilon \to 0$ as $\varepsilon \to 0, \ i \in X$. Lemma 4 and its proof remain to be valid if any functions $q_{i\varepsilon}$ and p_ε, satisfying the assumption (**i**), would be used in the formulas (63) and (68) defining, respectively, the processes $\xi_{ji\varepsilon}(t), t \geq 0$ and $\xi_{j\varepsilon}(t), t \geq 0$, and in the expression $(1 - \psi_i^{(\varepsilon)}(s/u_\varepsilon))/q_{i\varepsilon}$ used in the asymptotic relation (66). In this case, conditions **A** and **B** in Lemma 6 can be replaced by the simpler assumption (**i**) while condition **C** should remain.

The proof of Lemma 6 is based on the proposition about equivalence of weak convergence of the cyclic step sum-processes $\xi_{ji\varepsilon}(t)$, $t \geq 0$ introduced in (63) and the step sum-processes $\xi_{j\varepsilon}(t)$, $t \geq 0$ introduced in (68).

Let us now formulate the proposition about equivalence of the relation of weak convergence (70) for processes $\xi_{j\varepsilon}(t), t \geq 0$ and the following asymptotic relation formulated in terms of averaged Laplace transforms $\varphi^{(\varepsilon)}(s)$,

$$\frac{1 - \varphi^{(\varepsilon)}(s/u_\varepsilon)}{p_\varepsilon} \to \varsigma(s) \text{ as } \varepsilon \to 0, \ s \geq 0, \tag{88}$$

where (**j**) $\varsigma(s) > 0$ for $s > 0$.

Lemma 7. *Let conditions* **B**, **C** *hold, and* $\eta_0^{(\varepsilon)} = j$. *Then*: ($\boldsymbol{\iota}$) *the relation of weak convergence* (65) *holds, for given* $i, j \in X$, *if and only if the relation of weak convergence* (70) *holds, for the same* j, ($\boldsymbol{\kappa}$) *the limit process* $\xi(t)$,

$t \geq 0$ is the same in relations (65) and (70); ($\boldsymbol{\lambda}$) the assumption that the relation of weak convergence (70) holds for some $j \in X$ implies that this relation holds for every $j \in X$; ($\boldsymbol{\mu}$) the limit process $\xi(t), t \geq 0$ in (70) is the same for any $j \in X$; ($\boldsymbol{\nu}$) $\xi(t), t \geq 0$ is a non-zero and non-decreasing homogenous process with independent increments; ($\boldsymbol{\xi}$) relation (70) holds for given $j \in X$ if and only if relation (88) holds; ($\boldsymbol{\pi}$) the limit function $\varsigma(s)$ in (88) is a cumulant of the process $\xi(t), t \geq 0$, i.e. $\mathsf{E} e^{-s\xi(t)} = e^{-\varsigma(s)t}, \ s, t \geq 0$; ($\boldsymbol{\rho}$) conditions $\mathbf{E}$ and $\mathbf{F}$ are necessary and sufficient for relation (88) to hold; ($\boldsymbol{\sigma}$) cumulant $\varsigma(s) = a(s)$ in this case.

Proof. The statements ($\boldsymbol{\iota}$) – ($\boldsymbol{\nu}$) have been already verified in the proof of Lemma 4.

Let us introduce conditional distribution functions for sojourn times $\varkappa_n^{(\varepsilon)}$ for the semi-Markov process $\eta^{(\varepsilon)}(t)$,

$$G_{ij}^{(\varepsilon)}(t) = \mathsf{P}\left\{\varkappa_1^{(\varepsilon)} \leq t / \eta_0^{(\varepsilon)} = i, \eta_1^{(\varepsilon)} = j\right\}, \ t \geq 0, \ i, j \in X.$$

Obviously

$$Q_{ij}^{(\varepsilon)}(t) = p_{ij}^{(\varepsilon)} G_{ij}^{(\varepsilon)}(t), \ t \geq 0, \ i, j \in X,$$

and

$$G_i^{(\varepsilon)}(t) = \sum_{j=1}^{m} Q_{ij}^{(\varepsilon)}(t) = \sum_{j=1}^{m} p_{ij}^{(\varepsilon)} G_{ij}^{(\varepsilon)}(t), \ t \geq 0, \ i, j \in X,$$

Note that one can choose $G_{ij}^{(\varepsilon)}(t)$ as arbitrary distribution functions concentrated on the positive half-line if $p_{ij}^{(\varepsilon)} = 0$. This does not affect transition probabilities $Q_{ij}^{(\varepsilon)}(t)$ and distribution functions $G_i^{(\varepsilon)}(t)$.

As is known from the theory of semi-Markov Processes that the sojourn times $\varkappa_n^{(\varepsilon)}$ are conditionally independent with respect to the values of the embedded Markov chain $\eta_n^{(\varepsilon)}$. More precisely this means that, for any $t_1, \ldots, t_n \geq 0, i_0, i_1, \ldots, i_n, n = 1, 2, \ldots,$

$$\mathsf{P}\left\{\varkappa_1^{(\varepsilon)} \leq t_1, \ldots, \varkappa_k^{(\varepsilon)} \leq t_n / \eta_0^{(\varepsilon)} = i_0, \ldots, \eta_n^{(\varepsilon)} = i_n\right\}$$
$$= G_{i_0 i_1}^{(\varepsilon)}(t_1) \times \cdots \times G_{i_{n-1} i_n}^{(\varepsilon)}(t_n). \tag{89}$$

As in the proof of Lemma 4, we assume that $\eta_0^{(\varepsilon)} = j$.

It follows from relation (89) that the process $\xi_{j\varepsilon}(t)$ has, for every $j \in X$, the same finite-dimensional distribution as the following process $\breve{\xi}_{j\varepsilon}(t)$ (we use the symbol $\overset{d}{=}$ to show this stochastic equality),

$$\xi_{j\varepsilon}(t) = \sum_{n=1}^{[tp(\varepsilon)^{-1}]} \frac{\varkappa_n^{(\varepsilon)}}{u_\varepsilon}, t \geq 0 \ \overset{d}{=} \ \breve{\xi}_{j\varepsilon}(t), \ t \geq 0, \tag{90}$$

where

$$\breve{\xi}_{j\varepsilon}(t) = \sum_{n=1}^{[tp(\varepsilon)^{-1}]} \frac{\varkappa_n^{(\varepsilon)}\left(\eta_{n-1}^{(\varepsilon)}, \eta_n^{(\varepsilon)}\right)}{u_\varepsilon}, \ t \geq 0, \tag{91}$$

and

($\mathbf{k}_1$) $\left\{\eta_n^{(\varepsilon)}, n = 1, 2, \ldots\right\}$ is a Markov chain with a state space X and the matrix of transition probabilities $\left\|p_{ij}^{(\varepsilon)}\right\|$;

($\mathbf{k}_2$) $\varkappa_n^{(\varepsilon)}(i,j), i,j \in X, n \geq 1$ are mutually independent random variables;

($\mathbf{k}_3$) $\mathsf{P}\left\{\varkappa_n^{(\varepsilon)}(i,j) \leq t\right\} = G_{ij}^{(\varepsilon)}(t), t \geq 0$ for $i, j \in X$, $n \geq 1$;

($\mathbf{k}_4$) the set of random variables $\left\{\varkappa_n^{(\varepsilon)}(i,j), i,j \in X, n \geq 1\right\}$ and the Markov chain $\left\{\eta_n^{(\varepsilon)}, n = 1, 2, \ldots\right\}$ are independent.

It follows from the stochastic equality (90) that ($\mathbf{l}$) the relation of weak convergence (69), treated in Lemma 4, is equivalent to the following relation,

$$\breve{\xi}_{j\varepsilon}(t), t \geq 0 \Rightarrow \xi(t), t \geq 0 \text{ as } \varepsilon \to 0, \tag{92}$$

where ($\mathbf{e}$) $\xi(t), t \geq 0$ is a non-zero and non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us define, for every $j, i, k \in X$, the counting random variables for the random sequence $\bar{\eta}_n^{(\varepsilon)} = \left(\eta_{n-1}^{(\varepsilon)}, \eta_n^{(\varepsilon)}\right), n = 1, 2, \ldots$,

$$\nu_{jn}^{(\varepsilon)}(i,k) = \sum_{r=1}^{n} \chi\left\{\left(\eta_{r-1}^{(\varepsilon)}, \eta_r^{(\varepsilon)}\right) = (i,k)\right\}, \ n = 0, 1, \ldots.$$

It follows from the ($\mathbf{k}_1$) - ($\mathbf{k}_4$) that the process $\breve{\xi}_{j\varepsilon}(t)$ has, for every $j \in X$, the same finite-dimensional distribution as the following process $\tilde{\xi}_{j\varepsilon}(t)$,

$$\breve{\xi}_{j\varepsilon}(t), t \geq 0 \ \stackrel{d}{=} \ \tilde{\xi}_{j\varepsilon}(t), t \geq 0, \tag{93}$$

where

$$\tilde{\xi}_{j\varepsilon}(t) = \sum_{(i,k) \in \widetilde{X}} \ \sum_{n=1}^{\nu_{j[tp_\varepsilon^{-1}]}^{(\varepsilon)}(i,k)} \frac{\varkappa_n^{(\varepsilon)}(i,k)}{u_\varepsilon}, t \geq 0. \tag{94}$$

and

$$\widetilde{X}_\varepsilon = \left\{(i,k) \in X : p_{ik}^{(\varepsilon)} > 0\right\}.$$

Note that, due to $\mathbf{C}$,

$$\widetilde{X}_\varepsilon \subseteq \widetilde{X}_0$$

for all ε small enough.

Note that the definition of the process $\tilde{\xi}_{j\varepsilon}(t)$ takes into account that random variables $\nu^{(\varepsilon)}_{jn}(i,k) = 0, n = 0, 1, \ldots$ with probability 1, if $p^{(\varepsilon)}_{ik} = 0$.

The stochastic equalities (90) and (93) let us replace the processes $\xi_{j\varepsilon}(t)$ by the processes $\tilde{\xi}_{j\varepsilon}(t)$ when we study their weak convergence.

It follows from the stochastic equality (93) that the relation of weak convergence (69), treated in Lemma 5, is also equivalent to the following relation,

$$\tilde{\xi}_{j\varepsilon}(t), t \geq 0 \Rightarrow \xi(t), t \geq 0 \text{ as } \varepsilon \to 0, \tag{95}$$

where (**e**) $\xi(t), t \geq 0$ is a non-zero and non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us also introduce the following step sum-processes,

$$\hat{\xi}_\varepsilon(t) = \sum_{(i,k)\in \widetilde{X}_0} \sum_{n=1}^{\left[t\pi^{(0)}_i p^{(\varepsilon)}_{ik} p_\varepsilon^{-1}\right]} \frac{\varkappa^{(\varepsilon)}_n(i,k)}{u_\varepsilon}, \ t \geq 0. \tag{96}$$

We are also interested in the following relation of weak convergence,

$$\hat{\xi}_\varepsilon(t), \ t \geq 0 \Rightarrow \xi(t), \ t \geq 0 \text{ as } \varepsilon \to 0, \tag{97}$$

where (**e**) $\xi(t), t \geq 0$ is a non-zero, non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us prove the equivalence of relations (95) and (97). This means that (**m**) the relation (95) holds for some $j \in X$ if and only if the relation (97) holds, and, moreover, the limit process can be taken the same in both relations.

We display the proof for one-dimensional distributions. The proof for multi-dimensional distributions is similar.

Let us prove that ($\mathbf{m}_1$) the assumption that relation (97), assumed to hold for every $t > 0$, implies that relation (95) holds for every $t > 0$ and $j \in X$, moreover the limit random variable $\xi(t)$ can be taken the same in both relations.

The law of large numbers for ergodic Markov chains in triangular array settings (see, for example, Silvestrov (1974)) implies that, under conditions **C** and **D**, for every $t > 0$ and $j, i, k \in X$,

$$\frac{\nu^{(\varepsilon)}_{j[tp(\varepsilon)^{-1}]}(i,k)}{p_\varepsilon^{-1}} \xrightarrow{\mathsf{P}} \pi^{(0)}_i p^{(0)}_{ik} t \text{ as } \varepsilon \to 0. \tag{98}$$

Let us choose an arbitrary $t > 0$ and a sequence $0 < c_n < t, n = 1, 2, \ldots$ such that $c_n \to 0$ as $n \to \infty$.

The processes $\sum_{n=1}^{[tp_\varepsilon^{-1}]} \varkappa^{(\varepsilon)}_n(i,k)/u_\varepsilon, \ t \geq 0$ and $p_\varepsilon \nu^{(\varepsilon)}_{j[tp_\varepsilon^{-1}]}(i,k), \ t \geq 0$ are non-negative and non-decreasing, for every $j, i, k \in X$. Taking into account this fact, and representation (94), we get, for every $t > 0$, $j \in X$, any real-valued x, and $n \geq 1$,

$$\begin{aligned}
\mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t) > x\right\} &= \mathsf{P}\{\tilde{\xi}_{j\varepsilon}(t) > x, \bigcap_{(i,k)\in\widetilde{X}_\varepsilon} A^{(\varepsilon)}_{jik}(t, t + c_n)\} \\
&\quad + \mathsf{P}\{\tilde{\xi}_{j\varepsilon}(t) > x, \bigcup_{(i,k)\in\widetilde{X}_\varepsilon} \bar{A}^{(\varepsilon)}_{jik}(t, t + c_n)\} \\
&\leq \mathsf{P}\left\{\hat{\xi}_\varepsilon(t + c_n) > x\right\} \\
&\quad + \sum_{(i,k)\in\widetilde{X}_\varepsilon} \mathsf{P}\left\{\bar{A}^{(\varepsilon)}_{jik}(t, t + c_n)\right\}, \qquad (99)
\end{aligned}$$

where

$$A^{(\varepsilon)}_{jik}(t, s) = \left\{\nu^{(\varepsilon)}_{j[tp_\varepsilon^{-1}]}(i, j) \leq s\pi^{(0)}_i p^{()}_{ik} p_\varepsilon^{-1}\right\}, \ t, s > 0, \ j, i, k \in X.$$

Note that (98) implies that, for every $0 < t < s$ and $j \in X$, $(i, k) \in \widetilde{X}_\varepsilon \subseteq \widetilde{X}_0$,

$$\mathsf{P}\left\{A^{(\varepsilon)}_{jik}(s, t)\right\} + \mathsf{P}\left\{\bar{A}^{(\varepsilon)}_{jik}(t, s)\right\} \to 0 \text{ as } \varepsilon \to 0. \qquad (100)$$

Let Y_t be the set of continuity points for the distribution functions of the limit random variables $\xi(t)$ and $\xi(t \pm c_n), n = 1, 2, \ldots$ in (97). This set is the real line R except at most a countable set of points.

Using the estimate (99), relation (100), and the assumptions that relation (97) holds for one-dimensional distributions, for every $t > 0$, and that the limit process $\xi(t)$ in (97) is stochastically continuous, we get, for every $t > 0$ and $j \in X$,

$$\begin{aligned}
\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t) > x\right\} &\leq \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \Big(\mathsf{P}\left\{\hat{\xi}_\varepsilon(t + c_n) > x\right\} \\
&\quad + \sum_{(i,k)\in\widetilde{X}_\varepsilon} \mathsf{P}\left\{\bar{A}^{(\varepsilon)}_{jik}(t, t + c_n)\right\}\Big) \\
&= \lim_{n\to\infty} \mathsf{P}\left\{\xi(t + c_n) > x\right\} \\
&= \mathsf{P}\left\{\xi(t) > x\right\}, \ x \in Y_t, \qquad (101)
\end{aligned}$$

or, equivalently,

$$\underline{\lim}_{\varepsilon\to 0} \mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t) \leq x\right\} \geq \mathsf{P}\left\{\xi(t) \leq x\right\}, \ x \in Y_t. \qquad (102)$$

Similarly, we can get, for every $t > 0$ and $j \in X$,

$$\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t) \le x\right\} \le \mathsf{P}\left\{\xi(t) \le x\right\}, \ x \in Y_t. \tag{103}$$

Relations (102) and (103) implies that $\mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t) \le x\right\} \to \mathsf{P}\left\{\xi(t) \le x\right\}$ as $\varepsilon \to 0, x \in Y_t$, for every $j \in X$. Since the set Y_t is dense in R, this relation implies that, for every $t > 0$ and $j \in X$,

$$\tilde{\xi}_{j\varepsilon}(t) \Rightarrow \xi(t) \text{ as } \varepsilon \to 0. \tag{104}$$

We omit details in the proof of an inverse proposition that ($\mathbf{m}_2$) the assumption that relation (95), assumed to hold for every $t > 0$ and given $j \in X$, implies that relation (97) holds for every $t > 0$ and, moreover the limit random variable $\xi(t)$ can be taken the same in both relations.

Let us choose an arbitrary $t > 0$ and a sequence $0 < d_n < t, n = 1, 2, \dots$ such that $d_n \to 0$ as $n \to \infty$.

Analogously to (99), we get the following "inverse" to (99) estimate, for any every $t > 0$, real-valued x and $n \ge 1$,

$$\begin{aligned}
\mathsf{P}\left\{\hat{\xi}_{\varepsilon}(t) > x\right\} &= \mathsf{P}\left\{\hat{\xi}_{\varepsilon}(t) > x, \bigcap_{(i,k)\in\widetilde{X}_\varepsilon} \bar{A}^{(\varepsilon)}_{jik}(t+d_n, t)\right\} \\
&\quad + \mathsf{P}\left\{\hat{\xi}_{\varepsilon}(t) > x, \bigcup_{(i,k)\in\widetilde{X}_\varepsilon} A^{(\varepsilon)}_{jik}(t+d_n, t)\right\} \\
&\le \mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t+d_n) > x\right\} \\
&\quad + \sum_{(i,k)\in\widetilde{X}_\varepsilon} \mathsf{P}\left\{A^{(\varepsilon)}_{jik}(t+d_n, t)\right\}.
\end{aligned} \tag{105}$$

Let Z_t be the set of continuity points for the distribution functions of the limit random variables $\xi(t)$ and $\xi(t \pm d_n), n = 1, 2, \dots$ in (95). This set is the real line R except at most a countable set of points.

Using the estimate (105), relation (100), and the assumptions that relation (95) holds for one-dimensional distributions, for every $t > 0$ and given $j \in X$, and that the limit process $\xi(t)$ in (95) is stochastically continuous, we get, for every $t > 0$ and $x \in Z_t$,

$$\begin{aligned}
\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\left\{\hat{\xi}_{\varepsilon}(t) > x\right\} &\le \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \Big(\mathsf{P}\left\{\tilde{\xi}_{j\varepsilon}(t+d_n) > x\right\} \\
&\quad + \sum_{(i,k)\in\widetilde{X}_\varepsilon} \mathsf{P}\left\{A^{(\varepsilon)}_{jik}(t+d_n, t)\right\}\Big) \\
&= \lim_{n\to\infty} \mathsf{P}\left\{\xi(t+d_n) > x\right\} \\
&= \mathsf{P}\{\xi(t) > x\}.
\end{aligned} \tag{106}$$

The continuation of the proof for the proposition ($\mathbf{m}_2$) is analogous to those given above in the proof of the proposition ($\mathbf{m}_1$).

Let us introduce now the step sum-process,

$$\breve{\xi}^*_\varepsilon(t) = \sum_{n=1}^{[tp(\varepsilon)^{-1}]} \frac{\varkappa^{*(\varepsilon)}_n\left(\eta'^{(\varepsilon)}_n, \eta''^{(\varepsilon)}_n\right)}{u_\varepsilon}, \ t \geq 0, \tag{107}$$

where

($\mathbf{n}_1$) $\left\{\bar{\eta}^{*(\varepsilon)}_n = \left(\eta'^{(\varepsilon)}_n, \eta''^{(\varepsilon)}_n\right), n = 1, 2, \ldots\right\}$ a sequence of i.i.d. random vectors which takes values (i, j) with probabilities $\pi^{(0)}_i p^{(\varepsilon)}_{ij}$ for $i, j \in X$;

($\mathbf{n}_2$) $\varkappa^{*(\varepsilon)}_n(i, j), i, j \in X, n \geq 1$ are mutually independent random variables;

($\mathbf{n}_3$) $\mathsf{P}\left\{\varkappa^{*(\varepsilon)}_n(i, j) \leq t\right\} = G^{(\varepsilon)}_{ij}(t), t \geq 0$ for $i, j \in X$, $n \geq 1$;

($\mathbf{n}_4$) the set of random variables $\left\{\varkappa^{*(\varepsilon)}_n(i, j), i, j \in X, n \geq 1\right\}$ and the random sequence $\left\{\bar{\eta}^{*(\varepsilon)}_n, n = 1, 2, \ldots\right\}$ are independent.

We are interested in the following relation of weak convergence,

$$\breve{\xi}^*_\varepsilon(t), \ t \geq 0 \Rightarrow \xi(t), \ t \geq 0 \text{ as } \varepsilon \to 0, \tag{108}$$

where ($\mathbf{e}$) $\xi(t), t \geq 0$ is a non-zero and non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us define, for every $i, k \in X$, the counting random variables for the random sequence $\bar{\eta}^{*(\varepsilon)}_n = \left(\eta'^{(\varepsilon)}_n, \eta''^{(\varepsilon)}_n\right), n = 1, 2, \ldots$,

$$\nu^{*(\varepsilon)}_n(i, k) = \sum_{r=1}^{n} \chi\left\{\left(\eta'^{(\varepsilon)}_r, \eta''^{(\varepsilon)}_r\right) = (i, k)\right\}, \ n = 0, 1, \ldots.$$

It follows from the properties ($\mathbf{n}_1$) - ($\mathbf{n}_4$) that the process $\breve{\xi}^*_\varepsilon(t)$ has, for every $j \in X$, the same finite-dimensional distribution as the following process $\tilde{\xi}^*_\varepsilon(t)$,

$$\breve{\xi}^*_\varepsilon(t), t \geq 0 \ \overset{d}{=} \ \tilde{\xi}^*_\varepsilon(t), \ t \geq 0, \tag{109}$$

where

$$\tilde{\xi}^*_\varepsilon(t) = \sum_{(i,k)\in\widetilde{X}} \sum_{n=1}^{\nu^{*(\varepsilon)}_{[tp_\varepsilon^{-1}]}(i,k)} \frac{\varkappa^{*(\varepsilon)}_n(i, k)}{u_\varepsilon}, \ t \geq 0. \tag{110}$$

It follows from stochastic equality (110) that ($\mathbf{o}$) the relation of weak convergence (108) is equivalent to the following relation,

$$\tilde{\xi}^*_\varepsilon(t), \ t \geq 0 \Rightarrow \xi(t), \ t \geq 0 \text{ as } \varepsilon \to 0, \tag{111}$$

where (**e**) $\xi(t), t \geq 0$ is a non-zero and non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

Let us also introduce the following step sum-processes,

$$\hat{\xi}^*_\varepsilon(t) = \sum_{(i,k)\in \widetilde{X}} \sum_{n=1}^{\left[t\pi_i^{(0)} p_{ik}^{(0)} p_\varepsilon^{-1}\right]} \frac{\varkappa_n^{*(\varepsilon)}(i,k)}{u_\varepsilon}, \ t \geq 0. \tag{112}$$

Let us also consider the following relation of weak convergence,

$$\hat{\xi}^*_\varepsilon(t), \ t \geq 0 \Rightarrow \xi(t), \ t \geq 0 \text{ as } \varepsilon \to 0, \tag{113}$$

where (**e**) $\xi(t), t \geq 0$ is a non-zero, non-decreasing and stochastically continuous process with the initial value $\xi(0) = 0$.

We state that relations (111) and (113) are equivalent. This means that (**p**) the assumption that relation (111) holds if and only if relation (113), moreover the limit stochastic process $\xi(t), t \geq 0$ can be taken the same in both relations.

By the definition, $\chi\{(\eta_r'^{(\varepsilon)}, \eta_r''^{(\varepsilon)}) = (i,k)\}, \ r = 1, 2, \ldots$ are i.i.d. random variables taking value 1 and 0 with probabilities $\pi_i^{(0)} p_{ik}^{(\varepsilon)}$ and $1 - \pi_i^{(0)} p_{ik}^{(\varepsilon)}$. Thus, under condition **D**, due to standard weak law of large number for binary random variables, for every $t > 0$ and $i, k \in X$,

$$\frac{\nu^{*(\varepsilon)}_{[tp_\varepsilon^{-1}]}(i,k)}{p_\varepsilon^{-1}} \xrightarrow{\mathsf{P}} \pi_i^{(0)} p_{ik}^{(0)} t \text{ as } \varepsilon \to 0. \tag{114}$$

The careful analysis of the proof of the proposition (**l**) about the equivalence of the relations of weak convergence (95), for processes $\tilde{\xi}_\varepsilon(t), t \geq 0$, and (97), for processes $\hat{\xi}_\varepsilon(t), t \geq 0$, shows that conditions ($\mathbf{n}_2$) - ($\mathbf{n}_4$) were used in this proof plus the asymptotic relation (98), which is a weak law of large numbers for the corresponding frequency random variables for the random sequence $\eta_n^{(\varepsilon)}$. Condition ($\mathbf{n}_1$) was used together with condition **C** only as conditions providing the asymptotic relation (98).

These remarks let us state that the proof given for the proposition (**m**) can be just replicated in order to prove the proposition (**p**). Indeed, conditions ($\mathbf{n}_2$) - ($\mathbf{n}_4$) replace, in this case, conditions ($\mathbf{k}_2$) - ($\mathbf{k}_4$), and the asymptotic relation (114), implied by the condition ($\mathbf{n}_1$), replaces the asymptotic relation (98).

Now let us use the following stochastic equality that obviously follows from comparison of conditions ($\mathbf{k}_2$) - ($\mathbf{k}_4$) and ($\mathbf{n}_2$) - ($\mathbf{n}_4$),

$$\hat{\xi}_\varepsilon(t), t \geq 0 \ \stackrel{d}{=} \ \hat{\xi}^*_\varepsilon(t), \ t \geq 0. \tag{115}$$

The propositions (**m**) and (**p**) combined with the stochastic equalities (90), (93), (109), and (115) implies that (**q**) the assumption that relation of weak convergence (69), treated in Lemma 5, holds if and only if the relation (108) holds, moreover the limit stochastic process $\xi(t)$ can be taken the same in both relations.

We are now in position to make the last step in the proof. Conditions $(\mathbf{n}_2)$ - $(\mathbf{n}_4)$ imply that $\varkappa_n^{*(\varepsilon)}\left(\eta_n'^{(\varepsilon)}, \eta_n''^{(\varepsilon)}\right), n = 1, 2, \ldots$ are i.i.d. random variables. Moreover, the corresponding distribution has the following form,

$$\begin{aligned} \mathsf{P}\left\{\varkappa_1^{*(\varepsilon)}\left(\eta_1'^{(\varepsilon)}, \eta_1''^{(\varepsilon)}\right) \leq t\right\} &= \sum_{i,k \in X} G_{ik}^{(\varepsilon)}(t)\pi_i^{(0)} p_{ik}^{(\varepsilon)} \\ &= \sum_{i \in X} \pi_i^{(0)} \sum_{j \in X} G_{ik}^{(\varepsilon)}(t) p_{ik}^{(\varepsilon)} \\ &= \sum_{i \in X} \pi_i^{(0)} G_i^{(\varepsilon)}(t) = G^{(\varepsilon)}(t),\ t \geq 0. \quad (116) \end{aligned}$$

The statements $(\boldsymbol{\xi})$ and $(\boldsymbol{\pi})$ follows, in an obvious way, from the proposition $(\mathbf{q})$. Indeed, $\breve{\xi}_\varepsilon^*(t), t \geq$ is the step sum-process based on i.i.d. random variables, and, therefore,

$$\mathsf{E} \exp\left\{-s\breve{\xi}_\varepsilon^*(t)\right\} = \varphi^{(\varepsilon)}(s/u_\varepsilon)^{[tp_\varepsilon^{-1}]},\ s, t \geq 0. \quad (117)$$

Relation (117) implies that, for given $t > 0$ the random variables $\breve{\xi}_\varepsilon^*(t)$ converge weakly to some non-zero limit random variable if and only if relation (88) holds and, in this case,

$$\begin{aligned} \mathsf{E} \exp\left\{-s\breve{\xi}_\varepsilon^*(t)\right\} &= \varphi^{(\varepsilon)}(s/u_\varepsilon)^{[tp_\varepsilon^{-1}]} \\ &\sim \exp\left\{-(1 - \varphi^{(\varepsilon)}(s/u_\varepsilon))tp_\varepsilon^{-1}\right\} \\ &\to \exp\left\{-\varsigma(s)t\right\} \text{ as } \varepsilon \to 0,\ s \geq 0, \quad (118) \end{aligned}$$

where $\varsigma(s) > 0$ for $s > 0$.

The random variable $\xi(t)$ has, for every $t > 0$, an infinitely divisible distribution, as a week limit of sums of i.i.d. random variables, and $\varsigma(s)t$ is the cumulant of the process $\xi(t)$.

As was pointed out in the proof of Theorem 1 relation (118) is equivalent to **E** and **F** and in this case $\varsigma(s) \equiv a(s)$. This remark completes the proof of statements $(\boldsymbol{\rho})$ and $(\boldsymbol{\sigma})$ of Lemma 7. □

Remark 3. The proof presented above shows that the only property of the quantities p_ε, used in the proof of Lemma 5, was $(\mathbf{r})$ $0 < p_\varepsilon \to 0$ as $\varepsilon \to 0$. Lemma 5 and its proof remain to be valid if any function p_ε, satisfying the assumption $(\mathbf{r})$, will be used in the formulas (63) and (68) defining, respectively, the process $\xi_{j\varepsilon}(t), t \geq 0$, and in the expression $\left(1 - \varphi^{(\varepsilon)}(s/u_\varepsilon)\right)/p_\varepsilon$ used in the asymptotic relation (88). In this case, condition **B** in Lemma 5 can be replaced by the simpler assumption (**r**).

Remark 4. The proof presented above can be applied to any sum-process of conditionally independent random variables $\breve{\xi}^*_\varepsilon(t), t \geq 0$ defined by formula (107) under the assumption that ($\mathbf{s}_1$) conditions ($\mathbf{n}_2$) - ($\mathbf{n}_4$) hold. Condition ($\mathbf{n}_1$) can be replaced by a general assumption that ($\mathbf{s}_2$) $\{\bar{\eta}^{*(\varepsilon)}_n = (\eta'^{(\varepsilon)}_n, \eta''^{(\varepsilon)}_n)$, $n = 1, 2, \ldots\}$ is a sequence of random vectors taking values in the space $X \times X$ such that the weak law of large numbers in the form of the asymptotic relation (114). Also, ($\mathbf{s}_3$) the positivity of $\pi^{(0)}_i$ is not needed, and ($\mathbf{s}_4$) any function satisfying assumption (**r**) can be taken as p_ε. Under the assumptions ($\mathbf{s}_1$) - ($\mathbf{s}_4$), the asymptotic relation (88) is necessary and sufficient condition for weak convergence of processes $\breve{\xi}^*_\varepsilon(t), t \geq 0$. The limit process is a non-negative homogeneous process with independent increments with the cumulant $\varsigma(s)$ which appears in (88). Moreover conditions **E** and **F** are necessary and sufficient for relation (88) to hold, and cumulant $\varsigma(s) = a(s)$ in this case.

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