Theory of Stochastic Processes
Vol.13 (29), no.1-2, 2007, pp.281-293

GRYGORIY TORBIN

PROBABILITY DISTRIBUTIONS WITH INDEPENDENT Q-SYMBOLS AND TRANSFORMATIONS PRESERVING THE HAUSDORFF DIMENSION

The paper is devoted to the study of connections between fractal properties of one-dimensional singularly continuous probability measures and the preservation of the Hausdorff dimension of any subset of the unit interval under the corresponding distribution function. Conditions for the distribution function of a random variable with independent $Q-$ digits to be a transformation preserving the Hausdorff dimension (DP-transformation) are studied in details. It is shown that for a large class of probability measures the distribution function is a DP-transformation if and only if the corresponding probability measure is of full Hausdorff dimension.

1. Introduction

It is well known (see, e.g., [9]) that fractal analysis of singularly continuous probability distributions allows us to study many essential properties of such distributions. The first stage of such an analysis is the study of metric, topological and fractal properties of the spectrum (topological support) of the distribution, which leads to the metric-topological classification of singular measures (see, e.g., [4], [9]). It should be mentioned here that the determination of the Hausdorff-Besocovitch dimension even for the topological support is often a very non-trivial problem (see, e.g., [1],[8], [9],[10]). On the other hand, the topological support is a rather "rough" characteristic for a measure with a complicated local structure. For instance, for any pair of real numbers $p_1 \in (0, \frac{1}{2})$ and $p_2 \in (0, \frac{1}{2}), p_1 \neq p_2$ the distributions of the following random variables $\xi^{(1)}$ and $\xi^{(2)}$ are mutually singular and they are singular w.r.t. Lebesgue measure, where $\xi^{(1)} = \sum\limits_{k=1}^{\infty} \frac{\xi_k^{(1)}}{2^k}, \xi^{(2)} = \sum\limits_{k=1}^{\infty} \frac{\xi_k^{(2)}}{2^k}$; $\xi_k^{(1)}$

2000 *Mathematics Subject Classifications* 60G30, 28A80, 11K55.
Key words and phrases. Singularly continuous probability distributions, Hausdorff dimension of probability measures, Hausdorff-Billingsley dimension, fractals, DP-transformations.

$(\xi_k^{(2)})$ is a sequence of independent random variables taking the values 0 and 1 with probabilities $p_1(p_2)$ and $1-p_1(1-p_2)$ correspondingly. Nevertheless, the topological supports of the above distributions coincide with the unit interval.

The second level of the fractal analysis of singular measures is the determination of fractal properties of the density supports and essential density supports, i.e., of the following sets $N_\mu = \{x : F'(x) \neq 0\}$ and $N_\mu^\infty = \{x : F'(x) = +\infty\}$. These sets describe properties of singularly continuous measures essentially better than the topological support. In particular, it is known (see, e.g., [6]) that a probability measure μ is singular if and only if $\mu(N_\mu^\infty) = 1$. In [6] it has been constructed an example of *absolutely continuous* probability distribution function such that the set N_μ^∞ is everywhere dense of full Hausdorf dimension, which show that usual derivative is "too sensitive" to describe only essential properties of singular measures.

The next level in the fractal analysis of a singularly continuous measure is the investigation of dimensionally minimal supports of the measure. For a probability measure μ one can define the Hausdorff dimension $\alpha_0(\mu)$ of the measure as follows: $\alpha_0(\mu) = \inf\limits_{E:\mu(E)=1}\{\alpha_0(E)\}$. If μ is discrete, then $\alpha_0(\mu) = 0$. If μ is absolutely continuous, then $\alpha_0(\mu) = 1$; and $\alpha_0(\mu) \in [0, 1]$ if μ is singularly continuous. Main problems of this level are the determination of the Hausdorff dimension of the measure itself and the study of supports whose Hausdorff dimension coincides with the Hausdorff dimension of the measure (see, e.g., [1]).

In the present paper we show how results on the latter level of fractal analysis of singularly continuous probability measures can be applied to the study of continuous transformations preserving the Hausdorff dimensions (DP-transformations) of any subset of real line. The development of a general theory of DP-transformations is important from theoretical as well as from applied point of view (see, e.g., [2], [3]). This paper is devoted to the development of ideas and methods for the investigation of continuous transformations which were proposed in [2], [3] and [5]. We show that the problem of the study of continuous DP-transformations of the real line is equivalent to the the investigation of the DP-properties of strictly increasing probability distribution functions. Moreover we show that for a large class of probability measures of the Jessen-Wintner type (see, e.g., [9]) the DP-property is equivalent to the superfractality of the corresponding probability measure (i.e., $\alpha_0(\mu) = 1$). The main results of the paper are Theorem 1 and Theorem 2 which give necessary conditions and sufficient conditions for the distribution function of a random variable with independent $Q-$symbols to be a DP-transformation. Corollaries after Theorem 2 show that the above theorems are essential generalizations of results from [2], [3] and [5].

2. Q-REPRESENTATION OF REAL NUMBERS AND RANDOM VARIABLES WITH INDEPENDENT Q-SYMBOLS.

Let us recall shortly the notion of the Q-representation of real numbers. For any $s > 1, s \in N$ let $N^0_{s-1} = \{0, 1, ..., s-1\}$, and let $Q = (q_0, q_1, ..., q_{s-1})$ be a stochastic vector with strictly positive coordinates. By using the vector Q, we define the Q- partition of the unit interval $[0, 1]$.

I step. We divide $[0, 1]$ from the left to the right into s closed intervals Δ^Q_0, Δ^Q_1, ..., Δ^Q_{s-1}, with $|\Delta^Q_i| = q_i$. We call the $\Delta^{Q^*}_i$ "first rank intervals".

*II step.*We divide (from the left to the right) each of the closed first rank interval $\Delta^Q_{i_1}$ into s closed intervals, called second rank intervals, whose lengths are in the following proportion : $q_0 : q_1 : ... : q_{(s-1)}$.

n-th step. We divide (from the left to the right) each of the closed (n-1)-th rank interval $\Delta^Q_{i_1 i_2 ... i_{n-1}}$ into s closed intervals of the n-th rank, whose lengths are in the same proportion : $q_0 : q_1 : ... : q_{(s-1)}$.

It is easy to see that $|\Delta^Q_{i_1 i_2 ... i_n}| = q_{i_1} \cdot q_{i_2} \cdot ... \cdot q_{i_n} \to 0 \quad (n \to \infty)$, and the sequence of imbedded closed intervals $\Delta^Q_{i_1} \supset \Delta^Q_{i_1 i_2} \supset ... \supset \Delta^Q_{i_1 i_2 ... i_n} \supset ...$ has a unique common point x.

Conversely, if a point x is not an end-point of any closed interval of the above partition, then for the point x there is a unique sequence of imbedded intervals $\Delta^Q_{\alpha_1(x)} \supset \Delta^Q_{\alpha_1(x)\alpha_2(x)} \supset ... \supset \Delta^Q_{\alpha_1(x)...\alpha_n(x)} \supset ...$ containing the point x.

Symbolically we write

$$x = \Delta^Q_{\alpha_1(x)\alpha_2(x)...\alpha_n(x)...}. \tag{1}$$

(1) is called the Q-representation of x. If a point x is an end-point of some closed interval of the above partition, then x has two Q-representations.

Actually, this representation can also be thought as the representation, which generates by the dynamical system on the unit interval with the following transformation T (see, e.g., [11] for details):

$$T(x) = \frac{1}{q_k} x - \frac{\sum\limits_{i=0}^{k-1} q_i}{q_k},$$

for $x \in [\sum\limits_{i=0}^{k-1} q_i, q_k + \sum\limits_{i=0}^{k-1} q_i)$, with $\sum\limits_{i=0}^{-1} q_i := 0$.

For the convenience of a reader we give one more explanation of the Q-representation: the cylinders of this representation are images of usual s-adic cylinders under the distribution function F_ψ of the random variable ψ with independent identically distributed s-adic digits, i.e.,

$$\psi = \sum_{k=1}^{\infty} \frac{\psi_k}{s^k},$$

where ψ_k are independent identically distributed random variables taking the values $0, 1, ..., s-1$ with probabilities $q_0, q_1, ..., q_{s-1}$ correspondingly.

Let $\{\eta_k\}$ be a sequence of independent random variables taking the values $0, 1, ..., s-1$ with probabilities $p_{0k}, p_{1k}, ..., p_{(s-1)k}$ correspondingly, and let us consider the random variable ξ:

$$\xi = \Delta^Q_{\eta_1\eta_2...\eta_k...}. \tag{2}$$

ξ is said to be a random variable with independent Q-digits. The corresponding probability measure μ_ξ can be obtained in the following way. Let $\Omega_k = \{0, 1, ..., s-1\}$, $\mathcal{F}_k = 2^{\Omega_k}$. We define a discrete measure ν_k on the $\mathcal{F}_k$ by $\nu_k(i) = p_{ik}$, $i \in N^0_{s-1}$, and consider the infinite product of probability spaces: $(\Omega, \mathcal{F}, \nu) = \prod_{k=1}^{\infty}(\Omega_k, \mathcal{F}_k, \nu_k)$ and the bimeasurable mapping $f : \Omega \to [0, 1]$, defined for any element $\omega = (\omega_1, \omega_2, ..., \omega_k, ...) \in \Omega$ as follows:

$$f(\omega) = \Delta^Q_{\omega_1\omega_2...\omega_k...}. \tag{3}$$

For any Borel subset $E \subset [0, 1]$ we define the image measure ν^* by the following relation: $\nu^*(E) = \nu(f^{-1}(E))$, where $f^{-1}(E) = \{\omega : \ \omega \in \Omega, \ \ f(\omega) \in E\}$. The measure ν^* coincides with the probability measure μ_ξ.

If $p_{ik} = p_i \ \forall k \in N$, $i \in N^0_{s-1}$ (i.e., ξ is a random variable with independent identically distributed Q-digits), then the measure μ_ξ is the self-similar measure associated with the list $(S_0, ..., S_{s-1}, p_0, ..., p_{s-1})$, where S_i is a similarity with the ratio q_i, and the list $(S_0, ..., S_{s-1})$ satisfies the open set condition. More precisely, μ_ξ is the unique Borel probability measure on $[0, 1]$ such that $\mu_\xi = \sum_{i=0}^{s-1} p_i \cdot \mu_\xi \circ S_i^{-1}$, (see, e.g., [8] for details). If we define discrete measures ν_k in the following way: $\nu_k(i) = q_i$, $i \in N^0_{s-1}$, $k \in N$, then the measure ν^* coincides with Lebesgue measure on $[0, 1]$.

The distribution of the random variable ξ is of pure type. In [10] it has been proved that it is of the discrete type iff

$$\prod_{k=1}^{\infty} \max_i p_{ik} > 0; \tag{4}$$

it is of absolutely continuous type iff

$$\sum_{k=1}^{\infty}\left(\sum_{i=0}^{s-1}(1 - \frac{p_{ik}}{q_i})^2\right) < \infty; \tag{5}$$

it is of singularly continuous type iff the infinite product (4) and series (5) diverge. It is not hard to see that in this situation the singularity plays a "generic" role.

3. DP-transformations with independent Q-symbols

A transformation f of R^n (in the sense of a bijective mapping of R^n into itself) is said to be transformation preserving the Hausdorff dimension (DP-transformation for short), if for any subset $E \subset R^n$ and its image $E' = f(E)$ the following condition holds

$$\alpha_0(E) = \alpha_0(E').$$

From the countable stability of the Hausdorff dimension it follows that a transformation f is a DP-transformation on R^1 if and only if f preserves the Hausdorff dimension of all subsets of any interval (a, b). So, to study the DP-transformations of R^1 it is sufficient to study DP-transformations of intervals. Without loss of generality we shall consider the unit segment. It is easy to see that a continuous function f is a transformation of $[0, 1]$ if and only if it is either a strictly increasing distribution function (in the sense of probability theory) F on $[0, 1]$ or it is of the form $1 - F$.

Our main aim in this Section is to find conditions for the distribution functions of random variables with independent Q-symbols to be DP-transformations.

The following theorems are the main results of this paper and they are essential generalizations of results from [2], [3] and [5].

Let $h_k = -\sum\limits_{i=0}^{s-1} p_{ik} \ln p_{ik}$, and let $b_k = -\sum\limits_{i=0}^{s-1} p_{ik} \ln q_i$.

Theorem 1. *Let* $\inf\limits_{i,j} p_{ij} > 0$. *If*

$$\lim_{n\to\infty} \frac{h_1 + h_2 + ... + h_n}{b_1 + b_2 + ... + b_n} = 1, \tag{6}$$

then the distribution function F_ξ of a random variable ξ with independent Q-symbols preserves the Hausdorff dimension of any subset of the unit interval.

Proof. It is not hard to prove (see, e.g., [12]) that for any two probability vectors $\overrightarrow{p} = (p_0, p_1, ..., p_{s-1})$ and $\overrightarrow{q} = (q_0, q_1, ..., q_{s-1})$ with $q_i > 0, \forall i \in N^0_{s-1}$ the following condition holds:

$$p_0^{p_0} \cdot p_1^{p_1} \cdot ... \cdot p_{s-1}^{p_{s-1}} \geq q_0^{p_0} \cdot q_1^{p_1} \cdot ... \cdot q_{s-1}^{p_{s-1}}, \tag{7}$$

and the equality holds if and only if $p_i = q_i, \forall i \in N^0_{s-1}$. Therefore,

$$h_k = -\ln(p_{0k}^{p_{0k}} \cdot p_{1k}^{p_{1k}} \cdot ... \cdot p_{(s-1)k}^{p_{(s-1)k}}) \leq b_k = -\ln(q_0^{p_{0k}} \cdot q_1^{p_{1k}} \cdot ... \cdot q_{s-1}^{p_{(s-1)k}}), \tag{8}$$

and condition (6) is equivalent to the existence of the following limit:

$$\lim_{n\to\infty} \frac{h_1 + h_2 + ... + h_n}{b_1 + b_2 + ... + b_n} = 1. \tag{9}$$

Let ε be an arbitrary positive number such that $\varepsilon < \min_i q_i$ and let us consider the following sets:

$$T^+_{\varepsilon,k} = \left\{ j: \quad j \in N,\ j \leq k, \quad |p_{ij} - q_i| \leq \varepsilon, \quad \forall i \in N^0_{s-1} \right\},$$

$$T^-_{\varepsilon,k} = \left\{ j: \quad j \in N,\ j \leq k, \quad |p_{ij} - q_i| > \varepsilon \quad \text{for some } i \in N^0_{s-1} \right\}.$$

Now we need the following lemma, which describes "how dense" the sets $T^+_{\varepsilon,k}$ is.

Lemma. *If condition (6) holds, then* $\lim_{k\to\infty} \frac{|T^+_{\varepsilon,k}|}{k} = 1$.

Proof. Suppose, contrary to our claim, that $\lim_{k\to\infty} \frac{|T^+_{\varepsilon,k}|}{k} \neq 1$. Since $| T^+_{\varepsilon,k} | \leq k$, the latter assumption is equivalent to the existence of a sequence $\{k_n\}$ such that $\lim_{k_n\to\infty} \frac{|T^+_{\varepsilon,k_n}|}{k_n} = a_0 < 1$. From inequalities (8) and (7) it follows that for any $\varepsilon > 0$ there exists a positive constant $\delta_0 = \delta_0(\varepsilon)$ such that $h_j \leq (1-\delta_0)b_j$ for any $j \in T^-_{\varepsilon,k}$. Therefore,

$$\frac{\sum_{j=1}^{k} h_j}{\sum_{j=1}^{k} b_j} = \frac{\sum_{j\in T^+_{\varepsilon,k}} h_j + \sum_{j\in T^-_{\varepsilon,k}} h_j}{\sum_{j=1}^{k} b_j} \leq \frac{\sum_{j\in T^+_{\varepsilon,k}} b_j + (1-\delta_0) \sum_{j\in T^-_{\varepsilon,k}} b_j}{\sum_{j=1}^{k} b_j} = 1 - \delta_0 \frac{\sum_{j\in T^-_{\varepsilon,k}} b_j}{\sum_{j=1}^{k} b_j}.$$

Let $\overrightarrow{p_j} = (p_{0j}, p_{1j}, ..., p_{(s-1)j})$ and $\overrightarrow{r} = (\ln \frac{1}{q_0}, \ln \frac{1}{q_1}, ..., \ln \frac{1}{q_{s-1}})$. Since $b_j = \overrightarrow{p_j} \cdot \overrightarrow{r}$, we conclude that

$$b_j \leq |\ \overrightarrow{p_j}\ | \cdot |\ \overrightarrow{r}\ | \leq 1 \cdot (\sum_{i=0}^{s-1} \ln^2 q_i)^{\frac{1}{2}} \leq d_1 = d_1(s, q_0, ..., q_{s-1}).$$

Since $|\ \overrightarrow{r}\ | = const$, all coordinates of the vector $\overrightarrow{r}$ are strictly positive and all coordinates of the vector $\overrightarrow{p_j}$ are non-negative, from $b_j = \overrightarrow{p_j} \cdot \overrightarrow{r}$ it follows that there exists a positive constant $d_0 = d_0(s, q_0, ..., q_{s-1})$ such that $b_j \geq d_0, \forall j \in N$. So, $0 < d_0 \leq b_j \leq d_1 < \infty, \forall j \in N$, and, therefore, there exist constants $C_{\varepsilon,k} \in [d_0, d_1]$ and $D_{\varepsilon,k} \in [d_0, d_1]$ such that

$$\sum_{j\in T^-_{\varepsilon,k}} b_j = |\ T^-_{\varepsilon,k}\ | \cdot C_{\varepsilon,k}; \qquad \sum_{j=1}^{k} b_j = k \cdot D_{\varepsilon,k}.$$

Hence,

$$\frac{\sum_{j=1}^{k_n} h_j}{\sum_{j=1}^{k_n} b_j} \leq 1 - \delta_0 \frac{\sum_{j\in T^-_{\varepsilon,k_n}} b_j}{\sum_{j=1}^{k_n} b_j} = 1 - \delta_0 \frac{|\ T^-_{\varepsilon,k_n}\ | \cdot C_{\varepsilon,k_n}}{k_n \cdot D_{\varepsilon,k_n}} \leq 1 - \frac{\delta_0 d_0}{d_1} \frac{|\ T^-_{\varepsilon,k_n}\ |}{k_n}.$$

Therefore,

$$1=\lim_{n\to\infty}\frac{h_1+h_2+...+h_{k_n}}{b_1+b_2+...+b_{k_n}}\leq\lim_{n\to\infty}(1-\frac{\delta_0 d_0}{d_1}\frac{\mid T^-_{\varepsilon,k_n}\mid}{k_n})=1-\frac{\delta_0 d_0}{d_1}(1-a_0)<1,$$

which is impossible. Let $\Delta^Q_{\alpha_1(x)...\alpha_k(x)}$ be the cylinder of the Q-representation containing the point x, let $\mu=\mu_\xi$, and let λ be the Lebesgue measure.

Let $p_{min}=\inf_{ij}p_{ij}, q_{min}=\min_i q_i, q_{max}=\max_i q_i$, and let

$$N_i(x,k)=\#\{j:j\leq k,\alpha_j(x)=i\};$$

$$N_i(\varepsilon,x,k)=\#\{j:j\leq k,\alpha_j(x)=i,j\in T^+_{\varepsilon,k}\}.$$

Then for any $x\in[0,1]$, for any $k\in N$, and for any $\varepsilon<\frac{1}{2}q_{min}$ we have:

$$-\ln\mu(\Delta^Q_{\alpha_1(x)...\alpha_k(x)})=-(\ln[\prod_{j=1}^{k}p_{\alpha_j(x)j}])=-(\sum_{j\in T^-_{\varepsilon,k}}\ln p_{\alpha_j(x)j}+\sum_{j\in T^+_{\varepsilon,k}}\ln p_{\alpha_j(x)j})$$

$$=\sum_{j\in T^-_{\varepsilon,k}}\ln\frac{1}{p_{\alpha_j(x)j}}+\sum_{i=0}^{s-1}(\sum_{\alpha_j(x)=i,\ \ j\in T^+_{\varepsilon,k}}\ln\frac{1}{p_{\alpha_j(x)j}})\leq$$

$$\leq\sum_{j\in T^-_{\varepsilon,k}}\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}(\sum_{\alpha_j(x)=i,\ \ j\in T^+_{\varepsilon,k}}\ln\frac{1}{q_i-\varepsilon})=$$

$$=\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\ln\frac{1}{q_i-\varepsilon}=$$

$$=\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\left(\ln\frac{1}{q_i}+\ln(1+\frac{\varepsilon}{q_i-\varepsilon})\right)\leq$$

$$\leq\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\left(\ln\frac{1}{q_i}+\frac{\varepsilon}{q_i-\varepsilon}\right)\leq$$

$$\leq\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\ln\frac{1}{q_i}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\frac{2\varepsilon}{q_i}\leq$$

$$\leq\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\ln\frac{1}{q_i}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\frac{2\varepsilon}{q_{min}}=$$

$$=\mid T^-_{\varepsilon,k}\mid\ln\frac{1}{p_{min}}+\sum_{i=0}^{s-1}N_i(\varepsilon,x,k)\ln\frac{1}{q_i}+\mid T^+_{\varepsilon,k}\mid\frac{2\varepsilon}{q_{min}}.$$

Therefore, for any $x \in [0,1]$ and for any $\varepsilon < \frac{1}{2}q_{min}$ we have:

$$\varlimsup_{k\to\infty} \frac{\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})} \leq$$

$$\leq \varlimsup_{k\to\infty} \frac{\mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{min}} + \sum\limits_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} + \mid T^+_{\varepsilon,k} \mid \frac{2\varepsilon}{q_{min}}}{-\ln[\prod\limits_{j=1}^{k} q_{\alpha_j(x)}]} =$$

$$= \varlimsup_{k\to\infty} \frac{\mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{min}} + \sum\limits_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} + \mid T^+_{\varepsilon,k} \mid \frac{2\varepsilon}{q_{min}}}{\sum\limits_{i=0}^{s-1} N_i(x,k) \ln \frac{1}{q_i}} \leq$$

$$\leq 1 + \varlimsup_{k\to\infty} \frac{\mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{min}} + \mid T^+_{\varepsilon,k} \mid \frac{2\varepsilon}{q_{min}}}{\sum\limits_{i=0}^{s-1} N_i(x,k) \ln \frac{1}{q_i}} \leq$$

$$\leq 1 + \varlimsup_{k\to\infty} \frac{\mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{min}} + \mid T^+_{\varepsilon,k} \mid \frac{2\varepsilon}{q_{min}}}{k \ln \frac{1}{q_{max}}} = 1 + \frac{2\varepsilon}{q_{min} \cdot \ln \frac{1}{q_{max}}}.$$

On the other hand we have: $-\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)}) =$

$$= -(\ln[\prod_{j=1}^{k} p_{\alpha_j(x)j}]) = -(\sum_{j\in T^-_{\varepsilon,k}} \ln p_{\alpha_j(x)j} + \sum_{j\in T^+_{\varepsilon,k}} \ln p_{\alpha_j(x)j}) =$$

$$= \sum_{j\in T^-_{\varepsilon,k}} \ln \frac{1}{p_{\alpha_j(x)j}} + \sum_{i=0}^{s-1} \left(\sum_{\alpha_j(x)=i,\ \ j\in T^+_{\varepsilon,k}} \ln \frac{1}{p_{\alpha_j(x)j}} \right) \geq$$

$$\geq \sum_{j\in T^-_{\varepsilon,k}} \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} \left(\sum_{\alpha_j(x)=i,\ \ j\in T^+_{\varepsilon,k}} \ln \frac{1}{q_i+\varepsilon} \right) =$$

$$= \mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i+\varepsilon} =$$

$$= \mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \left(\ln \frac{1}{q_i} + \ln(1 - \frac{\varepsilon}{q_i+\varepsilon}) \right) \geq$$

$$\geq \mid T^-_{\varepsilon,k} \mid \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \left(\ln \frac{1}{q_i} - \frac{\varepsilon}{q_i+\varepsilon} \right) \geq$$

$$\geq | T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} - \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \frac{\varepsilon}{q_i} \geq$$

$$\geq | T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} - \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \frac{\varepsilon}{q_{min}} =$$

$$= | T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} + \sum_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} - | T^+_{\varepsilon,k} | \frac{\varepsilon}{q_{min}}.$$

Therefore, for any $x \in [0,1]$ and for any $\varepsilon < q_{min}$ we have:

$$\lim_{k\to\infty} \frac{\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})} \geq$$

$$\geq \lim_{k\to\infty} \frac{| T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} + \sum\limits_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} - | T^+_{\varepsilon,k} | \frac{\varepsilon}{q_{min}}}{-\ln[\prod\limits_{j=1}^{k} q_{\alpha_j(x)}]} =$$

$$= \lim_{k\to\infty} \frac{| T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} + \sum\limits_{i=0}^{s-1} N_i(\varepsilon,x,k) \ln \frac{1}{q_i} - | T^+_{\varepsilon,k} | \frac{\varepsilon}{q_{min}}}{\sum\limits_{i=0}^{s-1} N_i(x,k) \ln \frac{1}{q_i}} =$$

$$= 1 + \lim_{k\to\infty} \frac{| T^-_{\varepsilon,k} | \ln \frac{1}{p_{max}} - | T^+_{\varepsilon,k} | \frac{\varepsilon}{q_{min}}}{\sum\limits_{i=0}^{s-1} N_i(x,k) \ln \frac{1}{q_i}} \geq$$

$$\geq 1 + \lim_{k\to\infty} \frac{| T^-_{\varepsilon,k} | \ln \frac{1}{p_{min}} - | T^+_{\varepsilon,k} | \frac{\varepsilon}{q_{min}}}{k \ln \frac{1}{q_{min}}} = 1 - \frac{\varepsilon}{q_{min} \cdot \ln \frac{1}{q_{min}}}.$$

So, for any $x \in [0,1]$ and for any $\varepsilon < \frac{1}{2} q_{min}$ we have:

$$1 - \frac{\varepsilon}{q_{min} \cdot \ln \frac{1}{q_{min}}} \leq \underline{\lim}_{k\to\infty} \frac{\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})} \leq$$

$$\leq \overline{\lim_{k\to\infty}} \frac{\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})} \leq 1 + \frac{2\varepsilon}{q_{min} \cdot \ln \frac{1}{q_{max}}}.$$

Therefore, for any $x \in [0,1]$ the following condition holds:

$$\lim_{k\to\infty} \frac{\ln \mu(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \lambda(\Delta^Q_{\alpha_1(x)\ldots\alpha_k(x)})} = 1. \tag{10}$$

Finally, from formula (10) and from Billingsley's theorem ([7], p.142), we have for all $E \subset [0,1]$:

$$\alpha_\lambda(E) = 1 \cdot \alpha_\mu(E),$$

where $\alpha_\lambda(E)$ and $\alpha_\mu(E)$ are the Hausdorff-Billingsley dimensions with respect to measures λ and μ correspondingly (see, e.g., [7] or [12] for details).

The Hausdorff-Billingsley dimension with respect to the Lebesgue measure coincides with the classical Hausdorff dimension: $\alpha_\lambda(E) = \alpha_0(E), \forall E \subset [0,1]$. From $\inf_{ij} p_{ij} = p_{min} > 0$ and from Theorem 1 of the paper [1] it follows that the Hausdorff-Billingsley dimension of an arbitrary subset $E \subseteq [0,1]$ with respect to the measure μ coincides with the Hausdorff dimension of the set $E' = F_\xi(E) : \alpha_\mu(E) = \alpha_0(F_\xi(E))$.

So, F_ξ is a DP-transformation on the unit interval.

Remark. The condition $\inf_{i,j} p_{ij} > 0$ plays an essential role in the theorem 1. The following example shows that there exists a random variable ξ with independent Q-adic digits such that condition (11) holds but the distribution function F_ξ does not preserve the Hausdorff dimension.

Example. Let $s = 3$ and $q_0 = q_1 = q_2 = \frac{1}{3}$, i.e., $\xi = \sum_{k=1}^{\infty} \frac{\eta_k}{3^k}$. Let $p_{1k} = \frac{1}{3}, p_{2k} = \frac{2}{3} - p_{0k}$, and let

$$p_{0k} = \begin{cases} \frac{1}{3}, & \text{if } \ k \neq 3^n,\ n \in N; \\ 7^{-7^n} & \text{if } \ k = 3^n,\ n \in N. \end{cases}$$

It is easy to see that $h_j = \ln 3$ for $j \neq 3^k$, $0 < h_j < \ln 3$ for $j = 3^k$, and $b_j = \ln 3$ for any $j \in N$.

Let $L(k) = \{i : i = 3^n, i \leq k, n \in N\}$, and $l(k) =| L(k) |$. Then

$$\lim_{k\to\infty} \frac{h_1 + h_2 + ... + h_k}{b_1 + b_2 + ... + b_k} = \lim_{k\to\infty} \frac{(k - l(k)) \ln 3 + \sum\limits_{j \in L(k)} h_j}{k \ln 3} = 1,$$

because $\frac{l(k)}{k} \to 0$ as $k \to \infty$.

Let us consider the set $T(Q) =$

$$= \left\{ x : \ x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}; \ \alpha_k = 0 \text{ if } k = 3^n; \ \alpha_k \in \{0,1,2\} \text{ if } k \neq 3^n,\ n \in N \right\}.$$

The set $T(Q)$ is the topological support of a specially constructed random variable ξ^* with independent Q-digits ($q_0^* = q_1^* = q_2^* = \frac{1}{3}$; $p_{0k}^* = 1$ if $k = 3^n$, $p_{0k}^* = p_{1k}^* = p_{2k}^* = \frac{1}{3}$ if $k \neq 3^k$). From Theorem 2 of the paper [1] it follows that the Hausdorff dimension of the set $T(Q)$ is equal to 1.

If $x \in T$, then

$$\lim_{k\to\infty} \frac{\ln \lambda(\Delta^Q_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)})}{\ln \mu(\Delta^Q_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)})} = \lim_{k\to\infty} \frac{\ln\left(\frac{1}{3}\right)^k}{\ln(p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2}\cdot \ldots \cdot p_{\alpha_k(x)k})} =$$

$$= \lim_{k\to\infty} \frac{k\ln\frac{1}{3}}{(k-l(k))\ln\frac{1}{3} + \sum\limits_{j=1}^{l(k)} 7^j \ln\frac{1}{7}} = 0.$$

Therefore, $\alpha_\mu(T(Q)) = 0 \cdot \alpha_\lambda(T(Q))$ which is equivalent to the condition $\alpha_0(F_\xi(T(Q))) = 0$, and we conclude that F_ξ does not preserve the Hausdorff dimension. Moreover, F_ξ transforms the superfractal set $T(Q)$ into the anomalously fractal set $T^{'}(Q) = F_\xi(T(Q))$.

The following theorem gives us general necessary conditions for the distribution function F_ξ to be a DP-transformation.

Theorem 2. *If the distribution function F_ξ of a random variable ξ with independent Q-symbols preserves the Hausdorff dimension of any subset of the unit interval, then*

$$\lim_{n\to\infty} \frac{h_1+h_2+\ldots+h_n}{b_1+b_2+\ldots+b_n} = 1. \tag{11}$$

Proof. Let A_ξ be the set of all possible "supports" of the distribution of the random variable ξ, i.e. $A_\xi = \{E: \quad E \in \mathcal{B}, \quad \mu_\xi(E) = 1\}$. The number $\alpha_0(\xi) = \inf\limits_{E\in A_\xi}\{\alpha_0(E)\}$ is said to be the Hausdorff dimension of the probability distribution ξ and a set M with $\alpha_0(M) = \alpha_0(\xi)$ is said to be the minimal dimensional support of the measure μ_ξ. Generally speaking, the Hausdorff dimension of a probability distribution is less or equal to the Hausdorff dimension of the topological support (minimal closed support) of the distribution. Usually, the fractal analysis of minimal dimensional supports is a rather nontrivial problem.

In [1] an explicit formula for the determination of the Hausdorff dimension of probability distributions with independent Q^*-symbols has been found (under the restriction $\inf\limits_{i,j} q_{ij} > 0$). If we put $q_{ij} = q_i, \forall i \in N^0_{s-1}$, then the above mentioned formula gives us the exact value for the Hausdorff dimension of our probability distributions ξ with independent Q-symbols:

$$\alpha_0(\xi) = \lim_{n\to\infty} \frac{h_1+h_2+\ldots+h_n}{b_1+b_2+\ldots+b_n}.$$

If $\alpha_0(\xi) < 1$, then there exists a support E such that such that $\alpha_0(\xi) \le \alpha_0(E) < 1$. Since $\mu_\xi(E) = 1$, we conclude that $\alpha_0(F_\xi(E)) = 1 \ne \alpha_0(E)$, which contradicts the assumption of the theorem.

Corollary 1. *Let* $\inf_{i,j} p_{ij} = p > 0$. *Then the distribution function* F_ξ *of a random variable* ξ *with independent Q-symbols is a DP-transformation of [0,1] if and only if*

$$\lim_{n\to\infty} \frac{h_1 + h_2 + ... + h_n}{b_1 + b_2 + ... + b_n} = 1,$$

i.e., if and only if the Hausdorff dimension of the measure μ_ξ *is equal to 1.*

Corollary 2. *If* $q_i = s^{-1}, \forall i \in N^0_{s-1}$ *(i.e.* ξ *is a random variable with independent s-adic digits), and* $\inf_{i,j} p_{ij} = p > 0$, *then the distribution function* F_ξ *is a DP-transformation if and only if*

$$\lim_{n\to\infty} \frac{h_1 + h_2 + ... + h_n}{n \ln s} = 1.$$

Corollary 3. *If* $\lim_{k\to\infty} p_{ik} = q_i, (\forall i \in N^0_{s-1})$, *then the distribution function* F_ξ *is a DP-transformation of the unit interval.*

Acknowledgment. This work was partly supported by Alexander von Humboldt Foundation and by DFG 436 UKR 113/78,80 projects.

Bibliography

1. Albeverio, S.; Torbin, G., *Fractal properties of singular continuous probability distributions with independent Q* -digits,* Bull. Sci. Math., **129** (2005), no.4, 356-367.
2. Albeverio, S.; Pratsiovytyi, M.; Torbin G., *Transformations preserving the fractal dimension and distribution functions*, submitted to Journal of the London Math. Soc., (SFB 611 Preprint, University of Bonn).
3. Albeverio, S.; Pratsiovytyi, M.; Torbin, G., *Fractal probability distributions and transformations preserving the Hausdorff-Besicovitch dimension,* Ergodic Theory and Dynamical Systems, **24**(2004), No.1, 1-16.
4. Albeverio, S.; Koshmanenko, V.; Torbin, G., *Fine structure of singular continuous spectrum*, Methods Funct. Anal. Topology, **9**(2003), No. 2, 101-119.
5. Albeverio, S.; Pratsiovytyi, M.; Torbin, G., *Fractal probability distributions and transformations preserving the Hausdorff-Besicovitch dimension,* (SFB 256 Preprint, University of Bonn).
6. Goncharenko, Ya.; Pratsiovytyi, M.; Torbin G., *Fractal properties of the set of points of nondifferentiability for absolutely continuous and singular distribution functions,* Theor. Probability and Math. Statist, **65**(2001), 27-34.
7. Billingsley, P., *Hausdorff dimension in probability theory II,* Ill. J. Math., **5**(1961), 291-198.
8. Falconer, K.J., *Fractal geometry*, John Wiley & Sons, 1990.

9. Pratsiovytyi M., *Fractal approach to investigations of singular distributions,* National Pedagogical Univ., Kyiv, 1998.

10. Pratsiovytyi, M., *Distributions of random variables with independent Q-symbols,* Asymptotic and applied problems in the theory of random evolutions, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1990, 92–101.

11. Schweiger, F., *Ergodic theory of fibred systems and metric number theory,* Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

12. Turbin, A.F.; Pratsiovytyi M.V., *Fractal sets, functions, and distributions,* Naukova Dumka, Kiev, 1992.

Institut für Angewandte Mathematik, Universität Bonn, Bonn, Germany; National Pedagogical University, Kyiv, Ukraine; Institute for Mathematics of NASU, Kyiv.
E-mail: torbin@wiener.iam.uni-bonn.de