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INTERPOLATION OF HOMOGENEOUS AND ISOTROPIC RANDOM FIELD IN THE CENTER OF THE SPHERE BY UNIFORM DISTRIBUTED OBSERVATIONS

We consider interpolation of homogeneous and isotropic random field in the center of the sphere by uniform distributed observations on the sphere. The asymptotic behavior of the mean-square interpolation error is investigated. The degree of convergence to zero of the mean-square interpolation error is obtained. Efficient volume of the set of observations is given.

1. Introduction

In the paper [1] the problem of interpolation of homogeneous and isotropic random field in in the center of the circle by observations on the circle was considered. In the paper we consider a particle case of the problem of interpolation of homogeneous and isotropic random field in an arbitrary point inside of the sphere by observations on the sphere which had considered in paper [2]. That is the problem of interpolation of homogeneous and isotropic random field in the center of the sphere by uniform distributed observations on the sphere. Namely, we investigate the order of convergence of the interpolation error to zero. We obtain also efficient volume of the set of observations.

So, we will use the results and notions of the aforementioned papers to investigate our problem.

Give some initial notions.

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2. The interpolation problem

At first, give the following notations.

Let $\xi(x), x \in \mathbb{R}^n$, be a mean-square continuous, homogeneous and wide-isotropic centered random field at the Euclidian space $\mathbb{R}^n$, that is, $E\xi(x) = 0$, $E|\xi(x)|^2 < \infty$ and the correlation function $\varphi(|x-y|) = E\xi(x)\overline{\xi(y)}$ depends only on the distance $|x-y|$ between the points x and y.

Denote by $(r, \overline{\varphi}) = (\rho, \varphi_1, \ldots, \varphi_{n-1})$ the spherical coordinates of the point x and let $S_m^l(\overline{\varphi})$ be orthonormalized spherical harmonics of order m, $h(m,n) = (2m+n-2)\frac{(m+n-3)!}{(n-2)!m!}$ is their quantity.

The following spectral representation takes place [3].

$$\begin{aligned} \xi(x) &= c_n \sum_{m\geq 0} \sum_{l=1}^{h(m,n)} S_m^l(\overline{\varphi})\zeta_m^l(r), \\ \zeta_m^l(r) &= \int_0^\infty \frac{J_{m+\frac{n-2}{2}}(\lambda r)}{(\lambda r)^{\frac{n-2}{2}}} Z_m^l(d\lambda), \end{aligned} \tag{1}$$

where $c_n^2 = 2^{n-1}\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}$, and $Z_m^l(\cdot)$ is a sequence of uncorrelated random measures on $(0, +\infty)$ with spectral measure Φ:

$$\begin{aligned} &EZ_m^l(S) = 0 \\ &EZ_m^l(S_1)\overline{Z_p^q(S_2)} = \delta_m^p \delta_l^q \Phi(S_1 \cap S_2) \end{aligned} \tag{2}$$

for all $S, S_1, S_2 \in \mathcal{B}(0, +\infty)$ and for all $m, l, p, q \geq 0$.

$$b_m(r,\rho) = \int_0^\infty \frac{J_{m+\frac{n-2}{2}}(\lambda r)}{(\lambda r)^{\frac{n-2}{2}}} \cdot \frac{J_{m+\frac{n-2}{2}}(\lambda \rho)}{(\lambda \rho)^{\frac{n-2}{2}}} d\Phi(\lambda), \tag{3}$$

where J_m is the Bessel function of order m, and $b_m(r) = b_m(r,r)$.

From (2)it follows that $\zeta_m^l(r)$ are uncorrelated random variables for different m and l:

$$E\zeta_m^l(r)\overline{\zeta_p^q(r)} = \delta_m^p \delta_l^q b_m(r). \tag{4}$$

Consider the problem of interpolation of field $\xi(x)$ in an arbitrary point y inside of the sphere S_n of radius r by observations on the sphere. This problem reduces to that of finding of the projection $\widehat{\xi(y)}$ of element $\xi(y)$ at the linear space $H_\xi(r)$ formed by the mean-square closure of the linear span of random variables of the form $\xi(x) : |x| = r$:

$$\widehat{\xi(y)} \in H_\xi(r) = Cl\left\{\sum \alpha_k \xi(x_k), |x_k| = r\right\}.$$

Such interpolation formula by Theorem 3 ([3], IV, § 1) has the form

$$\widehat{\xi(y)} = c(y) \int_{\mathrm{S_n}} \xi(x) d\mu_n, \tag{5}$$

where

$$c(y) = \frac{1}{\omega_n} \cdot \frac{b_0(r,\rho)}{b_0(r)}, \tag{6}$$

μ_n is Lebesgue measure on the unit sphere in Euclidian space $\mathbb{R}^n$ and ω_n is that sphere surface square.

The corresponding interpolation error $\sigma^2 = E(\xi(y) - \widehat{\xi(y)})^2$ equals

$$\sigma^2 = \varphi(0) - \frac{c_n^2}{\omega_n} \cdot \frac{b_0^2(r,\rho)}{b_0(r)}. \tag{7}$$

When using the interpolation formula (5) in practice, the integral $\int\limits_{\mathrm{S_n}} \xi(x) d\mu_n$ is replaced by the corresponding integral sum. Hence, we consider, instead of the problem of interpolation based on the space

$$H_\xi(r) = Cl \left\{ \sum \alpha_k \xi(x_k), |x_k| = r \right\},$$

the problem of interpolation of the value $\xi(y)$ from observations on the space

$$H_X(r) = \left\{ \sum \alpha_k \xi(x_k), x_k \in X \right\}, \tag{8}$$

where $X = \{x_1, \ldots, x_N\}$ is some finite subset of the sphere S_n formed by observation points.

So, the integral (5) replaced by finite linear combination (8) of values of the field $\xi(x)$ in points of a certain set X. Therefore, we obtain the problem of accuracy of that replacing, which equivalent to that one of accuracy of the interpolation formula (5) approximation.

Fix a set $X = \{x_1, \ldots, x_N\}$. Since $H_X \subset H_\xi$, the properties of mean-square projection imply that for any estimate $\widehat{\xi_X(y)} \in H_X$ the identity

$$\sigma_X^2 \equiv E(\xi(y) - \widehat{\xi_X(y)})^2 = \sigma^2 + \widehat{\sigma_X^2} \equiv E(\xi(y) - \widehat{\xi(y)})^2 + E(\widehat{\xi(y)} - \widehat{\xi_X(y)})^2 \tag{9}$$

holds.

Definition 1. According to (9), a set $X \subset S_n$ will be called *efficient* if the error of approximation of the integral (5) by the integral sum does not exceed the error of the corresponding interpolation, that is, $\widehat{\sigma_X^2} \leq \sigma^2$. In this case the total variance of the approximate solution of the interpolation problem does not exceed the value $\sigma_X^2 \leq 2\sigma^2$, where σ^2 is evaluated in (7).

The best mean-square interpolation of the value $\xi(y)$ from values from X is of the form

$$\widehat{\xi(y)} = \sum_{k=1}^{N} \alpha_k \xi(x_k), \tag{10}$$

where $\alpha_k, k = 1, \ldots, N$ are weight coefficients.

Denote by

$$F_N(\overline{\varphi}) = \sum_{k:\overline{\varphi_k}\le\overline{\varphi}} \alpha_k, \qquad \varphi_1 \in [0, 2\pi], \quad \varphi_i \in [0, \pi], i = \overline{2, n-1},$$

the cumulative "distribution function" of weights $\{\alpha_1, \ldots, \alpha_N\}$ of points $\{x_1, \ldots, x_N\}$ on the sphere. Here the inequality $\overline{\varphi_k} \le \overline{\varphi}$ means the system of step-by-step inequalities $\varphi_k^{(i)} \le \varphi_i, i = \overline{1, n-1}$.

In the paper [2] we get the following result.

Corollary. *For any sequence of series of sets $X_N = \{x_1, \ldots, x_N\}$, where $x_k = (r, \overline{\varphi_k}), 0 < \varphi_1^{(i)} < \varphi_2^{(i)} < \cdots < \varphi_N^{(i)} = 2\pi, 0 < \varphi_1^{(i)} < \varphi_2^{(i)} < \cdots < \varphi_N^{(i)} = \pi, i = \overline{2, n-1}$, under condition*

$$\sum_{m\ge 0} \sum_{l=1}^{h(m,n)} \left| c_m^l(y) \right|^2 < +\infty, \tag{11}$$

the mean-square approximating error $\widehat{\sigma_X^2}$ tends to zero as $N \to \infty$, provided that weights

$$\alpha_k = C(y, x_k) \prod_{i=1}^{n-1} (\varphi_k^{(i)} - \varphi_{k-1}^{(i)}),$$

and $\max | \prod_{i=1}^{n-1} (\varphi_k^{(i)} - \varphi_{k-1}^{(i)})| \to 0$ *as* $N \to \infty$.

Here

$$C(y, x) = \frac{1}{\omega_n(n-2)} \sum_{m\ge 0} \frac{b_m(r, \rho)}{b_m(r)} (2m + n - 2) C_m^{\frac{n-2}{2}}(\cos\theta), \tag{12}$$

θ is angle between vectors x and y and $C_m^\nu(x), m \ge 0$, are Gegenbouer polynomials which can be defined as coefficients in representation of function

$$(1 - 2tx + t^2)^{-\nu} = \sum_{m=0}^{\infty} C_m^\nu(x) t^m, \tag{13}$$

that is generatrices for those polynomials.

3. Asymptotic behavior of the error for the uniform distribution

Let us consider the case where the points of the set $X_N = \{x_1, \ldots, x_N\}$ are uniformly distributed on the sphere S_n, that is, $\varphi_k^{(1)} = \frac{2\pi(k_1-1)}{M}$, $\varphi_k^{(i)} = \frac{\pi(k_i-1)}{M}, i = \overline{2, n}, k_i = \overline{1, M}$. So, we have $N = M^{n-1}$ points. From Corollary 1 follows, that choice of interpolation weights of the form $\alpha_k =$

$\frac{\varphi(r)l(\overline{\varphi}_k)}{c_n^2 b_0(r)}\prod_{i=1}^{n-1}(\varphi_k^{(i)}-\varphi_{k-1}^{(i)})$, where $l(\overline{\varphi})=(\sin\varphi_2)^{n-2}(\sin\varphi_3)^{n-3}\dots\sin\varphi_{n-1}$ and $\varphi(r) = 2^{\frac{n-2}{2}}\Gamma(n/2)\int\limits_0^\infty \mathrm{J}_{\frac{n-2}{2}}(\lambda r)(\lambda r)^{\frac{2-n}{2}}d\Phi(\lambda)$ is correlation function, minimizes (moreover, reduces to zero) limit error $\widehat{\sigma_\infty^2}$. Let's investigate the degree of convergence of $\widehat{\sigma_X^2}$ to zero in this case.

Since the random variables ζ_m^l are uncorrelated in accordance with (4) and interpolation formula for $\widehat{\xi(y)}$ has the form (10), taking into account the spectral representation (1), we have

$$\begin{aligned}&\widehat{\sigma_X^2} = E(\widehat{\xi(y)}-\widehat{\xi_X(y)})^2 =\\ &= c_n^2\sum_{m\geq 0}\sum_{l=1}^{h(m,n)}\left|\sum_{k=1}^N \alpha_k S_m^l(\overline{\varphi_k})-\frac{b_m(r,\rho)}{b_m(r)}S_m^l(\overline{\psi})\right|^2 E|\zeta_m^l(r)|^2\end{aligned} \tag{14}$$

Evaluate equality in (14), taking into account the fact that the field is extrapolating in the center of the sphere (it means that $\rho=0$ and $b_m(r,\rho)=\delta_m^0\frac{\omega_n}{c_n^2}\varphi(r)$).

$$\begin{aligned}&\widehat{\sigma_X^2} = E(\widehat{\xi(y)}-\widehat{\xi_X(y)})^2 = c_n^2\sum_{m\geq 0}\sum_{l=1}^{h(m,n)}\left|\sum_{k=1}^N \alpha_k S_m^l(\overline{\varphi_k})-\frac{b_m(r,\rho)}{b_m(r)}S_m^l(\overline{\psi})\right|^2 b_m(r)\\ &= \frac{c_n^2}{\omega_n}(\sum_{k=1}^N\alpha_k-\frac{\omega_n\varphi(r)}{c_n^2 b_0(r)})^2 b_0(r)+\\ &+\frac{c_n^2}{\omega_n}\frac{2}{(n-2)}\sum_{k,j=1}^N\alpha_k\alpha_j\int\limits_0^\infty\sum_{m\geq 1}(m+\frac{n-2}{2})\frac{J^2_{m+\frac{n-2}{2}}(\lambda\ r)}{(\lambda\ r)^{n-2}}C_m^{\frac{n-2}{2}}(\cos\theta_{kj})d\Phi(\lambda)=\\ &=\frac{c_n^2}{\omega_n}(S(\alpha)-\frac{\omega_n\varphi(r)}{c_n^2 b_0(r)})^2 b_0(r)-\frac{c_n^2}{\omega_n}b_0(r)S^2(\alpha)+\\ &+\frac{c_n}{\sqrt{\omega_n}}\sum_{k,j=1}^N\alpha_k\alpha_j\int\limits_0^\infty\frac{J_{\frac{n-2}{2}}(\lambda R_{kj})}{(\lambda R_{kj})^{\frac{n-2}{2}}}d\Phi(\lambda),\end{aligned} \tag{15}$$

where $S(\alpha)=\sum\limits_{k=1}^N\alpha_k$, $R_{kj}=2r\sin\frac{\theta_{kj}}{2}$, and θ_{kj} are angles between vectors x_k and x_j.

Substitute values of interpolation weights α_k in last equality.

$$\begin{aligned}&\widehat{\sigma_X^2} = \frac{c_n^2}{\omega_n}(\frac{\omega_n\varphi(r)}{c_n^2 b_0(r)}-\frac{\omega_n\varphi(r)}{c_n^2 b_0(r)})^2 b_0(r)-\frac{c_n^2}{\omega_n}b_0(r)\left(\frac{\omega_n\varphi(r)}{c_n^2 b_0(r)}\right)^2+\\ &+\frac{c_n}{\sqrt{\omega_n}}\sum_{k,j=1}^N\left(\frac{2\pi^{n-1}\varphi(r)}{c_n^2 N b_0(r)}\right)^2 l(\overline{\varphi}_k)l(\overline{\varphi}_j)\int\limits_0^\infty\frac{J_{\frac{n-2}{2}}(2\lambda r\sin\frac{\theta_{kj}}{2})}{(2\lambda r\sin\frac{\theta_{kj}}{2})^{\frac{n-2}{2}}}d\Phi(\lambda)=\\ &=-\frac{\omega_n\varphi^2(r)}{c_n^2 b_0(r)}+\frac{c_n}{\sqrt{\omega_n}}\sum_{k,j=1}^N\left(\frac{\varphi(r)2\pi^{n-1}}{b_0(r)c_n^2 N}\right)^2 l(\overline{\varphi}_k)l(\overline{\varphi}_j)\int\limits_0^\infty\frac{J_{\frac{n-2}{2}}(2\lambda r\sin\frac{\theta_{kj}}{2})}{(2\lambda r\sin\frac{\theta_{kj}}{2})^{\frac{n-2}{2}}}d\Phi(\lambda).\end{aligned}$$

Denote by $F(\overline{\varphi})=\frac{\varphi(r)}{b_0(r)c_n^2}\int_0^{\overline{\varphi}}\mathrm{l}(\overline{\mathrm{u}})d\overline{\mathrm{u}}$ distribution function of weights $\{\alpha_1,\dots,$

$\alpha_N\}$. If we pass in (15) to the limit as $N \to \infty$ we obtain

$$\frac{\omega_n \varphi^2(r)}{c_n^2 b_0(r)} = \frac{c_n}{\sqrt{\omega_n}} \int_0^\infty \left(\int_{\Pi_n} \int_{\Pi_n} \frac{J_{\frac{n-2}{2}}(2\lambda r\sqrt{2(1-\cos\theta)})}{(2\lambda r\sqrt{2(1-\cos\theta)})^{\frac{n-2}{2}}} dF(\overline{u}) dF(\overline{v}) \right) d\Phi(\lambda) =$$
$$= \frac{c_n}{\sqrt{\omega_n}} \left(\frac{\varphi(r)}{b_0(r) c_n^2} \right)^2 \int_0^\infty \left(\int_{\Pi_n} \int_{\Pi_n} l(\overline{u}) l(\overline{v}) \frac{J_{\frac{n-2}{2}}(2\lambda r\sqrt{2(1-\cos\theta)})}{(2\lambda r\sqrt{2(1-\cos\theta)})^{\frac{n-2}{2}}} d\overline{u} d\overline{v} \right) d\Phi(\lambda),$$

where θ is the angle between $\overline{u}$ and $\overline{v}$.

Recall that $l(\overline{u}) = (\sin u_2)^{n-2} (\sin u_3)^{n-3} \ldots \sin u_{n-1}$, and

$$\begin{aligned}
\cos\widehat{(\overline{u}, \overline{v})} &= \cos u_2 \cos v_2 + \sin u_2 \sin v_2 \cos u_3 \cos v_3 + \\
&+ \sin u_2 \sin v_2 \sin u_3 \sin v_3 \cos u_4 \cos v_4 + \ldots + \\
&+ (\cos u_1 \cos v_1 + \sin u_1 \sin v_1) \prod_{i=2}^{n-1} \sin u_i \sin v_i = \\
&= \cos u_2 \cos v_2 + \sin u_2 \sin v_2 \cos u_3 \cos v_3 + \\
&+ \sin u_2 \sin v_2 \sin u_3 \sin v_3 \cos u_4 \cos v_4 + \ldots + \\
&+ \cos(u_1 - v_1) \prod_{i=2}^{n-1} \sin u_i \sin v_i
\end{aligned}$$

depends on $\cos(u_1 - v_1)$ and variables $\check{u} = (u_2, \ldots, u_{n-1})$ and $\check{v} = (v_2, \ldots, v_{n-1})$.

Denote $H(\cos(u_1 - v_1), \check{u}, \check{v}) = \frac{c_n}{\sqrt{\omega_n}} \int_0^\infty l(\overline{u}) l(\overline{v}) \frac{J_{\frac{n-2}{2}}(2\lambda r\sqrt{2(1-\cos\theta)})}{(2\lambda r\sqrt{2(1-\cos\theta)})^{\frac{n-2}{2}}} d\Phi(\lambda) = l(\overline{u}) l(\overline{v}) \varphi(2r\sqrt{2(1-\cos\theta)})$, $\check{k} = (k_2, \ldots, k_{n-1})$, $\Pi_{n-1} = [0, \pi]^{n-2}$, and write

$$\widehat{\sigma_X^2} = \left(\frac{\varphi(r)}{b_0(r) c_n^2} \right)^2 \left(\left(\frac{\pi^{n-2}}{M^{n-2}}\right)^2 \sum_{\check{k}, \check{j}=1}^{M} \frac{(2\pi)^2}{M^2} \sum_{k_1, j_1=1}^{M} H(\cos(\varphi_k^{(1)} - \varphi_j^{(1)}), \check{\varphi}_k, \check{\varphi}_j) - \right.$$
$$\left. - \int_{\Pi_{n-1}} \int_{\Pi_{n-1}} \int_0^{2\pi} \int_0^{2\pi} H(\cos(u_1 - v_1), \check{u}, \check{v}) du_1 dv_1 d\check{u} d\check{v} \right). \tag{16}$$

Taking into account evenness and summeriness of the function $H(\cos(u_1 - v_1), \check{u}, \check{v})$ write

$$\frac{(2\pi)^2}{M^2} \sum_{k_1, j_1=1}^{M} H(\cos(\varphi_k^{(1)} - \varphi_j^{(1)}), \check{\varphi}_k, \check{\varphi}_j) = \frac{(2\pi)^2}{M^2} \sum_{k_1, j_1=1}^{M} H(\cos(k-j)\tfrac{2\pi}{M}, \check{\varphi}_k, \check{\varphi}_j) =$$
$$= \frac{(2\pi)^2}{M} \sum_{|s|<M} (1 - \tfrac{|s|}{M}) H(\cos s\tfrac{2\pi}{M}, \check{\varphi}_k, \check{\varphi}_j) = 2\frac{(2\pi)^2}{M} \sum_{s=0}^{M-1} (1 - \tfrac{s}{M}) H(\cos s\tfrac{2\pi}{M}, \check{\varphi}_k, \check{\varphi}_j),$$

and also

$$\int_0^{2\pi} \int_0^{2\pi} H(\cos(u_1 - v_1), \check{u}, \check{v}) du_1 dv_1 = 2(2\pi)^2 \int_0^1 (1 - x) H(\cos 2\pi x, \check{u}, \check{v}) dx.$$

Denote $G(x,\check u,\check v)=(1-x)H(\cos 2\pi x,\check u,\check v)$, $\Pi_{\check k}=\prod\limits_{i=2}^{n-1}[\frac{(k_i-1)\pi}{M},\frac{k_i\pi}{M}]$, $\Pi_{1/M}=[\frac{\pi}{M},\frac{\pi}{M}]^{n-2}$.

Evaluate the equality in (16).

$$\widehat{\sigma_X^2}=2\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\left((\frac{\pi^{n-2}}{M^{n-2}})^2\sum_{\check k,\check j=1}^{M}\frac{1}{M}\sum_{s=0}^{M-1}G(\tfrac{s}{M},\check\varphi_k,\check\varphi_j)-\right.$$

$$\left.-\int_{\Pi_{n-1}}\int_{\Pi_{n-1}}\int_0^1 \mathrm{G(x,\check u,\check v)dxd\check ud\check v}\right)=$$

$$=-2\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\sum_{\check k,\check j=1}^{M}\sum_{s=0}^{M-1}\int_{\Pi_{\check k}}\int_{\Pi_{\check j}}\int_{\frac{s}{M}}^{\frac{s+1}{M}}\left(\mathrm{G(x,\check u,\check v)-G(\tfrac{s}{M},\check\varphi_k,\check\varphi_j)}\right)\mathrm{dxd\check ud\check v}=$$

$$=-2\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\sum_{\check k,\check j=0}^{M-1}\sum_{s=0}^{M-1}\int_{\Pi_{1/M}}\int_{\Pi_{1/M}}\int_0^{\frac{1}{M}}\left(\mathrm{G(x+\tfrac{s}{M},\check u+\check k\phi,\check v+\check j\phi)}-\right.$$

$$\left.-G(\tfrac{s}{M},\check k\phi,\check j\phi)\right)dxd\check ud\check v,$$

where $\phi=(\underbrace{\pi/M,\dots,\pi/M}_{n-2})$.

$$\widehat{\sigma_X^2}=-2\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\sum_{\check k,\check j=0}^{M-1}\sum_{s=0}^{M-1}\int_{\Pi_{\frac{1}{M}}}\int_{\Pi_{\frac{1}{M}}}\int_0^{\frac{1}{M}}\left(\mathrm{G'(\tfrac{s}{M},\check k\phi,\check j\phi)(x,\check u,\check v)+o_1(\tfrac{1}{M})}\right)\mathrm{dxd\check ud\check v}.$$

Since

$$\int_{\Pi_{1/M}}\int_{\Pi_{1/M}}\int_0^{\frac{1}{M}}\mathrm{G'_{u_i}((\tfrac{s}{M},\check k\phi,\check j\phi)u_idxd\check ud\check v=G'_{u_i}(\tfrac{s}{M},\check k\phi,\check j\phi)\frac{\pi}{2M^2}}\left(\frac{\pi^{\mathrm{n-2}}}{\mathrm{M^{n-2}}}\right)^2,\quad i\geq 2,$$

(similarly for $\int_{\Pi_{1/M}}\int_{\Pi_{1/M}}\mathrm{G'_{v_i}(\overline{k}\phi;\overline{j}\phi)v_id\overline{u}d\overline{v}}$),

$$\int_{\Pi_{1/M}}\int_{\Pi_{1/M}}\int_0^{\frac{1}{M}}\mathrm{G'_x((\tfrac{s}{M},\check k\phi,\check j\phi)u_idxd\check ud\check v=G'_x(\tfrac{s}{M},\check k\phi,\check j\phi)\frac{1}{2M^2}}\left(\frac{\pi^{\mathrm{n-2}}}{\mathrm{M^{n-2}}}\right)^2,$$

we have

$$\widehat{\sigma_X^2}=-\frac{1}{M}\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\sum_{\check k,\check j=0}^{M-1}\sum_{s=0}^{M-1}\left(\frac{\pi}{M}\left(\frac{\pi^{n-2}}{M^{n-2}}\right)^2\sum_{i=2}^{n-1}(G'_{u_i}(\tfrac{s}{M},\check k\phi,\check j\phi)+\right.$$

$$\left.+G'_{v_i}(\tfrac{s}{M},\check k\phi,\check j\phi))+\frac{1}{M}\left(\frac{\pi^{n-2}}{M^{n-2}}\right)^2G'_x(\tfrac{s}{M},\check k\phi,\check j\phi)\right)+o_2(\tfrac{1}{M})=$$

$$=-\frac{1}{M}\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2\int_{\Pi_{n-1}}\int_{\Pi_{n-1}}\int_0^1\left\{\pi\sum_{\mathrm{i=2}}^{\mathrm{n-1}}(\mathrm{G'_{u_i}(x,\check u,\check v)+G'_{v_i}(x,\check u,\check v)})+\right.$$

$$\left.+G'_x(x,\check u,\check v)\right\}dxd\check ud\check v+o_2(\tfrac{1}{M}).$$

Note that for all $i\geq 2$ integral $\int_0^\pi \mathrm{G'_{u_i}(x,\check u,\check v)du_i}=\mathrm{G(x,\check u,\check v)}\mid_0^\pi=0$, since $\sin u_i$ is a component of the function $G(x,\check u,\check v)$.

(Similarly, $\int_0^{\pi} \mathrm{G}'_{\mathrm{v_i}}(\mathrm{x},\check{\mathrm{u}},\check{\mathrm{v}})\mathrm{dv_i} = 0.$)

$$\int_0^1 \mathrm{G}'_{\mathrm{x}}(\mathrm{x},\check{\mathrm{u}},\check{\mathrm{v}})\mathrm{du_i} = \mathrm{G}(1,\check{\mathrm{u}},\check{\mathrm{v}}) - \mathrm{G}(0,\check{\mathrm{u}},\check{\mathrm{v}}) = -\mathrm{H}(1,\check{\mathrm{u}},\check{\mathrm{v}})$$

So,

$$\widehat{\sigma_X^2} = \frac{1}{M}\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2 \int\limits_{\Pi_{n-1}}\int\limits_{\Pi_{n-1}} \mathrm{H}(1,\check{\mathrm{u}},\check{\mathrm{v}})\mathrm{d}\check{\mathrm{u}}\mathrm{d}\check{\mathrm{v}} + \mathrm{o}(\frac{1}{\mathrm{M}}). \qquad (17)$$

Taking into account obtained result we formulate the following theorem.

Theorem 1. *Let the sphere S_n be uniformly divided by the set $X_N = \{x_1,\ldots,x_N\}$, $N = M^{n-1}$, of points with spherical angles $\varphi_k^{(1)} = 2\pi(k_1 - 1)/M, \varphi_k^{(i)} = \pi(k_i - 1)/M, i = \overline{2, n-1}$. Let the coefficients $\alpha_k = \frac{\varphi(r)l(\overline{\varphi}_k)}{b_0(r)c_n^2}\frac{2\pi^{n-1}}{N}$, where $l(\overline{\varphi}) = (\sin\varphi_2)^{n-2}(\sin\varphi_3)^{n-3}\ldots\sin\varphi_{n-1}$. Then the asymptotic mean-square error of the estimate*

$$\widehat{\xi_X(0)} = \frac{\varphi(r)}{b_0(r)c_n^2}\cdot\frac{2\pi^{n-1}}{N}\sum_{k=0}^{N} l(\overline{\varphi}_k)\xi(x_k)$$

is

$$\widehat{\sigma_X^2} = \frac{1}{\sqrt[n-1]{N}}\left(\frac{\varphi(r)}{b_0(r)c_n^2}\right)^2 Var\int\limits_{\mathrm{S_n}} \xi(0,\check{\mathrm{u}})\mathrm{d}\mu_{\mathrm{n}}(\overline{\mathrm{u}}) + \mathrm{o}(\frac{1}{\sqrt[n-1]{N}}). \qquad (18)$$

Proof. Taking into account the value $H(\cos(u_1 - v_1), \check{u}, \check{v})$, from (17) obtain

$$\widehat{\sigma_X^2} = \frac{1}{M}\left(\frac{2\pi\varphi(r)}{b_0(r)c_n^2}\right)^2 \int\limits_{\Pi_{n-1}}\int\limits_{\Pi_{n-1}} \mathrm{l}(\overline{\mathrm{u}})\mathrm{l}(\overline{\mathrm{v}})\varphi(2\mathrm{r}\sqrt{2(1-\cos(\widehat{\check{\mathrm{u}},\check{\mathrm{v}}}))})\mathrm{d}\check{\mathrm{u}}\mathrm{d}\check{\mathrm{v}} + \mathrm{o}(\frac{1}{\mathrm{M}}) =$$
$$= \frac{1}{M}\left(\frac{\varphi(r)}{b_0(r)c_n^2}\right)^2 \int\limits_{\Pi_n}\int\limits_{\Pi_n} \mathrm{l}(\overline{\mathrm{u}})\mathrm{l}(\overline{\mathrm{v}})\varphi(|\overline{\mathrm{u}}-\overline{\mathrm{v}}|)\,|_{\mathrm{u_1=v_1=0}}\,\mathrm{d}\overline{\mathrm{u}}\mathrm{d}\overline{\mathrm{v}} + \mathrm{o}(\frac{1}{\mathrm{M}}) =$$
$$= \frac{1}{M}\left(\frac{\varphi(r)}{b_0(r)c_n^2}\right)^2 \int\limits_{\mathrm{S_n}}\int\limits_{\mathrm{S_n}} \varphi(|\overline{\mathrm{u}}-\overline{\mathrm{v}}|)\,|_{\mathrm{u_1=v_1=0}}\,\mathrm{d}\mu_{\mathrm{n}}(\overline{\mathrm{u}})\mathrm{d}\mu_{\mathrm{n}}(\overline{\mathrm{v}}) + \mathrm{o}(\frac{1}{\mathrm{M}}) =$$
$$= \frac{1}{M}\left(\frac{\varphi(r)}{b_0(r)c_n^2}\right)^2 Var\int\limits_{\mathrm{S_n}} \xi(0,\check{\mathrm{u}})\mathrm{d}\mu_{\mathrm{n}}(\overline{\mathrm{u}}) + \mathrm{o}(\frac{1}{\mathrm{M}}).$$

Corollary 1. *The uniformly distributed set $X_N = \{x_1,\ldots,x_N\}$, $N = M^{n-1}$, of observations on the sphere S_n with interpolation weights $\alpha_k = \frac{\varphi(r)l(\overline{\varphi}_k)}{b_0(r)c_n^2}\frac{2\pi^{n-1}}{N}$ is efficient starting from the volume*

$$N = \left\{\left(\frac{\varphi(r)}{b_0(r)c_n^2}\right)^2 \cdot \frac{Var\int\limits_{\mathrm{S_n}} \xi(0,\check{\mathrm{u}})\mathrm{d}\mu_{\mathrm{n}}(\overline{\mathrm{u}})}{\varphi(0) - \frac{\varphi(r)}{b_0(r)}}\right\}^{n-1}.$$

Proof. The proof follows directly from the Definition 1, by which $\widehat{\sigma_X^2} \simeq \sigma^2$, and from formulas (18) and (7).

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