Theory of Stochastic Processes
Vol. 13 (29), no. 3, 2007, pp. 65–79

UDC 519.21

ALEXEY V. RUDENKO

LOCAL TIME AS AN ELEMENT OF THE SOBOLEV SPACE

For a centered Gaussian random field taking its values in $\mathbb{R}^d$, we investigate the existence of a local time as a generalized functional, i.e an element of some Sobolev space. We give the sufficient condition for such an existence in terms of the field covariation and apply it in several examples: the self-intersection local time for a fractional Brownian motion and the intersection local time for two Brownian motions.

Introduction

Let $(T, \mathfrak{B})$ be a measurable space with finite measure ν on it. Let $\xi(t), t \in T$, be a centered Gaussian random field with values in $\mathbb{R}^d$. Consider the integrals $L_\varepsilon = \int_T f_\varepsilon(\xi(t))\nu(dt)$, where f_ε approximates, in a sense, δ_0 as $\varepsilon \to 0+$ (δ_0 denotes the measure with weight 1 concentrated in $0 \in \mathbb{R}^d$). If the limit of L_ε exists, we can say that the local time exists (in some sense) and we call this limit as the local time of ξ at $0 \in \mathbb{R}^d$. We say that L_ε is the approximation for the local time of ξ at $0 \in \mathbb{R}^d$. Our aim is to investigate the existence of local time so the question is: When does L_ε converge as $\varepsilon \to 0+$? The condition for L_2-convergence in terms of covariation is relatively easy to obtain (see [2],[9]). We consider the convergence in some Sobolev space $D_{2,\alpha}$ defined in the book of Watanabe [10]. The interest in this topic arises from the possibility to define the local time as an element of the Sobolev space with negative α, while it does not exist in the usual sense. It is possible to determine the exact smoothness of a local time in the sense of spaces $D_{2,\alpha}$. Sometimes the local time does not exist as an element of any Sobolev space, but if we modify the approximation by subtracting its mathematical expectation, we obtain the convergence to the so-called renormalized local time. A classical example of renormalization is the renormalization of the self-intersection local time for a two-dimensional Brownian motion. This result can be found, for instance, in [7]. We introduce some kind of renormalization, and our definition includes this classical setup.

In [5], Imkeller, Perez-Abreu, and Vives found an explicit form of the Itô–Wiener expansion for an approximation of the k-fold self-intersection local time of a Brownian motion. They estimated Hermite polynomials in this expansion and obtained the convergence of the approximation in some Sobolev space. Albeverio, Hu and Zhou [8] employed a similar approach, but their result is opposite. They proved that the renormalized self-intersection local time of a planar Brownian motion is not differentiable in the Watanabe–Meyer sense, i.e. it does not belong to the space $D_{2,1}$. Another question was considered in [11] where Nualart and Hu deal with the renormalization of the 2-fold self-intersection local time for a fractional Brownian motion. They proved the convergence of this renormalization in L_2 (which is same as in $D_{2,0}$) under some conditions.

2000 *AMS Mathematics Subject Classification.* Primary 60H07.

Key words and phrases. Local time, Itô–Wiener expansion, Sobolev spaces, Gaussian random field, fractional Brownian motion.

Research was partially supported by the Ministry of Education and Science of Ukraine, project GP/F13/0095.

Later on, they gave a sufficient condition for the existence of the same renormalized local time as an element of $D_{2,\alpha}$ for positive α [12].

In [2], we have found a suitable integral representation for the expectation of Itô–Wiener expansion terms for local time approximations and use it here to derive the condition for the convergence via the covariation of a field. This condition is both necessary and sufficient if the underlying Gaussian field is centered and we consider the local time at zero. We also get a similar condition for the convergence of a renormalized local time. In our context, the renormalization is the subtraction of several leading terms in the Itô–Wiener expansion. We also consider some applications including the 2-fold self-intersection local time for a fractional Brownian motion. We have to mention that our main ideas were already present in [2].

Let us mention that there are other ways to define the local time as an element of some generalized space of random variables when it does not exist in the usual sense. In [1,4], the authors consider the Hida distributions instead of Sobolev spaces.

The structrure of the exposition is the following. In the first section, the necessary and sufficient condition for the convergence is given in terms of a sequence of integrals. In the second section, this condition is rewritten in terms of one integral. In the third section, we introduce the renormalization using a generalization of results from the previous sections (with an additional condition on approximations). The last section is devoted to applications.

1. Necessary and sufficient condition for the existence of local time

Let's describe briefly the construction of Sobolev spaces and related objects (as it appears in [10]). To do this, we need to define a probability space more precisely and impose some conditions on ξ. Let B be a Banach space with Gaussian probability measure on it, which plays the role of a probability space for ξ. Denote the covariation matrix function for ξ as $K(s,t), s,t \in T$. Suppose that each $\xi(t)$ is linear as a function of $\omega \in B$, and $\sigma(\{\xi(t), t \in T\})$ coincides with $\mathfrak{B}(B)$, a Borel σ-algebra on B. Suppose also that ξ is jointly measurable (as a function of both $t \in T$ and $\omega \in B$) and that ξ is not degenerate ν-a.e.: $\nu(\{t | \det K(t,t) = 0\}) = 0$. Let $H_n, n = 0, 1, \dots,$ be a subspace of $L_2(B,\mu)$ generated by all polynomials on B with degree less or equal n. By polynomials on B, we mean functions of the form $P(l_1(x), l_2(x), \dots, l_k(x))$, where P is a polynomial in k variables, and $l_1, \dots, l_k \in B^*$. By G_n, we denote an orthogonal supplement of H_{n-1} in H_n for $n \in \mathbb{N}$. Let P_n be a projector on G_n. Then we have $\bigoplus\limits_{n=0}^{\infty} G_n = L_2(B,\mu)$ (polynomials are dense in $L_2(B,\mu)$). The corresponding decomposition $h = \sum\limits_{n=0}^{\infty} P_n h, h \in L_2(B,\mu)$ is called the Itô–Wiener expansion (or the chaos decomposition [6]). We can define a set of norms for each $h \in H_n$ as

$$\|h\|_{2,\alpha}^2 = \sum_{n=0}^{\infty} (1+n)^{\alpha} \|h\|_2^2$$

where $\|\cdot\|_2$ is the norm in $L_2(B,\mu)$. Then $D_{2,\alpha}$ is the completion of $\bigcup\limits_{n=0}^{\infty} H_n$ by the norm $\|\cdot\|_{2,\alpha}$.

We define an approximation for δ_0 as

$$f_\varepsilon(x) = \frac{1}{\varepsilon^d} f(\frac{x}{\varepsilon}), f \in S(\mathbb{R}^d), f \geqslant 0, \int_{\mathbb{R}^d} f(x)dx = 1.$$

We denote

$$G(s,t) = K^{-1/2}(t,t)K(s,t)K^{-1/2}(s,s),$$
$$I_n = \int_T\int_T\int_{S_d} \|G(s,t)x\|^n\,\sigma(dx)\frac{\nu(ds)}{\sqrt{\det K(s,s)}}\frac{\nu(dt)}{\sqrt{\det K(t,t)}},$$

where S_d is a unit sphere in $\mathbb{R}^d$, and σ is the uniform surface measure on S_d. The matrix G is defined as $\nu\times\nu$-a.e., thus the integral is well defined.

Theorem 1. *Fix $\alpha\in\mathbb{R}$. The following statements are equivalent:*

1. $L_\varepsilon \to L, \varepsilon\to 0+$ *in* $D_{2,\alpha}$,
2. $\overline{\lim\limits_{\varepsilon\to 0+}}\,\|L_\varepsilon\|_{2,\alpha} < +\infty$,
3. $\sum\limits_{n=0}^{\infty} I_{2n}(2n+1)^{\alpha+d/2-1} < +\infty$.

Proof. We will prove the following implications $1\Rightarrow 2\Rightarrow 3\Rightarrow 1$.

$1\Rightarrow 2$ is obvious.

$2\Rightarrow 3$. Let's write a formula for $\|L_\varepsilon\|_{2,\alpha}$ using the Itô–Wiener expansion. By definition, L_ε is bounded so $L_\varepsilon\in L_2(\Omega)$ and we may write the Itô–Wiener expansion $L_\varepsilon = \sum\limits_{n=0}^{\infty} a_n(\varepsilon)$. We have $\|L_\varepsilon\|_{2,\alpha} = \sum\limits_{n=0}^{\infty}(n+1)^\alpha E(a_n(\varepsilon))^2$. We know that (from [2])

$$E(a_n(\varepsilon))^2 = \\ = \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(y)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\varepsilon((x,u)+(y,w))}q_n(s,t,u,w)\,du\,dw\,dx\,dy\,\nu(ds)\nu(dt),$$

where

$$q_n(s,t,u,w) = (2\pi)^{-2d}\frac{(-1)^n}{n!}(K(s,t)u,w)^n e^{-\frac12(K(s,s)u,u)}e^{-\frac12(K(t,t)w,w)}.$$

From (2), we get $\overline{\lim\limits_{\varepsilon\to 0+}}\,E(a_0(\varepsilon))^2 < +\infty$. On the other hand,

$$E(a_0(\varepsilon))^2 = \left(\int_T\int_{\mathbb{R}^d} f(x)e^{-\frac12\varepsilon(K^{-1}(t,t)x,x)}\frac{(2\pi)^{-d/2}}{\sqrt{\det K(t,t)}}dx\,\nu(dt)\right)^2.$$

We can see that $E(a_0(\varepsilon))^2$ is a monotonous function of ε with maximum at $\varepsilon=0$. We conclude that

$$\int_T\frac{\nu(dt)}{\sqrt{\det K(t,t)}} < +\infty.$$

Let's estimate $E(a_n(\varepsilon))^2$:

$$(1)\quad E(a_n(\varepsilon))^2 \leqslant \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)f(y)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|q_n(s,t,u,w)|\,du\,dw\,dx\,dy\,\nu(ds)\nu(dt) = \\ = \int_T\int_T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|q_n(s,t,u,w)|\,du\,dw\,\nu(ds)\nu(dt) = J_n.$$

We want to transform J_n.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |q_n(s,t,u,w)|\, dudw =$$

$$(2\pi)^{-2d}\frac{1}{n!}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(K(s,t)u,w)|^n e^{-\frac{1}{2}(K(s,s)u,u)} e^{-\frac{1}{2}(K(t,t)w,w)} dudw =$$

$$= \frac{(2\pi)^{-2d}}{n!\sqrt{\det K(t,t)\det K(s,s)}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |(G(s,t)u,w)|^n e^{-\frac{1}{2}\|u\|^2-\frac{1}{2}\|w\|^2} dudw =$$

$$= \frac{(2\pi)^{-2d}}{n!\sqrt{\det K(t,t)\det K(s,s)}}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\|u\|^2}\left(\int_{\mathbb{R}^d} |(G(s,t)u,w)|^n e^{-\frac{1}{2}\|w\|^2} dw\right) du =$$

$$= \frac{(2\pi)^{-2d}}{n!\sqrt{\det K(t,t)\det K(s,s)}}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\|u\|^2}\, \|G(s,t)u\|^n \left(\int_{\mathbb{R}^d} |w_1|^n\, e^{-\frac{1}{2}\|w\|^2} dw\right) du =$$

$$= \frac{(2\pi)^{-2d}}{n!\sqrt{\det K(t,t)\det K(s,s)}}\int_{\mathbb{R}^d} |w_1|^n\, e^{-\frac{1}{2}\|w\|^2} dw\cdot$$

$$\cdot\int_{\mathbb{R}_+} e^{-\frac{1}{2}r^2} r^{n+d-1}\left(\int_{S_d} \|G(s,t)x\|^n\, \sigma(dx)\right) dr =$$

$$= \frac{(2\pi)^{-2d}}{n!\sqrt{\det K(t,t)\det K(s,s)}}\int_{\mathbb{R}^d} |w_1|^n\, e^{-\frac{1}{2}\|w\|^2} dw\cdot$$

$$\cdot\int_{\mathbb{R}_+} e^{-\frac{1}{2}r^2} r^{n+d-1} dr \int_{S_d} \|G(s,t)x\|^n\, \sigma(dx) =$$

$$= \frac{C(n,d)}{\sqrt{\det K(t,t)\det K(s,s)}}\int_{S_d} \|G(s,t)x\|^n\, \sigma(dx).$$

As we can see, $J_n = C(n,d)I_n$, where

$$C(n,d) = \frac{(2\pi)^{-2d}}{n!}\int_{\mathbb{R}^d} |w_1|^n\, e^{-\frac{1}{2}\|w\|^2} dw \int_{\mathbb{R}_+} e^{-\frac{1}{2}r^2} r^{n+d-1} dr =$$

$$= \frac{2}{n!}(2\pi)^{-(3d+1)/2}\int_0^\infty v_1^n e^{-v_1^2/2} dv \int_0^\infty r^{n+d-1} e^{-r^2/2} dr =$$

$$= \frac{2^{n+(d-1)/2}}{n!}(2\pi)^{-(3d+1)/2}\int_0^\infty x^{(n+d-2)/2} e^{-x} dx \int_0^\infty x^{(n-1)/2} e^{-x} dx =$$

$$= \frac{2^{n+(d-1)/2}}{n!}(2\pi)^{-(3d+1)/2}\Gamma(\frac{n+d}{2})\Gamma(\frac{n+1}{2}).$$

We need to find the asymptotics of $C(n,d)$ for large n. We can use the asymptotics of the gamma function (the Stirling formula),

$$\Gamma(x) \sim \sqrt{2\pi/x}\left(\frac{x}{e}\right)^x, x \to +\infty,$$

and the equality $n! = \Gamma(n+1)$. We have

$$C(n,d) = \frac{2^{n+(d-1)/2}}{\Gamma(n+1)}(2\pi)^{-(3d+1)/2}\Gamma(\frac{n+d}{2})\Gamma(\frac{n+1}{2}) \sim$$
$$\sim 2^{(d-1)/2}(2\pi)^{-(3d+1)/2}2^n\sqrt{\frac{8\pi}{n+d}}\cdot$$
$$\cdot\left(\frac{n+d}{2e}\right)^{(n+d)/2}\left(\frac{n+1}{2e}\right)^{(n+1)/2}\left(\frac{e}{n+1}\right)^{n+1} =$$
$$= 2^{(d-1)/2}(2\pi)^{-(3d+1)/2}2^{-(d+1)/2}(8\pi)^{1/2}e^{-(d+1)/2}\cdot$$
$$\cdot\sqrt{\frac{1}{n+d}}(1+\frac{d+1}{n+1})^{(n+1)/2}(n+d)^{(d-1)/2} \sim$$
$$\sim (2\pi)^{-3d/2}n^{d/2-1}, n \to +\infty.$$

The integral $\int_{S_d}\|G(s,t)x\|^n\,\sigma(dx)$ can be calculated explicitly in terms of eigenvalues of $G(s,t)$, but it is more convenient for us to leave it in the form of integral. Indeed, we can use that the expression taken to the n-th power under integral, which is $\|G(s,t)x\|$, is less or equal than 1 when $\|x\| = 1$ (from the definition of G). Therefore, the condition $I_0 < +\infty$ (this is our assumption) gives us $I_n < +\infty, n \in \mathbb{N}$. Using the Lebesgue theorem of dominated convergence, we get $\lim\limits_{\varepsilon\to 0+} E(a_n(\varepsilon))^2 = J_n$ for even n and $\lim\limits_{\varepsilon\to 0+} E(a_n(\varepsilon))^2 = 0$ for odd n. We can see now that

$$\sum_{n=0}^{+\infty} J_{2n}(2n+1)^\alpha \le \varlimsup_{\varepsilon\to 0+}\|L_\varepsilon\|_{2,\alpha} < +\infty.$$

Using the asymptotics of $C(n,d)$, it is easy to complete the proof of this implication.

3 $\Rightarrow$ 1. Using the Hölder inequality, we get $I_{2n+1} \leqslant M\sqrt{I_{2n}I_{2n+2}}, n = 0,1,\ldots$ (the constant is independent of n). So if we include I_{2n+1} into the infinite sum, then $\sum\limits_{n=0}^{\infty} I_n(n+1)^{\alpha+d/2-1} < +\infty$. Using the formula for $Ea_n(\varepsilon_1)a_n(\varepsilon_2)$ (see [2]) and the condition $I_n < +\infty$, we conclude that $Ea_n(\varepsilon_1)a_n(\varepsilon_2)$ is convergent to J_n for even n and to 0 for odd n. Thus, $a_n(\varepsilon)$ is convergent in $L_2(\Omega)$. Define $a_n(0) = L_2 - \lim\limits_{\varepsilon\to 0+} a_n(\varepsilon)$. From inequality (1), we get

$$E(a_n(\varepsilon))^2 \leqslant J_n = C(n,d)I_n$$

Consequently,

$$\sum_{n=0}^{\infty}(n+1)^\alpha E(a_n(0))^2 \leqslant \sum_{n=0}^{\infty}(n+1)^\alpha C(n,d)I_n < +\infty,$$

and there exists $L = \sum\limits_{n=0}^{\infty} a_n(0) \in D_{2,\alpha}$. Now all left to do is to use a uniform bound for tails of the Itô–Wiener expansion for approximations. We get

$$\|L - L_\varepsilon\|_{2,\alpha} = \sum_{n=0}^{\infty}(n+1)^\alpha E(a_n(0) - a_n(\varepsilon))^2 \to 0, \varepsilon \to 0+.$$

Note that if statement (3) in the theorem is not fulfilled, then it is not possible to construct the local time as an element of $D_{2,\alpha}$ using approximations of this kind and some Sobolev space with smaller α. Indeed, suppose that we have convergence to some

L in D_{2,α_0} with $\alpha_0 < \alpha$ (while there is no convergence for our fixed α). Then using same arguments as in theorem's proof, we can prove

$$\|L\|_{2,\alpha} = \sum_{n=0}^{+\infty} E(a_{2n}(0))^2(2n+1)^\alpha = \sum_{n=0}^{+\infty} J_{2n}(2n+1)^\alpha = \infty.$$

Thus L can not be an element of $D_{2,\alpha}$.

For some values of α, it is possible to simplify the condition of the theorem.

Corollary 1. *If $\alpha < -\frac{d}{2}$, then statements in the theorem above are equivalent to*

$$\int_T \frac{\nu(dt)}{\sqrt{\det K(t,t)}} < +\infty.$$

Proof. Note that $I_0 = \sigma(S_d)\Big(\int_T \frac{\nu(dt)}{\sqrt{\det K(t,t)}}\Big)^2$. Obviously, statement (3) of the theorem yields $I_0 < +\infty$. This proves one side of equivalence. Now suppose that $I_0 < +\infty$. As we have seen already, $I_n \leqslant I_0, n \in \mathbb{N}$. Taking into account that $\alpha + \frac{d}{2} - 1 < -1$, we have $\sum\limits_{n=0}^{\infty} I_{2n}(2n+1)^{\alpha+d/2-1} < +\infty$.

2. Integral form of the condition for local time existence

We want to give a different form of the condition for the local time existence from Theorem 1, using the following lemma.

Lemma 1. *Let $(E, \mathfrak{F})$ be ameasurable space with measure μ on it and let f be a measurable function on E with values in $[0,1]$. Denote:*

$$J_n = \int_E (f(x))^n \mu(dx),$$
$$p_\beta : [0,1] \to [1,\infty], \beta \in \mathbb{R},$$
$$p_\beta(z) = \begin{cases} z^\beta, & \beta < 0 \\ 1 - \ln z, & \beta = 0 \\ 1, & \beta > 0. \end{cases}$$

For any $\gamma \in \mathbb{R}$. the following statements are equivalent:

1. $\sum\limits_{n=0}^{\infty} J_n(n+1)^\gamma < +\infty$,
2. $\int_E p_{-\gamma-1}(1-f(x))\mu(dx) < +\infty$.

Proof. Note that if $\gamma < -1$, then both the sum and the integral converge when μ is finite or diverge when $\mu(E) = +\infty$. So it is enough to consider the case $\gamma \geqslant -1$. It is obvious that the sequence $(n+1)^\gamma$ may be replaced by the sequence of positive numbers $c_n, n \in \mathbb{N}$, if $c_n \sim (n+1)^\gamma$. We use this and take c_n to be coefficients near q^n in the Taylor series of $p_{-\gamma-1}(1-q)$ at the point $q = 0$. We can write an explicit form for c_n:

$$\gamma = -1 : p_{-\gamma-1}(1-q) = 1 - \ln(1-q) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} q^n = \sum_{n=0}^{\infty} c_n q^n$$

$$,\gamma > -1 : p_{-\gamma-1}(1-q) = (1-q)^{-\gamma-1} = \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} (\frac{\gamma}{k} + 1) \right) q^n = \sum_{n=0}^{\infty} c_n q^n.$$

All sums converge if $|q| < 1$. It can be verified that c_n satisfy the needed property. The condition $f \in [0,1]$ allows us to write

$$\sum_{n=0}^{\infty} J_n c_n = \int_E \sum_{n=0}^{\infty} (f(x))^n c_n \mu(dx) = \int_E p_{-\gamma-1}(1 - f(x))\mu(dx).$$

Lemma 1 is proved.

Now we apply Lemma 1 to get a different form of the condition of Theorem 1. Note that, for $\alpha < -\frac{d}{2}$, we have already done this above.

Corollary 2. *Fix $\alpha \in \mathbb{R}$. The following statements are equivalent:*

1. $L_\varepsilon \to L, \varepsilon \to 0+$ *in* $D_{2,\alpha}$,
2. $$\int_T \int_T \int_{\mathbb{R}^d} p_{-\alpha-d/2}(1 - \|G(s,t)x\|)\sigma(dx) \frac{\nu(ds)}{\sqrt{\det K(s,s)}} \frac{\nu(dt)}{\sqrt{\det K(t,t)}} < +\infty.$$

Proof. We choose $\mu(dx, ds, dt) = \sigma(dx)\frac{\nu(ds)}{\sqrt{\det K(s,s)}}\frac{\nu(dt)}{\sqrt{\det K(t,t)}}$ to be the measure and $\|G(s,t)x\|$ to be the function f from Lemma 1. Now Corollary 2 is a direct consequence of Theorem 1 and Lemma 1 (we also need the fact that we can include terms with odd numbers into sum (3) from Theorem 1, but we showed it in the proof of this Theorem).

3. Renormalization

We want to define the renormalization for local time approximations, but before we need to prove a more generalized form of Theorem 1. If we define local time approximations more precisely taking f to be a Gaussian density, $f(x) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|x\|^2}{2}}$, then we can prove a result similar to Theorem 1 in the case where the convergence of L_ε is in the sense of some seminorm on the space of square integrable random variables on our probability space (its values may be infinite). We define this seminorm as

$$S(\eta, \{c_n\}) = \sum_{n=0}^{\infty} c_n E\eta_n^2$$

where η is a square integrable random variable,

$$\eta = \sum_{n=0}^{\infty} \eta_n$$

is the Itô–Wiener expansion of η and $c_n \geqslant 0, n = 0, 1, \ldots,$ is an arbitrary sequence of non-negative numbers. Norms in Sobolev spaces can be considered as partial cases of this seminorm.

Theorem 2. *The following statements are equivalent*

1. $L_\varepsilon \to L, \varepsilon \to 0+$ *with respect to seminorm* $S(\cdot, \{c_n\})$,
2. $\overline{\lim\limits_{\varepsilon \to 0+}} S(L_\varepsilon, \{c_n\}) < +\infty$,
3. $\sum\limits_{n=0}^{\infty} I_{2n} c_n (2n+1)^{d/2-1} < +\infty$.

Proof. $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 3$. As in Theorem 1, we consider the Itô–Wiener expansion of $L_\varepsilon = \sum\limits_{n=0}^{\infty} a_n(\varepsilon)$. Using the explicit form of a_n, we get

$$E(a_n(\varepsilon))^2 =$$
$$= \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)f(y)e^{i\varepsilon((x,u)+(y,w))} q_n(s,t,u,w)dudwdxdy\nu(ds)\nu(dt) =$$
$$= (2\pi)^d \int_T \int_T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q_n(s,t,u,w)\exp(-\frac{\varepsilon}{2}(\|u\|^2 + \|w\|^2))dudw\nu(ds)\nu(dt).$$

We note that the function f is even and $q_n(s,t,u,-w) = (-1)^n q_n(s,t,u,w)$. As a consequence, the expression is zero for odd n. For even n, the function q_n is non-negative, so the expression above is a monotonous function of ε. We can see now that the same considerations as in Theorem 1 are valid in this case (we choose the smallest even n such that $c_n > 0$, then all reasonings are the same with the substitution of I_0 with I_n).

$3 \Rightarrow 1$. Because members of the Itô–Wiener expansion with odd index are zero, the proof is the same as that of Theorem 1.

Such a kind of seminorms is interesting because of the following considerations. If we take $c_n = 0, n < 2n_0; c_n = (1+n)^\alpha, n \geqslant 2n_0$, where α - arbitrary real number, n_0 - arbitrary integer, then the convergence with respect to seminorm is equivalent to the convergence of renormalized local time approximations L_{ε,n_0} (first $2n_0$ members of the Itô–Wiener expansion of L_ε are subtracted) in $D_{2,\alpha}$. We can also apply Lemma 1 to this case.

Corollary 3. *Fix $\alpha \in \mathbb{R}$, $n_0 \in \mathbb{N}$, and $\{c_n\}$ as above. Then the following statements are equivalent:*

1. $L_{\varepsilon,n_0} \to L, \varepsilon \to 0+$ *in* $D_{2,\alpha}$,
2.
$$\int_T \int_T \int_{\mathbb{R}^d} \|G(s,t)x\|^{2n_0}$$
$$p_{-\alpha-d/2}(1 - \|G(s,t)x\|)\sigma(dx)\frac{\nu(ds)}{\sqrt{\det K(s,s)}}\frac{\nu(dt)}{\sqrt{\det K(t,t)}} < +\infty.$$

Proof. We let $\mu(dx,ds,dt) = \|G(s,t)x\|^{2n_0}\sigma(dx)\frac{\nu(ds)}{\sqrt{\det K(s,s)}}\frac{\nu(dt)}{\sqrt{\det K(t,t)}}$ to be the measure from Lemma 1, and let the function f be the same as that in Corollary 2. Now we apply Theorem 2 together with Lemma 1. The only thing we need to check (like in Corollary 2) is why we can add terms with odd numbers to condition (3) for the convergence in Theorem 2. But here, because of the form of c_n, we can use the same arguments as we did in Theorem 1.

It is interesting to observe here some immediate conclusions about this integral condition. Possible singularities which come from $(\det K(s,s)\det K(t,t))^{-1/2}$ can be refined with $\|G(s,t)x\|^{2n_0}$. We can expect all kinds of situations here because $G(s,t), s,t \in T$ and $K(s,s), s \in T$ are in some sense independent (i.e. we can define one independently of the other). Also we can not obtain the convergence for bigger α using renormalization (it is obvious from the definition of renormalization, so we only confirm that) because we can not refine the singularities of $p_{-\alpha-d/2}(1 - \|G(s,t)x\|)$ with $\|G(s,t)x\|^{2n_0}$.

4. Applications

In this section, we apply our results to some Gaussian fields. First of all, we have to mention when the construction of the needed probability space for our Gaussian field is possible. If T is a separable metric space and our Gaussian random field is stochastically continuous, then we have the sequence of values $\{\xi(t_n), n \in \mathbb{N}\}$ which generate a σ-field of the field. Let H be a subspace of $L_2(\Omega)$ generated by linear combinations of $\{\xi(t_n)\}$. It is well known (see [3],[6]) that we can construct (as a part of the abstract Wiener space) a Banach space B and a probability Gaussian measure μ on it such that H is the space of admissible shifts for μ (subspace of all linear functionals on B). In this setting, $(B, \mathfrak{B}(B), \mu)$ is a replacement for the probability space, and our field will be defined naturally on it as linear functionals on B. That is exactly the construction we need. In all cases, we suppose that the function f from the approximation is a standart Gaussian density and use corollaries 2,3. All our assumptions can be easily verified for processes below. So, in proofs, we need to verify the corresponding integral conditions only.

We start from the case of fractional Brownian motion. Suppose $\{X(t) \in \mathbb{R}^d, t \in [0,1]\}$ is a fractional Brownian motion with the Hurst parameter $H \in (0,1)$. Coordinates are independent and each has the covariation function $r_H(s,t) = \frac{s^{2H}+t^{2H}-|s-t|^{2H}}{2}$.

Example. Let $T = [0,1]; \nu(dt) = dt; \xi(t) = X(t)$. In this case, the local time does not exist in any Sobolev space if $d \geq \frac{1}{H}$. If $d < \frac{1}{H}$, then the limit exists in $D_{2,\alpha}$ for $\alpha < \frac{1}{2H} - \frac{d}{2}$ and doesn't exist in other cases. The value of n_0 does not affect the convergence of L_{ε,n_0} (renormalization does not work).

Example. Let $T = [1/2,1]; \nu(dt) = dt; \xi(t) = X(t)$. The local time exists if $\alpha < \frac{1}{2H} - \frac{d}{2}$ (and doesn't exist in other cases). Renormalization does not work.

Proof. We have

$$K(s,t) = r_H(s,t)I$$
$$G(s,t) = r_H(s,t)(r_H(s,s)r_H(t,t))^{-1/2}I = \frac{|s|^{2H} + |t|^{2H} - |t-s|^{2H}}{2t^H s^H} I,$$

so the integral with respect to x in the integral condition (2) from Corollary 3 can be dropped. This condition now has form

$$\int_0^1 \int_0^1 \left(\frac{s^{2H} + t^{2H} - |s-t|^{2H}}{2s^H t^H} \right)^{2n_0} \cdot \cdot \left(\frac{|s-t|^{2H} - (s^H - t^H)^2}{2s^H t^H} \right)^{-\alpha - d/2} (st)^{-dH} dt ds.$$

We use the polar coordinates: $s = r\sin\phi, t = r\cos\phi$. We can extend or reduce the domain of integration:

$$\{s^2+t^2 < 1, s>0, t>0\} \subset \subset \{s<1, t<1, s>0, t>0\} \subset \subset \{s^2+t^2<2, s>0, t>0\}.$$

For both cases, we get similar integrals and the same conditions for their finiteness. We consider only the case of a reduced domain (the second one can be treated in the same way):

$$\int_0^1 \int_0^{\pi/2} \left(\frac{(\cos\phi)^{2H} + (\sin\phi)^{2H} - |\cos\phi - \sin\phi|^{2H}}{2(\cos\phi\sin\phi)^H} \right)^{2n_0} \cdot \cdot \left(\frac{|\cos\phi - \sin\phi|^{2H} - (\cos^H\phi - \sin^H\phi)^2}{2\cos^H\phi\sin^H\phi} \right)^{-\alpha-d/2} \cdot \cdot (\cos\phi\sin\phi)^{-dH} d\phi r^{1-2dH} dr < +\infty$$

Our integral becomes a product of two integrals. The integral with respect to r is finite, if (and only if) $dH < 1$. The integral with respect to ϕ can be infinite only near $\phi = 0$, $\phi = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$, because the integrand is continuous and finite away from these points. Note that the expression taken to power n_0 is bounded. Since

$$\frac{|cos\phi - sin\phi|^{2H} - (\cos^H \phi - sin^H \phi)^2}{2cos^H \phi sin^H \phi} =$$
$$= 1 + \frac{|\sin\phi|^H}{2|\cos\phi|^H} + \frac{|cos\phi - sin\phi|^{2H} - |\cos\phi|^{2H}}{2cos^H \phi sin^H \phi} \sim$$
$$\sim 1 - \frac{2H|\cos\phi|^{2H-1}\sin\phi}{2cos^H \phi sin^H \phi} \to 1$$

as $\phi \to 0+$, then integral with respect to ϕ is finite in a neighbourhood of $\phi = 0$ if $dH < 1$ (because only one important multiplier under the integral is $(\sin\phi)^{-dH}$). The behavior of the integral in a neighbourhood of $\phi = \frac{\pi}{2}$ is similar. Now we investigate the behaviour in a neighbourhood of $\phi = \frac{\pi}{4}$:

$$|\cos\phi - \sin\phi|^{2H} - (\cos^H \phi - \sin^H \phi)^2 \sim 2^H|\phi - \frac{\pi}{4}|^{2H} - H2^{2H-1}(\phi - \frac{\pi}{4})^2$$
$$\sim 2^H|\phi - \frac{\pi}{4}|^{2H}, \phi \to \frac{\pi}{4}.$$

From this relation, we get the sufficient condition for the integral to be finite $2H(-\alpha - \frac{d}{2}) > -1$. It is also necessary, because the expression taken to power n_0 can not refine this singularity, as we mentioned earlier.

Now let $T = [1/2, 1]$. In this case, the points $s = 0$ and $t = 0$ lie away from the integration domain. So, if we make same calculations as above, we get that $\phi = 0$ and $\phi = \frac{\pi}{2}$ are away from the integration domain. The only one condition for the finiteness here is $\alpha < \frac{1}{2H} - \frac{d}{2}$.

Now we turn to the interesting case of the self-intersection for a fractional Brownian motion. Here, we generalize the results from [5,8,11,12].

Example. Let $T = [0, 1]^2; \nu(dt) = dt_1dt_2; \xi(t) = X(t_1) - X(t_2)$. The renormalized local time exists as an element of the Sobolev space $D_{2,\alpha}$ if and only if

$$\alpha < \frac{1}{H} - \frac{d}{2}, d < \frac{3}{2H}, n_0 > \frac{dH - 1}{2(1 - H)}.$$

Example. Let $T = [0, 1/3] \times [2/3, 1]; \nu(dt) = dt_1dt_2; \xi(t) = X(t_1) - X(t_2)$. The local time exists in $D_{2,\alpha}$ for $\alpha < \frac{1}{H} - \frac{d}{2}$. Renormalization does not work.

Proof. The covariation in this case is given by $K(s, t) = \frac{1}{2}(|s_1 - t_2|^{2H} + |s_2 - t_1|^{2H} - |s_1 - t_1|^{2H} - |s_2 - t_2|^{2H})I$, and the correlation equals $G(s, t) = g(s, t)I$, where

$$g(s, t) = \frac{|s_1 - t_2|^{2H} + |s_2 - t_1|^{2H} - |s_1 - t_1|^{2H} - |s_2 - t_2|^{2H}}{2|s_1 - s_2|^H|t_1 - t_2|^H}.$$

The integral from Corollary 3 has the form

$$\int_0^1\int_0^1\int_0^1\int_0^1 |g(s, t)|^{2n_0}|1 - |g(s, t)||^{-\alpha-\frac{d}{2}}|s_1 - s_2|^{-dH}|t_1 - t_2|^{-dH}dt_1dt_2ds_1ds_2.$$

By introducing the new variables $x = s_1 - t_1, y = s_2 - s_1, z = t_1 - t_2$ and extending the integration domain, we simplify this integral to

$$\int_{-1}^1\int_{-1}^1\int_{-1}^1 \left|\frac{D_{2H}(x, y, z)}{2|yz|^H}\right|^{2n_0}\left|1 - \frac{|D_{2H}(x, y, z)|}{2|yz|^H}\right|^{-\alpha-\frac{d}{2}}|y|^{-dH}|z|^{-dH}dxdydz,$$

where $D_\gamma(x,y,z) = |x+y|^\gamma + |x+z|^\gamma - |x|^\gamma - |x+y+z|^\gamma$. This new integral is finite if and only if the starting integral is finite (because we can obtain a multiple of this integral by reducing the integration domain instead of extending it). Now we make the change of variables $x = ru, y = r\cos\phi, z = r\sin\phi$ and reduce the integration domain by an additional constraint $y^2 + z^2 < 1$ (we get almost the same integral by extending this domain, so this procedure is also two-way):

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\left|\frac{D_{2H}(u,\cos\phi,\sin\phi)}{2|\cos\phi\sin\phi|^H}\right|^{2n_0}\left|1-\frac{D_{2H}(u,\cos\phi,\sin\phi)}{2|\cos\phi\sin\phi|^H}\right|^{-\alpha-\frac{d}{2}}\cdot$$
$$\cdot|\cos\phi|^{-dH}|\sin\phi|^{-dH}\int_0^{\min(1/|u|,1)} r^{2-2dH}drd\phi du.$$

Integrating with respect to r, we obtain the necessary condition for finiteness, $dH < \frac{3}{2}$. To proceed, we need some inequalities describing the behavior of $D_{2H}(x,y,z)$.

Lemma 2.

1. *For all $\gamma \in (0,1)\cup(1,2]$, there exist the positive constants C_1 and C_2 such that if $3|y| < |x|, 3|z| < |x|$, then*
$$C_1|yz||x|^{\gamma-2} \le |D_\gamma(x,y,z)| \le C_2|yz||x|^{\gamma-2}.$$
If $\gamma = 1$, then, for the same x,y,z, we have $|D_\gamma(x,y,z)| = 0$, and so we can take $C_1 = C_2 = 0$.
2. *If $\gamma \in (1,2)$, then*
$$|D_\gamma(x,y,z)| \le 2^{1-\gamma}(||y|+|z||^\gamma - ||y|-|z||^\gamma).$$
Additionally, if $y,z>0$, then the inequality changes into an equality if and only if $x = -(y+z)/2$.
3. *If $\gamma \in (0,1)$, then*
$$|D_\gamma(x,y,z)| \le |y|^\gamma + |z|^\gamma - ||y|-|z||^\gamma.$$
Additionally, if $y,z>0$, then the inequality changes into an equality if and only if $x = -y,-z$.
4. *If $\gamma = 1$, then*
$$|D_\gamma(x,y,z)| \le |y|+|z|-||y|-|z||.$$
Additionally, if $y,z>0$, then the inequality changes into an equality if and only if $x \in [-\max(y,z), -\min(y,z)]$.

Proof. The first inequality is equivalent to

$$C_1|uv| \le |D_\gamma(1,u,v)| \le C_2|uv|; |u| < \frac{1}{3}, |v| < \frac{1}{3}.$$

Because u,v is small enough by our assumption, it is possible to use the representation

$$D_\gamma(x,y,z) = -\gamma(\gamma-1)\int_0^{|y|}\int_0^{|z|}(x+sign(y)v+sign(z)w)^{\gamma-2}dvdw$$

which is valid for all $x > \max(0,-y,-z,-y-z)$ (here we denote $sign(y) = 1$ if $y \ge 0$ and $sign(y) = -1$ if $y < 0$). We have

$$|D_\gamma(1,u,v)| = \left|\gamma(\gamma-1)\int_0^{|u|}\int_0^{|v|}(1+sign(u)y+sign(v)w)^{\gamma-2}dydw\right| \le$$
$$\le |\gamma(\gamma-1)uv|\sup_{|y|<1/3;|w|<1/3}(1+y+w)^{\gamma-2} \le 3^{2-\gamma}|\gamma(\gamma-1)||uv|$$

and similarly

$$|D_\gamma(1,u,v)| \geq (\frac{5}{3})^{2-\gamma}\,|\gamma(\gamma-1)|\,|uv|.$$

Note that, for $\gamma = 1$, we have $|D_\gamma(1,u,v)| = 0$ under the same assumptions on u, v.

To prove other inequalities, it is enough to consider the case $0 < y \leq z$. Indeed, if we have both y and z negative, then we can introduce new variables $\tilde{x} = -x, \tilde{y} = -y, \tilde{z} = -z$ and obtain the same inequality for positive variables. If, for example, $y < 0$ and $z > 0$, then another change of variables helps: $\tilde{x} = x + y, \tilde{y} = -y, \tilde{z} = z$. All other cases are similar because of the symmetry with respect to the exchange $y \leftrightarrow z$. Moreover, if $y = 0$, then all inequalities are trivial equalities, so we may suppose that $y \neq 0$. Now we can prove these inequalities by finding the maximum and minimum of $D_\gamma(x,y,z)$ for fixed y and z. Using the convexity of a power function, we can find the sign of the derivative of $g(x) = D_\gamma(x,y,z)$ with respect to x. For $\gamma \in (1,2)$, we have

$$\begin{aligned}
&g'(x) < 0, x \in (-\infty, -(y+z)/2),\\
&g'(x) = 0, x = -(y+z)/2,\\
&g'(x) > 0, x \in (-(y+z)/2, +\infty).
\end{aligned}$$

For $\gamma \in (0,1)$ (note that, in this case, the derivative may not exist at some points):

$$\begin{aligned}
&g'(x) > 0, x \in (-\infty, -y-z),\\
&g'(x) < 0, x \in (-y-z, -z),\\
&g'(x) > 0, x \in (-z, -(y+z)/2),\\
&g'(x) = 0, x = -(y+z)/2,\\
&g'(x) < 0, x \in (-(y+z)/2, -y),\\
&g'(x) > 0, x \in (-y, 0),\\
&g'(x) < 0, x \in (0, +\infty).
\end{aligned}$$

For $\gamma = 1$ (here the derivative also does not exist at some points):

$$\begin{aligned}
&g'(x) = 0, x < -y-z,\\
&g'(x) < 0, x \in (-y-z, -z),\\
&g'(x) = 0, x \in (-z, -y),\\
&g'(x) > 0, x \in (-y, 0),\\
&g'(x) = 0, x > 0.
\end{aligned}$$

We also know that $D_\gamma(x,y,z)$ is continuous and that $D_H(x,y,z) \to 0, |x| \to \infty$ (from the first inequality). As we can see for $\gamma \in [1,2)$, the function has its minimum in $x = \frac{-y-z}{2}$ and always negative. The right side of the inequality is exactly $|D_\gamma(\frac{-y-z}{2}, y, z)|$, so it is proved. For $\gamma \in (0,1]$ by same method, we find the maximum at $x = 0; x = -y-z$ and the minimum at $x = -y; x = -z$ (values at both points of each pair are the same). Comparing the moduli of the function values at these points, we get $|D_\gamma(x,y,z)| \leq |D_\gamma(-y,y,z)|$. If $y, z > 0$, then the maximum is achieved only at the points $x = \frac{-y-z}{2}$ for $\gamma \in (1,2)$ and at $x = -y; x = -z$ for $\gamma \in (0,1)$. For $\gamma = 1$, the maximum is achieved on the interval $x \in [-z, -y]$. Lemma 2 is proved.

Now let us look at the integral

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\left|\frac{D_{2H}(u,\cos\phi,\sin\phi)}{2|\cos\phi\sin\phi|^H}\right|^{2n_0}\left|1-\frac{D_{2H}(u,\cos\phi,\sin\phi)}{2|\cos\phi\sin\phi|^H}\right|^{-\alpha-\frac{d}{2}}\cdot$$
$$\cdot\,|\cos\phi|^{-dH}|\sin\phi|^{-dH}(\max(1,|u|))^{2dH-3}d\phi du$$

once more. We split the domain of integration into several parts by restricting ourself to the case $\phi \in (0, \frac{\pi}{4})$ (other parts of the domain can be treated similarly):

$$\begin{aligned}
&D_1 = \{|u| > M\},\\
&D_2 = \{|u| < M; \phi \geqslant \varepsilon, \frac{\pi}{4} - \phi \geqslant \varepsilon\},\\
&D_3 = \{|u| < M, |u| \geqslant \varepsilon, |1+u| \geqslant \varepsilon; \phi < \varepsilon\},\\
&D_4 = \{|u|^2 + |\sin\phi|^2 < 4\varepsilon^2\},\\
&D_5 = \{|1+u|^2 + |\sin\phi|^2 < 4\varepsilon^2\},\\
&D_6 = \{|u| < M, |u + \frac{1}{\sqrt{2}}| \geqslant \varepsilon; \frac{\pi}{4} - \phi < \varepsilon\},\\
&D_7 = \{|\frac{u}{\cos\phi} + 1|^2 + |\,tg\,\phi - 1|^2 < 16\varepsilon^2\}.
\end{aligned}$$

Obviously, the union of these parts covers a selected part of the domain if we choose $\varepsilon < \frac{1}{16}$ and M such that, for $|u| > M$, we have $C_2|u|^{2H-2} < \frac{1}{2}$, where C_2 is the constant from the first inequality above for $H \neq \frac{1}{2}$. We use this inequality for D_1 and immediately obtain that the finiteness of our integral is equivalent to the finiteness of two integrals $\int_M^\infty u^{2n_0(2H-2)+2dH-3}du$ and $\int_0^{\pi/4} |sin\phi|^{2n_0(1-H)-dH} d\phi$. Both integrals are finite if and only if $n_0 > \frac{dH-1}{2(1-H)}$. If $H = \frac{1}{2}$, we can choose M such that $D_{2H}(u, cos\phi, sin\phi) = 0$ on D_1. In this case, it is easy to see that the condition $dH < 1$ or $n_0 > 0$ (it is true if the declared conditions for the local time existence hold) is sufficient for the integral over D_1 to be finite.

For D_2, D_3, D_4, D_5, D_6, we have that $1 - \frac{D_{2H}(u,cos\phi,sin\phi)}{2|cos\phi sin\phi|^H}$ is not equal to zero on the closure of these sets. Indeed, using the inequalities from Lemma 2, we get for $H \in (0, \frac{1}{2})$:

$$\begin{aligned}
\frac{|D_{2H}(u,\cos\phi,\sin\phi)|}{2|\cos\phi\sin\phi|^H} &\leqslant \frac{|\cos\phi|^{2H} + |\sin\phi|^{2H} - |\cos\phi - \sin\phi|^{2H}}{2|\cos\phi\sin\phi|^H} =\\
&= 1 + \frac{(|\cos\phi|^H - |\sin\phi|^H)^2 - |\cos\phi - \sin\phi|^{2H}}{2|\cos\phi\sin\phi|^H} \leqslant\\
&= 1 - (|\sin\phi|^H + |\cos\phi - \sin\phi|^H - |\cos\phi|^H)\cdot\\
&\cdot\frac{(|\cos\phi|^H - |\sin\phi|^H + |\cos\phi - \sin\phi|^H))}{2|\cos\phi\sin\phi|^H} =\\
&= 1 - (1 + |\,ctg\,\phi - 1|^H - |\,ctg\,\phi|^H)\cdot\\
&\cdot(1 - |\,tg\,\phi|^H + |1 - tg\,\phi|^H) \leqslant\\
&\leqslant 1 - (1 + |\,ctg(\pi/4) - 1|^H - |\,ctg(\pi/4)|^H)\cdot\\
&\cdot(1 - |\,tg(\pi/4)|^H + |1 - tg(\pi/4)|^H) = 1.
\end{aligned}$$

For $H \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
\frac{|D_{2H}(u,\cos\phi,\sin\phi)|}{2|\cos\phi\sin\phi|^H} &\leqslant 2^{-2H}\frac{|\cos\phi + \sin\phi|^{2H} - |\cos\phi - \sin\phi|^{2H}}{|\cos\phi\sin\phi|^H} =\\
&= 2^{-2H}(|tg^2\phi + ctg^2\phi + 2|^H - |tg^2\phi + ctg^2\phi - 2|^H) \leqslant\\
&\leqslant 2^{-2H}(|tg^2(\pi/4) + ctg^2(\pi/4) + 2|^H - \left|tg^2(\pi/4) + ctg^2(\pi/4) - 2\right|^H) = 1.
\end{aligned}$$

For $H = \frac{1}{2}$, both inequalities are true. As we can see, the equality in these inequalities is possible only if $\phi = \pi/4, u = -\frac{1}{\sqrt{2}}$ (from Lemma 2 and the inequalities above). But the function we consider is continuous on D_2, D_3, D_4, D_5, D_6 and therefore is not equal to zero. So, this expression taken to the power $-\alpha - d/2$ can be omitted in the integral.

For D_2, the integrand is bounded, so the integral is always finite. For D_3, D_4, D_5, we have to deal only with the singularity of $|sin\phi|^{-dH}$ which can be refined by the renormalization term.

For D_3, we have the bound $|D_{2H}(u, cos\phi, sin\phi)| \leq C|\phi|$, where C is a constant depending only on ε, and the condition $n_0 > \frac{dH-1}{2(1-H)}$ is sufficient for the integral to be finite. Note that, for D_4, D_5, this inequality is not true (for $2H < 1$), because the derivative of $D_{2H}(u, cos\phi, sin\phi)$ with respect to ϕ at $\phi = 0$ blows up if $u = 0$ or $u = -1$. Instead, we have to use a change of coordinates. For D_3, we can also prove that, for $H = 1/2$, the condition $n_0 > \frac{dH-1}{2(1-H)}$ is necessary. It is enough to note that $|D_{2H}(u, \cos\phi, \sin\phi)| = 2\sin\phi$ on $0 \leqslant \phi < \varepsilon; u \in (-\cos\phi, -\sin\phi)$.

For D_4, let $sin\phi = rsin\theta, u = rcos\theta$. We obtain the following integral:

$$\int_0^{2\varepsilon}\int_0^{2\pi} ||rcos\theta + rsin\theta|^{2H} + |rcos\theta + \sqrt{1-(rsin\theta)^2}|^{2H} - |rcos\theta|^{2H} - \\ - |rcos\theta + rsin\theta + \sqrt{1-(rsin\theta)^2}|^{2H}|^{2n_0}|sin\theta|^{-dH-2n_0H} r^{1-dH-2n_0H} d\theta dr.$$

Here, we already dropped multipliers under the integral bounded above and below. The expression taken to $2n_0$ power can be bounded (using derivatives) by the expression $Cr^{min(2H,1)}|sin\theta|$. Thus, our integral is bounded by the product of two integrals $\int_0^{2\varepsilon} r^{2n_0 min(2H,1)+1-dH-2n_0H}dr$ and $\int_0^{2\pi}|sin\theta|^{2n_0-dH-2n_0H}d\theta$. The conditions for their finiteness are $2n_0(min(2H,1) - H) + 2 - dH > 0$ and $2n_0(1-H) - dH + 1 > 0$. Recall that we have already the conditions $dH < \frac{3}{2}$ and $n_0 > \frac{dH-1}{2(1-H)}$ as necessary and that $n_0 \geq 0$. We can see that these conditions are sufficient in this case. The case of D_5 is almost similar. We have to let $sin\phi = rsin\theta, u + 1 = rcos\theta$ and obtain a similar bound for the integrand.

For D_6, the integrand is bounded as we have proved above. For D_7, the only unbounded part under the sign of integral is $\left|1 - \frac{D_{2H}(u,cos\phi,sin\phi)}{2|cos\phi sin\phi|^H}\right|^{-\alpha-\frac{d}{2}}$ and

$$2|cos\phi sin\phi|^H - D_{2H}(u, cos\phi, sin\phi) = \\ = |cos\phi|^{2H}(2|1+w|^H - |1+v+w|^{2H} - |v-1|^{2H} + |v|^{2H} + |v+w|^{2H}),$$

where $w = tg\phi - 1, v = \frac{u}{cos\phi}$. We introduce w and v as new variables of integration and note that the expression above is equivalent to $|v|^{2H} + |v+w|^{2H}$ (multiplied by a constant) when $v^2 + w^2 \to 0+$. Using the polar coordinates, we obtain two integrals $\int_0^{4\varepsilon} r^{1-2H(\alpha+\frac{d}{2})}dr$ and $\int_0^{2\pi}(|cos\theta|^{2H} + |cos\theta + sin\theta|^{2H})^{-\alpha-\frac{d}{2}}d\theta$. The second integral is always finite and first gives us the necessary and sufficient condition $\alpha + \frac{d}{2} < \frac{1}{H}$.

If we set $T = [0, 1/3] \times [2/3, 1]$, then we can get the same integral but on a different domain. This domain can be treated exactly like the union of D_2, D_6, D_7 (for example, the singularities near $s_1 = s_2$ and $t_1 = t_2$ and, consequently, near $\sin\phi = 0$ and $\cos\phi = 0$ are outside the integration domain), and we have only the condition on α as necessary and sufficient for the integral to be finite.

In all previous examples, we had the covariation matrix proportional to the identity one. In the next one, the situation is different. Suppose that $\{W_1(t), W_2(t) \in \mathbb{R}^d; t \in [0,1]\}$ are two Brownian motions which are not necessarily independent. Suppose that $EW_1(s)W_2(t) = min(s,t)Q$, where Q is some $d \times d$ matrix. We consider the intersection local time of these two processes. By $\{\lambda_i, i = 1, \dots, d\}$, we denote eigenvalues of the self-adjoint matrix $\frac{Q+Q^T}{2}$. Here, we study only the condition from Corollary 1.

Example. Let $T = [0,1]^2; \nu(dt) = dt_1dt_2; \xi(t) = W_1(t_1) - W_2(t_2)$. The local time exists in $D_{2,\alpha}$ for all $\alpha < -\frac{d}{2}$ if and only if the number of eigenvalues of the matrix $\frac{Q+Q^T}{2}$ which are equal to 1 is smaller than 2 ($\#\{i : \gamma_i = 1\} < 2$) and $d < 4$.

Example. Let $T = [\frac{1}{2}, 1]^2; \nu(dt) = dt_1 dt_2; \xi(t) = W_1(t_1) - W_2(t_2)$. The local time exists in $D_{2,\alpha}$ for $\alpha < -\frac{d}{2}$ if and only if $\#\{i : \gamma_i = 1\} < 2$.

Example. Let $T = [\frac{1}{4}, \frac{1}{2}] \times [\frac{3}{4}, 1]; \nu(dt) = dt_1 dt_2; \xi(t) = W_1(t_1) - W_2(t_2)$. The local time always exists in $D_{2,\alpha}$ for $\alpha < -\frac{d}{2}$.

Proof. The integral we have to study has the form

$$\int_0^1 \int_0^1 \prod_{i=1}^d (t_1 + t_2 - 2min(t_1, t_2)\lambda_i)^{-1/2} dt_1 dt_2.$$

Here, we used that

$$\det K(t,t) = \det((t_1 + t_2)I - min(t_1, t_2)(Q + Q^T)) = \prod_{i=1}^d (t_1 + t_2 - 2min(t_1, t_2)\lambda_i).$$

Note that $t_1 + t_2 - 2min(t_1, t_2)\lambda_i = (t_1 + t_2)(1 - \lambda_i) + |t_1 - t_2|\lambda_i$. If $\lambda_i \neq 1$ and consequently $\lambda_i < 1$, we have only one singularity near $t_1 = t_2 = 0$. Using the polar coordinates, we conclude that the necessary condition for the integral to be finite is $2 - \frac{d}{2} > 0$. If, for some i, we have $\lambda_i = 1$, we have to study the singularity near $t_1 = t_2$. Using the polar coordinates, we conclude that multipliers with $\lambda_i \neq 1$ can be omitted in the integral with respect to ϕ, and we get the necessary condition $1 - \frac{k}{2} > 0$, where $k = \#\{i : \gamma_i = 1\}$. Two necessary conditions we obtained are also sufficient, because we do not have other singularities in the integral. For $T = [\frac{1}{2}, 1]^2$ we do not have singularity near $t_1 = t_2 = 0$ and for $T = [\frac{1}{4}, \frac{1}{2}] \times [\frac{3}{4}, 1]$ we also do not have singularity near $t_1 = t_2$ with obvious consequences.

Bibliography

1. A.A. Dorogovtsev, V.V. Bakun, *Random mappings and a generalized additive functional of a Wiener process*, Theory of Stoch. Proc. **48** (2003), no. 1.
2. A. Rudenko, *Existence of generalized local times for Gaussian random fields*, Theory of Stoch. Proc. **12(28)** (2006), no. 1-2, 142–154.
3. H.H. Kuo, *Gaussian Measures in Banach Spaces*, Springer, Berlin, 1975.
4. H. Watanabe, *The local time of self-intersections of brownian motions as generalized brownian functionals*, Lett. in Math. Phys. **23** (1991), 1–9.
5. P. Imkeller, V. Perez-Abreu, J. Vives, *Chaos expansions of double intersection local time of brownian motion in $\mathbb{R}^d$ and renormalization*, Stoch. Proc. and Appl. **56** (1995), 1–34.
6. P. Malliavin, *Stochastic Analysis*, Springer, Berlin, 1997.
7. J. Rosen, *A renormalized local time for multiple intersections of planar brownian motion*, Seminaire de Probabilities XX **20** (1986), 515–531.
8. S. Albeverio, Y. Hu, X.Y. Zhou, *A remark on non-smoothness of the self-intersection local time of planar brownian motion*, Stat. and Prob. Lett. **32** (1997), 57–65.
9. S. Orey, *Gaussian sample functions and the Hausdorff dimension of level crossings*, Wahrscheinlichkeitstheorie verw. Geb. **15** (1970), 249–256.
10. S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Springer, Berlin, 1984.
11. Y. Hu, D. Nualart, *Renormalized self-intersection local time for fractional brownian motion*, Annals of Prob. **33** (2005), no. 3, 948–983.
12. Y. Hu, D. Nualart, *Regularity of renormalized self-intersection local time for fractional Brownian motion*, J. of Commun. in Information and Systems (CIS) **7** (2007), no. 1, 21–30.

Institute of Mathematics, Kyiv, Ukraine
E-mail: arooden@yandex.ru