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ON THE REPRESENTATION OF SOLUTIONS OF ANTICIPATING LINEAR PARTIAL STOCHASTIC DIFFERENTIAL EQUATIONS

The unique solution of an anticipating linear partial stochastic differential equation is constructed by means of the Fourier transformation.

Denote, by K and L, linear partial differential operators with constant coefficients of the first and the second order, respectively:

$$K \equiv \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}; \quad L \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Operator L is assumed to be elliptic:

$$\sum_{i,j=1}^{n} a_{ij} \lambda_i \lambda_j \geq c|\lambda|^2, \ |\lambda|^2 = \sum_{i=1}^{n} \lambda_i^2, \ c > 0.$$

Consider the stochastic differential equation

$$du(t,x) = Lu(t,x)dt + Ku(t,x)dw(t), \tag{1}$$

where $w(t), \ 0 \leq t \leq 1$, is the standard Wiener process, $x \in \mathbb{R}^n$.

We will seek for a solution to Eq. (1) that satisfies the initial condition

$$u(0,x) = \alpha\varphi(x), \tag{2}$$

where $\alpha = f(w), \ w = w(\cdot)$, is a functional depending on a path of the Wiener process and $\varphi(x)$ is some real function, $x \in \mathbb{R}^n$.

By the solution of problem (1),(2), we mean a solution of the integral equation

$$u(t,x) = \alpha\varphi(x) + \int_0^t Lu(s,x)ds + \int_0^t Ku(s,x)dw(s), \tag{3}$$

where the stochastic integral is defined in the extended sense given by A.V. Skorokhod in [6].

Definition. A function of two variables $u(t,x), \ 0 \leq t \leq 1, \ x \in \mathbb{R}^n$ is said to be a solution of Eq. (3) if the following conditions are fulfilled:

1) $u(t,x)$ is continuous in t and has two continuous derivatives with respect to x for $0 \leq t \leq 1, \ x \in \mathbb{R}^n$;

2) $Ku(t,x)$ and $Lu(t,x)$ are continuous in t for $0 \leq t \leq 1, \ x \in \mathbb{R}^n$;

3) $Ku(s,x)$ belongs to the domain of definition of the extended stochastic integral in s for $0 \leq t \leq 1, \ x \in \mathbb{R}^n$;

4) relation (3) holds true for all $0 \leq t \leq 1, \ x \in \mathbb{R}^n$ with probability 1.

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The purpose of this paper is to formulate the condition for the existence of the solution of problem (1),(2). In order to solve the problem, it is possible to apply the approach proposed by A.A. Dorogovtsev in [1]. The solution of the problem (1), (2) can be found by means of the Fourier transformation (see [4]).

Denote by $\widehat{\varphi}(\lambda)$ the Fourier transform of the function $\varphi(x),\ x \in \mathbb{R}^n$, and $\check{\psi}(x)$ is the inverse Fourier transform:

$$\widehat{\varphi}(\lambda) = \int_{\mathbb{R}^n} e^{-i\langle \lambda, x\rangle}\varphi(x)dx,\ \ \check{\psi}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle}\psi(\lambda)d\lambda.$$

Here, $\langle x, \lambda\rangle = \sum_{i=1}^n x_i\lambda_i$ and $|\,x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ are the scalar product and norm of vectors from $\mathbb{R}^n$, $i^2 = -1$.

Consider the space of rapidly decreasing functions, on which the Fourier transformation is well defined:

$$\mathcal{G} = \{\varphi(\cdot):\ \varphi(\cdot) \in C^\infty(\mathbb{R}^n),\ \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \varphi(x)| < +\infty,\ \alpha, \beta \in \mathbb{N}\},$$

where $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n},\ \partial_j^{\alpha_j} = \partial^{\alpha_j}/\partial x_j^{\alpha_j},\ \alpha_j \in \mathbb{Z}_+,\ j = \overline{1,n}$.

Let $h(\lambda)$ and $g(\lambda)$ be the symbols of the operators L and K, respectively:

$$h(\lambda) = -\sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j;\ g(\lambda) = ip(\lambda),\ p(\lambda) = \sum_{j=1}^n b_j\lambda_j.$$

Apply the Fourier transformation to Eq. (1) and the initial condition (2) in a formal way assuming that all functions belong to $\mathcal{G}$ with respect to $x \in \mathbb{R}^n$. We get the following Cauchy problem:

$$dy(t,\lambda) = h(\lambda)y(t,\lambda)dt + ip(\lambda)y(t,\lambda)dw(t); \tag{4}$$

$$y(0,\lambda) = \alpha\widehat{\varphi}(\lambda). \tag{5}$$

Find the conditions, under which the solution of problem (1),(2) can be obtained as a result of the inverse Fourier transformation of problem (4),(5).

Theorem 1. *Let $\widehat{\varphi}(\lambda),\ y(t,\lambda) \in \mathcal{G}$ with respect to λ for each $t:\ 0 \le t \le 1$. If $y(t,\lambda)$ satisfies the equation*

$$y(t,\lambda) = \alpha\widehat{\varphi}(\lambda) + \int_0^t h(\lambda)y(s,\lambda)ds + i\int_0^t p(\lambda)y(s,\lambda)dw(s),$$

then the solution of problem (1),(2) can be found as a result of the inverse Fourier transformation of the function $y(t,\lambda):\ u(t,x) = \check{y}(t,x)$.

Proof. If $y(t,\cdot)$ belongs to $\mathcal{G}$, so do $h(\lambda)y(t,\lambda)$ and $p(\lambda)y(t,\lambda)$ with respect to λ. Set $u(t,x) = \check{y}(t,x)$. Since the Fourier transformation realizes an isomorphism $\mathcal{G}$ onto itself, $Lu(t,x)$ and $Ku(t,x)$, as well as $u(t,x)$, belong to $\mathcal{G}$ with respect to $x \in \mathbb{R}^n$. In addition,

$$\widehat{Lu}(t,\lambda) = h(\lambda)y(t,\lambda);\ \widehat{Ku}(t,\lambda) = ip(\lambda)y(t,\lambda).$$

To complete the proof, it still remains to prove the commutability between the inverse Fourier transformation and integration operations. Since the function $h(\lambda)y(s,\lambda)$ is integrable in the variable s, has a continuous modification, and belongs to $\mathcal{G}$ in λ, the inverse Fourier transformation commutes with the ordinary integration according to the Fubini theorem. Really,

$$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle}\int_0^t h(\lambda)y(s,\lambda)dsd\lambda = \int_{\mathbb{R}^n}\int_0^t (2\pi)^{-n}e^{i\langle x,\lambda\rangle}h(\lambda)y(s,\lambda)dsd\lambda =$$

$$= \int_0^t (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} h(\lambda) y(s,\lambda) d\lambda ds.$$

For the extended stochastic integral it also suffices to apply the Fubini theorem taking into account the definition of extended stochastic integral. Denote $V_t(s,\lambda) = 1\!\!\mathrm{I}_{[0,t]} p(\lambda) y(s,\lambda)$. For each $F \in \mathbb{D}^{1,2}$, we have

$$E\Big(F \int_0^1 (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} V_t(s,\lambda) d\lambda dw(s)\Big) =$$

$$= E\Big(\int_0^1 DF(s)(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} V_t(s,\lambda) d\lambda ds\Big) =$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} \Big(E \int_0^1 DF(s) V_t(s,\lambda) ds\Big) d\lambda =$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} E\Big(F \int_0^1 V_t(s,\lambda) dw(s)\Big) d\lambda =$$

$$= E\Big(F(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} \int_0^1 V_t(s,\lambda) dw(s)\Big) d\lambda.$$

Since F is an arbitrary element from $\mathbb{D}^{1,2}$,

$$\int_0^1 (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} V_t(s,\lambda) d\lambda dw(s) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\lambda\rangle} \int_0^1 V_t(s,\lambda) dw(s) d\lambda$$

with probability 1.

Consider now the question how to find the solution of problem (4),(5). It should be noted that problem (4),(5) can be solved under fixed λ. Set $a = h(\lambda)$, $b = p(\lambda)$, $z_0 = \alpha\widehat{\varphi}(\lambda)$. We have

$$\begin{cases} dz(t) = az(t)dt + ibz(t)dw(t); \\ z(0) = z_0. \end{cases} \tag{6}$$

In order to get the solution of problem (6), we write the system with respect to the real and imaginary parts of $z(t)$. Set $z(t) = z_1(t) + iz_2(t),\ z_k(t) \in \mathbb{R},\ k = \overline{1,2}$;

$$\vec{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix},\ \vec{z}_0 = \begin{pmatrix} \operatorname{Re} z_0 \\ \operatorname{Im} z_0 \end{pmatrix}.$$

We come to

$$\begin{cases} d\vec{z}(t) = aI\vec{z}(t)dt + B\vec{z}(t)dw(t); \\ \vec{z}(0) = \vec{z}_0. \end{cases} \tag{7}$$

In the case where α has a finite Itô–Wiener expansion, that is,

$$\alpha = \sum_{k=0}^{N} \int_{[0,1]^k} a_k(t_1,\dots,t_k) dw(t_1)\dots dw(t_k),\ N < +\infty,$$

the solution of problem (7) was obtained in [1]. To write this solution, it needs to introduce the following auxiliary system:

$$\begin{cases} dy_s^t = aIy_s^t dt + By_s^t dw(t); \\ y_s^s = I,\ s \le t. \end{cases} \tag{8}$$

The solution of problem (7) can be written in the form (see [1])

$$\vec{z}(t) = y_0^t \Big[\vec{z}_0 + \sum_{k=1}^{N} (-1)^k B^k \int_{\Delta_k(t)} D^k \vec{z}_0(t_1,\dots,t_k) dt_1 \dots dt_k\Big], \tag{9}$$

where $D^k \vec{z}_0(t_1, \dots, t_k),\ k \in \mathbb{N}$, are stochastic derivatives of $\vec{z}_0$; y_0^t is the solution of (8) when $s = 0$; $\Delta_k(t) = \{0 \le t_1 \le \dots \le t_k \le t\}$. Returning to the complex variable $z(t)$, we get

$$z(t) = e^{\left(a+\frac{1}{2}b^2\right)t+ibw(t)}\widehat{\varphi}(\lambda)\left[\alpha + \sum_{k=1}^{N}(-i)^k b^k \int_{\Delta_k(t)} D^k\alpha(t_1 \dots t_k)dt_1 \dots dt_k\right].$$

Since $a = h(\lambda),\ b = p(\lambda)$, the solution of (4), (5) becomes

$$\begin{aligned} y(t,\lambda) = e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t+ip(\lambda)w(t)}\widehat{\varphi}(\lambda)\Bigg[\alpha+ \\ +\sum_{k=1}^{N}(-i)^k p^k(\lambda)\int_{\Delta_k(t)} D^k\alpha(t_1 \dots t_k)dt_1 \dots dt_k\Bigg]. \end{aligned} \tag{10}$$

In order to extend the class of the initial conditions α, for which the solution of problem (4),(5) can be represented in the form (10), we use results from [3]. Suppose that the map $\alpha:\ C_0[0,1] \to \mathbb{R}$ is analytic, i.e. it can be expanded in the Taylor series for $w = 0:$

$$\alpha = \sum_{n=0}^{\infty} \frac{1}{n!}\alpha^{(n)}(0)w^n,\ \ w \in C_0[0,1], \tag{11}$$

where

$$\alpha^{(n)}(0) \in L(\overbrace{X, L(X, \dots, L(X}^{n}, \mathbb{R}))\dots),\ \ X \equiv C_0[0,1];$$
$$\alpha^{(n)}(0)w^n \equiv (\dots((\alpha^{(n)}(0)w)w)\dots)w.$$

Denote

$$\|\alpha^{(n)}(0)\| = \sup_{\|x_j\| \le 1, j=\overline{1,n}} |(\dots((\alpha^{(n)}(0)x_1)x_2)\dots)x_n|.$$

It is proved in [3] that, under the condition

$$\lim_{n\to\infty}(\|\alpha^{(n)}(0)\|/n!)^{1/n} = 0, \tag{12}$$

the Cauchy problem (7) in the one-dimensional case has a solution of the form (9) for $N = \infty$. That is, this solution can be expanded in an infinite power series. It should be noted that condition (12) is used only to ensure the convergence of the series, and the corresponding estimates are independent of the dimension. So that the solution of problem (7) is identical to (9) for $N = \infty$. Consequently, we come to the following result.

Theorem 2. *Let the functional $\alpha:\ C_0[0,1] \to \mathbb{R}$ can be expanded in the infinite power series (11), and let condition (12) be fulfilled. Then the solution of problem (4),(5) has the following form:*

$$\begin{aligned} y(t,\lambda) = e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t+ip(\lambda)w(t)}\widehat{\varphi}(\lambda)\Bigg[\alpha+ \\ +\sum_{k=1}^{\infty}(-i)^k p^k(\lambda)\int_{\Delta_k(t)} D^k\alpha(t_1 \dots t_k)dt_1 \dots dt_k\Bigg]. \end{aligned}$$

Find the conditions under which the solution $y(t,\lambda)$ of problem (4),(5) belongs to $\mathcal{G}$ with respect to λ for $0 \le t \le 1$.

Theorem 3. *Suppose that the following conditions are fulfilled:*
1) $-c+\frac{1}{2}(\overline{p})^2<0$, *where* $\overline{p}=\max_{|\lambda|=1}|p(\lambda)|$;
2) $\left|\int_{[0,t]^k} D^k\alpha(t_1,\dots,t_k)dt_1\dots dt_k\right|\le M^k,\ 0\le M<+\infty,\ 0\le t\le 1$;
3) $\varphi\in\mathcal{G}$.
Then $y(t,\cdot)\in\mathcal{G},\ 0\le t\le 1$.

Proof. The function $y(t,\lambda)$ has the form

$$y(t,\lambda)=$$
$$=e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t}e^{ip(\lambda)w(t)}\widehat{\varphi}(\lambda)\left[\alpha+\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}p^k(\lambda)\int_{[0,t]^k}D^k\alpha(t_1\dots t_k)dt_1\dots dt_k\right].$$

The following estimate holds true:

$$\left|\frac{\partial^r}{\partial\lambda_m^r}\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}p^k(\lambda)\int_{[0,1]^k}D^k\alpha(t_1\dots t_k)dt_1\dots dt_k\right|=$$
$$=\left|\sum_{k=r}^{\infty}\frac{(-i)^k}{k!}k(k-1)\cdot\ldots\cdot(k-r+1)p^{k-r}(\lambda)b_m^r\int_{[0,t]^k}D^k\alpha(t_1\dots t_k)dt_1\dots dt_k\right|\le$$
$$\le|b_m|^r\sum_{k=0}^{\infty}\frac{|p(\lambda)|^k}{k!}\left|\int_{[0,t]^k}D^k\alpha(t_1\dots t_k)dt_1\dots dt_k\right|\le$$
$$\le|b_m|^rM^re^{M\overline{p}|\lambda|},\ 1\le m\le n,\ r\in\mathbb{Z}_+.$$

Now the assertion of the theorem results from the explicit form of $y(t,\lambda)$, the assumption of the theorem, and the obtained estimate, because

$$\sup_{\lambda\in\mathbb{R}^n}\left|\lambda_j^l\frac{\partial^m}{\partial\lambda_k^m}y(t,\lambda)\right|<+\infty,\ 0\le t\le 1,\ 1\le k,j\le n,\ l,m\in\mathbb{Z}_+.$$

Consider now examples of random variables that satisfy the conditions of Theorem 3.

Example 1. Let $\alpha=f(w(1))$. Suppose that $f\in C^\infty(\mathbb{R})$ and $|f^{(k)}(x)|\le M^k,\ 0\le M<\infty,\ k\in\mathbb{Z}_+,\ x\in\mathbb{R}$. We have

$$D^k\alpha(t_1,\dots,t_k)=f^{(k)}(w(1))1\!\!\mathrm{I}_{[0,1]}(t_1)\cdot\ldots\cdot 1\!\!\mathrm{I}_{[0,1]}(t_k)=f^{(k)}(w(1));$$
$$\left|\int_{[0,1]^k}D^k\alpha(t_1,\dots,t_k)dt_1\dots dt_k\right|\le|f^{(k)}(w(1))|t^k\le M^k,\ 0\le t\le 1.$$

In this case,

$$y(t,\lambda)=e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t+ip(\lambda)w(t)}\widehat{\varphi}(\lambda)\left[\alpha+\sum_{k=1}^{\infty}\frac{(-i)^k}{k!}p^k(\lambda)f^{(k)}(w(1))t^k\right],\ 0\le t\le 1.$$

Example 2. Let

$$\alpha=f(w(h_1),\dots,w(h_m)),w(h_i)=\int_0^1h_i(s)dw(s),\ h_i\in L^2[0,1],\ i=\overline{1,m},\ m\ge 1.$$

Suppose that

$$f\in C^\infty(\mathbb{R}^n),\ |f^{(k)}_{i_1,\dots,i_m}(x)|\le M^k,\ 0\le M<+\infty,\ k\in\mathbb{Z}_+,\ x\in\mathbb{R}^n.$$

We have

$$D^k\alpha(t_1,\dots,t_k)=\sum_{i_1+\ldots+i_m=k}f^{(k)}_{i_1,\dots,i_m}(w(h_1),\dots,w(h_m))h_1(t_1)\cdot\ldots\cdot h_m(t_k);$$

$$\int_{[0,1]^k} D^k\alpha(t_1,\dots,t_k)dt_1\dots dt_k =$$

$$= \sum_{i_1+\dots+i_m=k} f^{(k)}_{i_1,\dots,i_m}(w(h_1),\dots,w(h_m)) \int_0^t h_1(t_1)dt_1\cdot\ldots\cdot\int_0^t h_m(t_k)dt_k;$$

$$\left|\int_{[0,1]^k} D^k\alpha(t_1,\dots,t_k)dt_1\dots dt_k\right| \le M^k\left(\sum_{i=1}^m \|h_i\|_{L^2[0,1]}\right)^k.$$

The conditions of Theorem 3 hold true. In this case,

$$y(t,\lambda) = e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t+ip(\lambda)w(t)}\widehat{\varphi}(\lambda)\Bigg[\alpha+$$

$$+\sum_{k=1}^\infty \frac{(-i)^k}{k!}p^k(\lambda)\sum_{i_1+\dots+i_m=k} f^{(k)}_{i_1,\dots,i_m}(w(h_1),\dots,w(h_m))\int_0^t h_1(t_1)dt_1\dots\int_0^t h_m(t_k)dt_k\Bigg].$$

Now replace the initial condition (2) by the relation

$$u(0,x) = \eta(x,\omega) \equiv \sum_{n=0}^\infty \varphi_n(x)I_n(f_n), \tag{13}$$

where

$$I_n(f_n) = \int_{[0,1]^n} f_n(t_1,\dots,t_n)dw(t_1)\dots dw(t_n),$$

$\varphi_n \in \mathcal{G}$. Consider problem (1), (13). Series (13) is assumed to be convergent in $L^2(\Omega)$ for every $x\in\mathbb{R}^n$. Apply the Fourier transformation to (1),(13), select the real and imaginary parts, fix λ, and denote $\vec{\varphi}_n = \left(\operatorname{Re}\widehat{\varphi}_n(\lambda), \operatorname{Im}\widehat{\varphi}_n(\lambda)\right)^\top$. We come to the following Cauchy problem:

$$\begin{cases} d\vec{z}(t) = aI\vec{z}(t)dt + B\vec{z}(t)dw(t) \\ \vec{z}(0) = \sum_{n=0}^\infty \vec{\varphi}_n I_n(f_n). \end{cases} \tag{14}$$

Suppose that

$$\sum_{n=0}^\infty \vec{\varphi}_n I_n(f_n) \text{ is convergent in } L^2(\Omega). \tag{15}$$

Let us approach the solution of (14) by the solution of the following problem:

$$\begin{cases} d\vec{z}_M(t) = aI\vec{z}_M(t)dt + B\vec{z}_M(t)dw(t) \\ \vec{z}_M(0) = \sum_{n=0}^M \vec{\varphi}_n I_n(f_n). \end{cases} \tag{16}$$

Find the requirements on φ_n and on f_n under which

$$\vec{z}(t) = L^2 - \lim_{M\to\infty}\vec{z}_M(t). \tag{17}$$

Taking the properties of multiple stochastic integrals and results given in [1] into consideration, we have

$$\vec{z}_M(t) = y_0^t\Bigg[\vec{z}_M(0) + \sum_{k=1}^M (-1)^k B^k \sum_{n=k}^M \vec{\varphi}_n \int_{\Delta_k(t)} D^k I_n(f_n)(t_1,\dots,t_k)dt_1\dots dt_k\Bigg] =$$

$$= y_0^t\sum_{m=0}^M I_m(\widetilde{f}_m),$$

where $\widetilde{f}_m(\cdot) = \vec{\varphi}_m f_m(\cdot) +$

$$+(1-\delta_{m,M})\sum_{k=1}^{M-m}(-1)^k\binom{k+m}{k}B^k\vec{\varphi}_{k+m}\int_{[0,t]^k} f_{k+m}(t_1,\dots,t_k,\cdot)dt_1\dots dt_k,$$

$\delta_{m,M}$ is the Kronecker symbol. Since

$$y_0^t = e^{aIt}\sum_{m=0}^{\infty}\frac{1}{m!}I_m(g_m), \qquad g_m = (B1\!\!\mathrm{I}_{[0,t]})^{\otimes m},$$

the valuable $\vec{z}_M(t)$ can be rewritten as

$$\vec{z}_M(t) = e^{aIt}\Big(\sum_{m=0}^{\infty}\frac{1}{m!}I_m(g_m)\Big)\Big(\sum_{k=0}^{M}I_k\big(\widetilde{f}_k\big)\Big) = e^{aIt}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{M}I_m(g_m)I_k\big(\widetilde{f}_k\big).$$

The last expression can be changed by means of the relation (see [6])

$$I_p(f_p)I_q(g_q) = \sum_{r=0}^{p\wedge q}r!\binom{p}{r}\binom{q}{r}I_{p+q-2r}(f_p \wr \otimes_r g_q), \text{ where } (f_p\otimes_r g_q)(t_1,\dots,t_{p+q-2r}) =$$

$$= \int_{[0,t]^r} f_p(t_1,\dots,t_{p-r},t^1,\dots,t^r)g_q(t_{p-r+1},\dots,t_{p+q-2r},t^1,\dots,t^r)dt^1\dots dt^r,$$

where $\wr$ is the symmetrization operation. As a result, we finally obtain

$$\vec{z}_M(t) = e^{aIt}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{M}\vec{\varphi}_n I_{m+n}\big(g_m \wr \otimes f_n\big) =$$

$$= e^{aIt}\Big[\sum_{l=0}^{M}I_l\Big(\sum_{n=0}^{l}\frac{\big(g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big)}{(l-n)!}\Big) + \sum_{l=M+1}^{\infty}I_l\Big(\sum_{n=0}^{M}\frac{\big(g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big)}{(l-n)!}\Big)\Big].$$

Now we are able to estimate constructively $E|\vec{z}_M(t)|^2$ using the properties of multiple stochastic integrals. Really,

$$E|\vec{z}_M(t)|^2 =$$

$$= e^{2aIt}\Big[\sum_{l=0}^{M}l!\Big\|\sum_{n=0}^{l}\frac{\big(g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big)}{(l-n)!}\Big\|_l^2 + \sum_{l=M+1}^{\infty}l!\Big\|\sum_{n=0}^{M}\frac{\big(g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big)}{(l-n)!}\Big\|_l^2\Big] \le$$

$$\le e^{2aIt}\Big[\sum_{l=0}^{M}l!\Big(\sum_{n=0}^{l}\frac{\big\|g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big\|_l}{(l-n)!}\Big)^2 + \sum_{l=M+1}^{\infty}l!\Big(\sum_{n=0}^{M}\frac{\big\|g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big\|_l}{(l-n)!}\Big)^2\Big].$$

It still remains to evaluate $\big\|g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big\|_l$. We have

$$\big\|g_{l-n}\wr\otimes(\vec{\varphi}_n f_n)\big\|_l^2 \le \int_{[0,1]^l}\big|\big((B1\!\!\mathrm{I}_{[0,t]})^{\otimes(l-n)}\otimes(\vec{\varphi}_n f_n)\big)(t_1,\dots,t_l)\big|^2 dt_1\dots dt_l =$$

$$= \big|B\big|^{2(l-n)}t^{(l-n)}\|\vec{\varphi}_n f_n\|_n^2 \le \big|B\big|^{2(l-n)}|\vec{\varphi}_n|^2\|f_n\|_n^2 \le 2^{(l-n)}\big|b\big|^{2(l-n)}|\vec{\varphi}_n|^2\|f_n\|_n^2.$$

Finally,

$$E|\vec{z}_M(t)|^2 \le e^{2at}\sqrt{2}\Big[\sum_{l=0}^{M}l!\Big(\sum_{n=0}^{l}\frac{\sqrt{2}^{(l-n)}\big|b\big|^{(l-n)}}{(l-n)!}|\vec{\varphi}_n|\|f_n\|_n\Big)^2 +$$

$$+\sum_{l=M+1}^{\infty}l!\Big(\sum_{n=0}^{M}\frac{\sqrt{2}^{(l-n)}\big|b\big|^{(l-n)}}{(l-n)!}|\vec{\varphi}_n|\|f_n\|_n\Big)^2\Big].$$

Set

$$\rho_l = l!\left(\sum_{n=0}^{l}\frac{\sqrt{2}^{(l-n)}|b|^{(l-n)}}{(l-n)!}|\vec{\varphi}_n|\|f_n\|_n\right)^2.$$

Now we are able to formulate the sufficient condition for (17) to be true. In order that (17) be valid, it is sufficient that

$$\sum_{l=0}^{\infty}\rho_l < +\infty. \tag{18}$$

If (18) holds true then $E|\vec{z}(t)|^2 \le \sum_{l=0}^{\infty}\rho_l < +\infty,\ 0 \le t \le 1$.

Consider the equation

$$\vec{z}_M(t) = \vec{z}_M(0) + \int_0^t aI\vec{z}_M(s)ds + \int_0^t B\vec{z}_M(s)dws,\ 0 \le t \le 1. \tag{19}$$

This equation is equivalent to problem (16). Estimate (18) makes it possible to pass to the limit in (19) under the ordinary integral sign because of the estimate

$$E\left|\int_0^t aI\vec{z}_M(s)ds\right|^2 \le E\int_0^t \left|aI\right|^2 ds\int_0^t\left|\vec{z}_M(s)\right|^2 ds \le$$

$$\le 2|a|^2 t\int_0^t E\left|\vec{z}_M(s)\right|^2 ds \le 2|a|^2 t\sup_{0\le t\le 1}\left|\vec{z}_M(s)\right|^2 t < +\infty,\ 0 \le t \le 1.$$

The possibility to pass to the limit in $L^2(\Omega)$ under the stochastic integral sign in (19) now follows from (19) because the stochastic integral can be expressed in the other terms of (19), for which the passage to the limit is proved, and acts closely, as it was shown in [2]. Hence, we can pass to the $L^2(\Omega)$-limit in (19) on the whole if condition (18) is fulfilled. Thus, the following result holds true.

Theorem 4. *If conditions (15) and (18) are fulfilled, then problem (14) has a solution of the form*

$$\vec{z}(t) = y_0^t\left[\vec{z}(0) + \sum_{k=1}^{\infty}(-1)^k B^k\int_{\Delta_k(t)} D^k\vec{z}(0)(t_1,\dots,t_k)dt_1\dots dt_k\right].$$

The correspondent solution of Eq. (4) with the initial condition

$$y(0,\lambda) = \widehat{\eta}(\lambda,\omega) \equiv \sum_{n=0}^{\infty}\widehat{\varphi}_n(\lambda)I_n(f_n) \tag{20}$$

has the form

$$y(t,\lambda) = e^{\left(h(\lambda)+\frac{1}{2}p^2(\lambda)\right)t+ip(\lambda)w(t)}\Bigg[\widehat{\eta}(\lambda,\omega)+$$

$$+\sum_{k=1}^{\infty}(-i)^k p^k(\lambda)\int_{\Delta_k(t)} D^k\widehat{\eta}(\lambda,\omega)(t_1\dots t_k)dt_1\dots dt_k\Bigg].$$

Remark. *Note that, as an example of fulfillment of (18), one can consider such $\vec{\varphi}_n$ and f_n that $|\vec{\varphi}_n|\,\|f_n\|_n \le \theta^n/n!$. We have*

$$\rho_l \le l!\left(\sum_{n=0}^{l}\frac{\sqrt{2}^{(l-n)}|b|^{(l-n)}\theta^n}{(l-n)!\,n!}\right)^2 =$$

$$= \frac{1}{l!}\Big(\sum_{n=0}^{l}\binom{l}{n}\sqrt{2}^{(l-n)}|b|^{(l-n)}\theta^n\Big)^2 = \frac{\big(\sqrt{2}\,|b|+\theta\big)^{2l}}{l!}.$$

$$\sum_{l=0}^{\infty}\rho_l \le \sum_{l=0}^{\infty}\frac{\big(\sqrt{2}\,|b|+\theta\big)^{2l}}{l!} = e^{(\sqrt{2}\,|b|+\theta)^2}.$$

$$E|\vec{z}_M(t)|^2 \le e^{2a|I|}\sum_{l=0}^{\infty}\frac{\big(|B|+1\big)^{2l}}{l!} \le e^{2a|I|}e^{(|B|+1)^2} \le e^{2a|I|+(|B|+1)^2}.$$

Clear up a question when the inverse Fourier transformation can be applied to the solution $y(t,\lambda)$ of Eq. (4) with the initial condition (20) to get $u(t,x)$. Taking the arguments from the proof of Theorem 1 into account, it is sufficient to put the condition

$$\int_{\mathbb{R}^n}|Q(\lambda)y(t,\lambda)|d\lambda < +\infty \ a.s., \quad 0\le t\le 1, \tag{21}$$

on $y(t,\lambda)$, where $Q(\lambda)$ is a polynomial. Evidently, condition (21) is fulfilled if

$$E\int_{\mathbb{R}^n}(1+|\lambda|^m)|y(t,\lambda)|^2 d\lambda < +\infty,\ m\in\mathbb{Z}_+, 0\le t\le 1.$$

Note that $\big(\operatorname{Re}y(t,\lambda),\,\operatorname{Im}y(t,\lambda)\big)^\top$ satisfies (14) for $a=h(\lambda)$, $b=p(\lambda)$ and $\vec{\varphi}_n=\vec{\varphi}_n(\lambda)=\big(\operatorname{Re}\widehat{\varphi}_n(\lambda),\,\operatorname{Im}\widehat{\varphi}_n(\lambda)\big)^\top$. According to the previous reasoning,

$$\begin{pmatrix}\operatorname{Re}y(t,\lambda)\\ \operatorname{Im}y(t,\lambda)\end{pmatrix} = e^{aIt}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{\infty}\vec{\varphi}_n I_{m+n}\big(g_m \wr\otimes f_n\big) =$$

$$= \sum_{l=0}^{\infty} I_l\Big(\sum_{n=0}^{l}\frac{\psi_{n,l}(t,\lambda)}{(l-n)!}\big((1\!\!\mathrm{I}_{[0,t]})^{\otimes(l-n)}\wr\otimes f_n\big)\Big), \quad J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

$$\psi_{n,l}(t,\lambda) = e^{h(\lambda)It}\big(p(\lambda)J\big)^{(l-n)}\vec{\varphi}_n(\lambda).$$

Suppose that, for each $m\in\mathbb{Z}_+, 0\le t\le 1$,

$$\Delta(m,t) \equiv \sum_{l=0}^{\infty} l!\int_{\mathbb{R}^n}(1+|\lambda|^m)\Big(\sum_{n=0}^{l}\frac{(\sqrt{2}t)^{(l-n)}|\psi_{n,l}(t,\lambda)|}{(l-n)!}\|f_n\|_n\Big)^2 d\lambda < +\infty. \tag{22}$$

Use the estimates for $E|y(t,\lambda)|^2$ from the proof of Theorem 4. We have

$$E\int_{\mathbb{R}^n}(1+|\lambda|^m)|y(t,\lambda)|^2 d\lambda = \int_{\mathbb{R}^n}(1+|\lambda|^m)E|y(t,\lambda)|^2 d\lambda \le$$

$$\le \int_{\mathbb{R}^n}(1+|\lambda|^m)\sum_{l=0}^{\infty} l!\Big(\sum_{n=0}^{l}\frac{(\sqrt{2}t)^{(l-n)}\psi_{n,l}(t,\lambda)}{(l-n)!}\|f_n\|_n\Big)^2 d\lambda \le$$

$$\le \sum_{l=0}^{\infty} l!\int_{\mathbb{R}^n}(1+|\lambda|^m)\Big(\sum_{n=0}^{l}\frac{(\sqrt{2}t)^{(l-n)}\psi_{n,l}(t,\lambda)}{(l-n)!}\|f_n\|_n\Big)^2 d\lambda = \Delta(m,t).$$

Denote

$$\gamma_{n,l}(m,t) = \int_{\mathbb{R}^n} 2(1+|\lambda|^m)e^{h(\lambda)t}|p(\lambda)|^{(l-n)}|\widehat{\varphi}_n(\lambda)|d\lambda.$$

Since $|\psi_{n,l}(t,\lambda)| \le 2e^{h(\lambda)t}|p(\lambda)|^{(l-n)}|\widehat{\varphi}_n(\lambda)|$ and $(\sum_{n=1}^{l}a_n)^2 \le l\sum_{n=1}^{l}a_n^2$, we come to

$$\Delta(m,t) \le \sum_{l=0}^{\infty} l!\,l\sum_{n=0}^{l}\frac{(\sqrt{2}t)^{2(l-n)}\gamma_{n,l}(m,t)}{\big((l-n)!\big)^2}\|f_n\|_n^2. \tag{23}$$

Hence, the convergence of the series in (23) is the sufficient condition for (22).

So, the following result is proved.

Theorem 5. *Let problem (14) have a solution for each $\lambda \in \mathbb{R}^n$. If condition (22) is fulfilled, then problem (1),(13) has the solution that can be found as*

$$u(t,x) = \check{y}(t,x),$$

where $y(t,\lambda)$ is a solution of problem (4),(20).

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