Theory of Stochastic Processes
Vol.13 (29), no.1-2, 2007, pp.152-165

YULIA MISHURA AND SERGIY POSASHKOV

EXISTENCE AND UNIQUENESS OF SOLUTION OF MIXED STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION AND WIENER PROCESS

The existence and uniqueness of solution of stochastic differential equation driven by standard Brownian motion and fractional Brownian motion with Hurst parameter $H \in (3/4, 1)$ is established.

1. Introduction

We will consider the following stochastic differential equation with nonhomogeneous coefficients, defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,T]), \mathbf{P})$:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t c(s, X_s)dB_s^H, \quad t \in [0,T], \tag{1}$$

where X_0 is $\mathcal{F}_0$-measurable random variable, $\mathsf{E}X_0^2 < \infty$, $W = (W_t, t \in [0,T])$ is standard Brownian motion (sBm), $B^H = (B_t^H, t \in [0,T])$ is fractional Brownian motion (fBm) with Hurst parameter $H \in (3/4, 1)$. Coefficients $a, b, c : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are deterministic functions, which satisfy well-known conditions of existence of integrals in the right-hand of (1), where the integral $\int_0^t c(s, X_s)dB_s^H$ is considered in pathwise sense.

If $b = 0$ we obtain the equation, which contains only fBm. The conditions of existence and uniqueness of solution of such equations were shown in [1]. The case $b(t,x) = bx$ and $c(t,x) = cx$ was considered in [7].

Using the similarity to sBm of process $M_t^{H,\varepsilon} := V_t + 1/\varepsilon B_t^H, \varepsilon > 0$, for $H \in (3/4, 1)$, which was proven in [2], and the representation of processes similar to sBm from [8], the existence and uniqueness of solution of auxiliary stochastic differential equation

$$X_t^N = X_0 + \int_0^t a(s, X_s^N)dt + \int_0^t b(s, X_s^N)dW_t$$
$$+ \int_0^t c(s, X_s^N)dB_s^H + \frac{1}{N}\int_0^t c(s, X_s^N)dV_s, t \in [0,T], \tag{2}$$

2000 *Mathematics Subject Classifications.* 60G15, 60H05, 60H10

Key words and phrases. Stochastic differential equation, fractional Brownian motion

where $V = (V_t, t \in [0,T])$ is another standard Brownian motion, independent on W and B^H, for any $N \in \mathbb{N}$, was proved in the paper [3].

The main aim of present work is to find the solution of equation (1) as the limit in some complete space of solutions of (2) when N tends to ∞.

2. Main part

Let the coefficients of equation (1) satisfy, in addition, the following assumptions

(A) There exists $A > 0$ such that $|a(t,x)| \le A$, $|b(t,x)| \le A$, $|c(t,x)| \le A$ $\forall t \in [0,T]$, $\forall x \in \mathbb{R}$.

(B) There exists $L > 0$ such that

$$(a(t,x)-a(t,y))^2+(b(t,x)-b(t,y))^2+(c(t,x)-c(t,y))^2 \le L^2(x-y)^2,$$

for $\forall t \in [0,T]$, $\forall x, y \in \mathbb{R}$.

(C) The function $c(t,x)$ is differentiable by c, and there exist constants $B > 0$ and $\beta \in (1-H,1)$ such that $\forall s,t \in [0,T]$, $\forall x \in \mathbb{R}$

$$|c(s,x)-c(t,x)| + |\partial_x c(s,x) - \partial_x c(t,x)| \le Bs-t|^{\beta}.$$

(D) Holder continuity of $\partial_x c(t,x)$ in x :

$$|\partial_x c(t,x) - \partial_x c(t,y)| \le D|x-y|^{\rho},$$

for $\forall t \in [0,T]$, $\forall x,y \in \mathbb{R}$ with some parameter $\rho \in (3/2-H,1)$.

Note that this conditions imply the conditions from [3] of existence and uniqueness of solution of equation (2).

Let consider for any $1-H < \alpha < \min(1/2, \beta, \rho-1/2)$ the space of Besov type

$$W_\alpha([0,T]) := \{Y = Y_t(\omega) : (t,\omega) \in [0,T] \times \Omega, ||Y||_\alpha < \infty\} \tag{3}$$

with the norm

$$||Y||_\alpha := \sup_{t\in[0,T]} \left(\mathsf{E}(Y_t)^2 + \mathsf{E}\left(\int_0^t \frac{|Y_t - Y_s|}{(t-s)^{1+\alpha}} ds \right)^2 \right), \tag{4}$$

and prove that the solution of SDE (2) belongs to this space for any $N \in \mathbb{N}$.

We shall denote different constants as C if it is unimportant for stated results.

From [5] we have for pathwise integral $\int_0^t f(s)dB_s^H$ the representation

$$\int_0^t f(s)dB_s^H = \int_0^t D_{0+}^{\alpha} f(s) D_{t-}^{1-\alpha} B_{t-}^H(s) ds,$$

therefore this integral can be estimated as

$$\left|\int_0^t f(s)dB_s^H\right| \le C_t(\omega)\left(\int_0^t \frac{|f(s)|}{s^\alpha}ds + \int_0^t\int_0^r \frac{|f(r)-f(u)|}{(r-u)^{1+\alpha}}dudr\right), \quad (5)$$

where

$$C_t(\omega) = \sup_{0\le u\le s\le t} |D_{s-}^{1-\alpha}B_{s-}^H(u)|. \quad (6)$$

The existence of all moments $\mathsf{E}|C_t(\omega)|^p$ for any $p \ge 1$ has also been proved in [1].

Lemma 1. *The process $|C_t(\omega)|$ is dominated by some continuous process.*

Proof. To prove this statement we estimate $|D_{t-}^{1-\alpha}B_{t-}^H(s)|$:

$$|D_{t-}^{1-\alpha}B_{t-}^H(s)| \le \frac{1}{\Gamma(\alpha)}\left(\frac{|B_t^H - B_s^H|}{(t-s)^{1-\alpha}} + (1-\alpha)\int_s^t \frac{|B_s^H - B_u^H|}{(s-u)^{2-\alpha}}du\right).$$

Using Garsia - Rodemich - Rumsey inequality from [1] that has a form

$$|f(t)-f(s)|^p \le C_{\lambda,p}|t-s|^{\lambda p-1}\int_0^t\int_0^t \frac{|f(x)-f(y)|^p}{|x-y|^{\lambda p+1}}dxdy \quad (7)$$

with $f \in C[0,T], t,s \in [0,T], p \ge 1$ and $\lambda > p^{-1}$, we obtain

$$|B_t^H - B_s^H| \le C_{\lambda,p}|t-s|^{\lambda-1/p}\left(\int_0^t\int_0^t \frac{|B_u^H - B_v^H|^p}{|u-v|^{\lambda p+1}}dudv\right)^{1/p}.$$

Let $\lambda = H - \varepsilon/2$, $p = 2/\varepsilon$ for $0 < \varepsilon < H + \alpha - 1$, then

$$|B_t^H - B_s^H| \le C_\varepsilon|t-s|^{H-\varepsilon}\left(\int_0^t\int_0^t \frac{|B_u^H - B_v^H|^{2/\varepsilon}}{|u-v|^{2H/\varepsilon}}dudv\right)^{\varepsilon/2},$$

note that $1-\alpha < H-\varepsilon$. Denote

$$\psi_{t,\varepsilon} = \left(\int_0^t\int_0^t \frac{|B_u^H - B_v^H|^{2/\varepsilon}}{|u-v|^{2H/\varepsilon}}dudv\right)^{\varepsilon/2}, \quad (8)$$

The process $\psi_{t,\varepsilon}$ is continuous and nondecreasing in t, and $\mathsf{E}\psi_{t,\varepsilon}^q < \infty$ for any $q > 0$ and any $t \in [0,T]$ [1].

So,

$$\frac{|B_t^H - B_s^H|}{(t-s)^{1-\alpha}} \le C_{H,\varepsilon}(t-s)^{H-\varepsilon-1+\alpha}\psi_{t,\varepsilon} \le C\psi_{t,\varepsilon},$$

and

$$\int_s^t \frac{|B_u^H - B_s^H|}{(u-s)^{2-\alpha}}du \le C_{H,\varepsilon}\int_s^t (u-s)^{H-\varepsilon-2+\alpha}\psi_{t,\varepsilon}du$$
$$\le C_{H,\varepsilon}(t-s)^{H+\alpha-1-\varepsilon}\psi_{t,\varepsilon} \le C\psi_{t,\varepsilon},$$

where $H+\alpha-1-\varepsilon>0$.

So $\sup\limits_{0\le u\le s\le t}|D_{s-}^{1-\alpha}B_{s-}^H(u)|\le C\psi_{t,\varepsilon}$, and the statement of lemma follows from continuity and strictly increasing property of $\psi_{t,\varepsilon}$. □

Introduce the random variable

$$\overline{C}(\omega):=\sup_{0\le t\le T}C_t(\omega). \tag{9}$$

Note that $\overline{C}(\omega)\le\psi_{T,\varepsilon}$ and $\mathsf{E}|\overline{C}(\omega)|^q<\infty$ for any $q\ge1$.

First of all we prove the Hölder continuity of the solution of equation (2).

Theorem 2. *For any* $\delta\in(0,1/2)$ *the solution of equation* (2) *is Hölder continuous with parameter* $1/2-\delta$.

Proof. At first, establish Hölder properties of the integral $\{\int_0^t b_s dW_s, t\in[0,T]\}$, where b_s is a predictable bounded process. For any $0<\delta<1/4$ put $p=\frac{2}{\delta}$, $\theta=1/2-\delta/2$ in Garsia - Rodemich - Rumsey inequality (7). Then

$$\left|\int_s^t b_u dW_u\right|\le C_\delta|t-s|^{1/2-\delta}\xi_{t,\delta}^b,$$

where

$$\xi_{t,\delta}^b:=\left(\int_0^t\int_0^t\frac{|\int_x^y b_u dW_u|^p}{|x-y|^{\theta p+1}}dxdy\right)^{1/p}=\left(\int_0^t\int_0^t\frac{|\int_x^y b_u dW_u|^{2/\delta}}{|x-y|^{1/\delta}}dxdy\right)^{\delta/2} \tag{10}$$

and for any $q>p$ from Hölder and Burkholder inequalities

$$\begin{aligned}\mathsf{E}(\xi_{t,\delta}^b)^q&=\mathsf{E}\left(\int_0^t\int_0^t\frac{|\int_x^y b_u dW_u|^{2/\delta}}{|x-y|^{1/\delta}}dxdy\right)^{q\delta/2}\\&\le\left(\int_0^t\int_0^t\frac{\mathsf{E}|\int_x^y b_u dW_u|^q dxdy}{|x-y|^{q/2}}\right)t^{q\delta-2}\\&\le C_q\left(\int_0^t\int_0^t\frac{|\int_x^y b_u^2 du|^{q/2}dxdy}{|x-y|^{q/2}}\right)t^{q\delta-2}\le\Theta_{t,q}.\end{aligned}$$

So, the process $\xi_{t,\delta}^b$ from (10) is continuous, strictly increasing and has the moments of any order.

Now consider $|X_r^N - X_z^N|$ for $0 < z < r < T$:

$$|X_r^N - X_z^N| \le |\int_z^r a(u, X_u)du| + |\int_z^r b(u, X_u)dW_u| + \frac{1}{N}|\int_z^r c(u, X_u)dV_u|$$
$$+ |\int_z^r c(u, X_u)dB_u^H| \le A(r-z) + C\xi_{r,\delta}^b |r-z|^{1/2-\delta} + \frac{C}{N}\xi_{r,\delta}^c |r-z|^{1/2-\delta}$$
$$+ C_r(\omega)\int_z^r \frac{|c(u, X_u^N)|du}{u^\alpha} + C_r(\omega)\int_z^r\int_z^u \frac{|c(u, X_u^N) - c(v, X_v^N)|}{(u-v)^{1+\alpha}}dvdu$$
$$\le C_r'(\omega)(r-z)^{1/2-\delta} + C_r'(\omega)\int_z^r\int_z^u \frac{|X_u^N - X_v^N|}{(u-v)^{1+\alpha}}dvdu,$$

where

$$C_t'(\omega) := C(\psi_{t,\varepsilon} \vee \xi_{t,\delta}^b \vee \xi_{t,\delta}^c \vee 1), \tag{11}$$

$\psi_{t,\varepsilon}$ is defined by (8), $C_t'(\omega) \le C_T'(\omega)$ and $C_T'(\omega)$ has the moments of any order.

Therefore, for $\delta < 1/2 - \alpha$

$$\phi_{r,s} := \int_s^r \frac{|X_r^N - X_z^N|}{(r-z)^{1+\alpha}}dz \le C_r'(\omega)\int_s^r (r-z)^{-1/2-\delta-\alpha}dz$$
$$+ C_r'(\omega)\int_s^r \frac{1}{(r-z)^{1+\alpha}}\int_z^r\int_z^u \frac{|X_u^N - X_v^N|}{(u-v)^{1+\alpha}}dvdudz$$
$$= C_r'(\omega)(r-s)^{1/2-\alpha-\delta} + C_r'(\omega)\int_s^r (r-u)^{-\alpha}\int_s^u \frac{|X_u^N - X_v^N|}{(u-v)^{1+\alpha}}dvdu$$
$$\le C_r'(\omega)(r-u)^{1/2-\alpha-\delta} + C_r'(\omega)\int_s^r (r-u)^{-\alpha}\phi_{u,s}du.$$

From modified Gronwall inequality (Lemma 7.6 [1])

$$\phi_{r,s} \le C_r'(\omega)(r-s)^{1/2-\alpha-\delta}\exp[C_r'(\omega)^{\frac{1}{1-\alpha}}].$$

Return to $|X_r^N - X_z^N|$:

$$|X_r^N - X_z^N| \le C_r'(\omega)(r-z)^{1/2-\delta}$$
$$+ C_r'(\omega)\exp[C_r'(\omega)^{\frac{1}{1-\alpha}}]\int_z^r (v-z)^{1/2-\alpha-\delta}dv \le \widetilde{C}_r(\omega)(r-z)^{1/2-\delta}, \tag{12}$$

where $\widetilde{C}_r(\omega) = C_r'(\omega)\exp[C_r'(\omega)^{\frac{1}{1-\alpha}}]$, and the theorem is proved for $0 < \delta < 1/2 - \alpha$ consequently for $0 < \delta < 1/2$. □

Introduce the random variable

$$\widetilde{C}(\omega) := \sup_{0 \le t \le T} \widetilde{C}_t(\omega), \tag{13}$$

it also has moments of any order.

Now prove that the solution of (2) belongs to space (3) with norm (4) for all $N \in \mathbb{N}$.

Theorem 3. *Under assumptions (A) - (D) the solution of equation* (2) *belongs to space of Besov type* (3) *with norm* (4) *for all* $N \in \mathbb{N}$.

Proof. To prove the statement of this theorem we estimate

$$\mathsf{E}(X_t^N)^2 + \mathsf{E}\left(\int_0^t \frac{|X_t^N - X_s^N|}{(t-s)^{1+\alpha}} ds\right)^2 =: A_1(t) + A_2(t). \tag{14}$$

At first, for $\mathsf{E}(X_t^N)^2$ we have

$$\mathsf{E}(X_t^N)^2 \leq 5\mathsf{E}(X_0)^2 + 5\mathsf{E}(\int_0^t a(s, X_s^N)ds)^2 + 5\mathsf{E}(\int_0^t b(s, X_s^N)dW_s)^2 \\ + 5\mathsf{E}(\int_0^t c(s, X_s^N)dB_s^H)^2 + 5\mathsf{E}(\frac{1}{N}\int_0^t c(s, X_s^N)dV_s)^2.$$

Evidently

$$\mathsf{E}\left(\int_0^t a(s, X_s^N)ds\right)^2 \leq A^2T^2,$$

$$\mathsf{E}\left(\int_0^t b(s, X_s^N)dW_s\right)^2 \leq A^2T,$$

$$\mathsf{E}\left(\frac{1}{N}\int_0^t c(s, X_s^N)dV_s\right)^2 \leq \frac{A^2T}{N^2} \leq A^2T.$$

Further, using the estimate (5), (12) and with the help of random variables defined by (9), (13), $\mathsf{E}\left(\int_0^t c(s, X_s^N)dB_s^H\right)^2$ can be estimated as

$$\mathsf{E}\left(\int_0^t c(s, X_s^N)dB_s^H\right)^2 \\ \leq \mathsf{E}\left(\overline{C}^2(\omega)\left(\int_0^t \frac{c(s, X_s^N)}{s^\alpha}ds + \int_0^t\int_0^s \frac{|c(s, X_s^N) - c(u, X_u^N)|}{(s-u)^{1+\alpha}}duds\right)^2\right) \\ \leq C\mathsf{E}\left(\overline{C}^2(\omega)\left(t\int_0^t \frac{A^2}{s^{2\alpha}}ds \right.\right. \\ \left.\left. + \left(\int_0^t\int_0^s \frac{B(s-u)^\beta + L\widetilde{C}(\omega)(s-u)^{1/2-\delta}}{(s-u)^{1+\alpha}}duds\right)^2\right)\right) \\ \leq C(A^2t^{2-2\alpha}\mathsf{E}\overline{C}^2(\omega) + B^2\mathsf{E}\overline{C}^2(\omega)t^{2(1-\alpha+\beta)} + L^2\mathsf{E}(\widetilde{C}^2(\omega)\overline{C}^2(\omega))T^{3-2\alpha-2\delta})).$$

So, we have

$$A_1(t) \le C(A^2T^2 + 2A^2T + A^2T^{2-2\alpha}\mathsf{E}\overline{C}^2(\omega) \\ + B^2\mathsf{E}\overline{C}^2(\omega)T^{2(1-\alpha+\beta)} + L^2\mathsf{E}(\widetilde{C}^2(\omega)\overline{C}^2(\omega))T^{3-2\alpha-2\delta}) < \infty. \quad (15)$$

Consider now $A_2(t)$. We have that

$$A_2(t) \le 4\mathsf{E}\left(\int_0^t \frac{|\int_s^t a(u, X_u^N)du|}{(t-s)^{1+\alpha}}ds\right)^2 \\ + 4\mathsf{E}\left(\int_0^t \frac{|\int_s^t b(u, X_u^N)dW_u|}{(t-s)^{1+\alpha}}ds\right)^2 + 4N^{-2}\mathsf{E}\left(\int_0^t \frac{|\int_s^t c(u, X_u^N)dV_u|}{(t-s)^{1+\alpha}}ds\right)^2 \\ + 4\mathsf{E}\left(\int_0^t \frac{|\int_s^t c(u, X_u^N)dB_u^H|}{(t-s)^{1+\alpha}}ds\right)^2.$$

Evidently,

$$\mathsf{E}\left(\int_0^t \frac{|\int_s^t a(u, X_u)du|}{(t-s)^{1+\alpha}}ds\right)^2 \le CA^2t^{2-2\alpha}.$$

Now, let $\gamma \in (\alpha, 1/2)$, then

$$\mathsf{E}\left(\int_0^t \frac{|\int_s^t b(u, X_u)dW_u|}{(t-s)^{1+\alpha}}ds\right)^2 \le Ct^{1-2\gamma}\int_0^t \frac{\mathsf{E}|\int_s^t b(u, X_u)dW_u|^2}{(t-s)^{2+2\alpha-2\gamma}}ds \\ \le Ct^{1-2\gamma}\int_0^t \frac{\int_s^t b^2(u, X_u)du}{(t-s)^{2+2\alpha-2\gamma}}ds \le Ct^{1-2\gamma}\int_0^t \frac{A^2}{(t-s)^{1+2\alpha-2\gamma}}ds \\ \le CA^2t^{1-2\alpha},$$

and similarly

$$\mathsf{E}\left(\int_0^t \frac{|\int_s^t c(u, X_u)dV_u|}{(t-s)^{1+\alpha}}ds\right)^2 \le CA^2t^{1-2\alpha}.$$

Now we estimate $\mathsf{E}\left(\int_0^t \frac{|\int_s^t c(u,X_u)dB_u^H|}{(t-s)^{1+\alpha}}ds\right)^2$.

$$\mathsf{E}\left(\int_0^t \frac{|\int_s^t c(u,X_u)dB_u^H|}{(t-s)^{1+\alpha}}ds\right)^2$$
$$\leq \mathsf{E}\left(\overline{C}(\omega)\int_0^t \frac{\int_s^t \frac{|c(u,X_u)|}{(u-s)^\alpha}du + \int_s^t\int_s^u \frac{|c(u,x_u^N)-c(r,X_r^N)|}{(u-r)^{1+\alpha}}drdu}{(t-s)^{1+\alpha}}ds\right)^2$$
$$\leq \mathsf{E}\left(\overline{C}(\omega)\int_0^t \frac{\int_s^t \frac{|c(u,X_u)|}{(u-s)^\alpha}du + \int_s^t\int_s^u \frac{B(u-r)^\beta + L\widetilde{C}(\omega)(u-r)^{1/2-\delta}}{(u-r)^{1+\alpha}}drdu}{(t-s)^{1+\alpha}}ds\right)^2$$
$$\leq \mathsf{E}\left(\overline{C}(\omega)\int_0^t \frac{A(t-s)^{1-\alpha} + B(t-s)^{1+\beta-\alpha} + L\widetilde{C}(\omega)(t-s)^{3/2-\delta-\alpha}}{(t-s)^{1+\alpha}}ds\right)^2$$
$$\leq C(A^2t^{2-4\alpha}\mathsf{E}\overline{C}^2(\omega) + B^2t^{2+2\beta-4\alpha}\mathsf{E}\overline{C}^2(\omega) + L^2t^{3-2\delta-4\alpha}\mathsf{E}\overline{C}^2(\omega)\widetilde{C}^2(\omega)).$$

Therefore $A_2(t)$ satisfies the inequality

$$A_2(t) \leq C(A^2T^{2-2\alpha} + A^2T^{1-2\alpha} + A^2T^{1-2\alpha}$$
$$+(A^2T^{2-4\alpha}\mathsf{E}\overline{C}^2(\omega) + B^2T^{2+2\beta-4\alpha}\mathsf{E}\overline{C}^2(\omega) + L^2T^{3-2\delta-4\alpha}\mathsf{E}\overline{C}^2(\omega)\widetilde{C}^2(\omega)) < \infty. \tag{16}$$

At last, the statement of our theorem follows from inequalities (15) and (16). □

Introduce for any $R > 1$ the stopping time τ_R by

$$\tau_R := \inf\{t : C_t'(\omega) \geq R\} \wedge T, \tag{17}$$

where $C_t'(\omega)$ is defined by (11). For any $\omega \in \Omega$ $\tau_R = T$, for any $R > R(\omega)$.

Define the processes $\{X_{t\wedge\tau_R}^N, N \in \mathbb{N}, t \in [0,T]\}$ as the solutions of equation (2) stopped an the moment τ_R, and prove that they are fundamental in the norm (4) of the space (3).

Theorem 4. *Under assumptions (A) - (D) the sequence $\{X_{t\wedge\tau_R}^N, N \geq 1, t \in [0,T]\}$ of solutions of equations (2) is fundamental in the norm (4).*

Proof. Consider

$$\mathsf{E}(X_{t\wedge\tau_R}^N - X_{t\wedge\tau_R}^M)^2 + \mathsf{E}\left(\int_0^t \frac{|X_{t\wedge\tau_R}^N - X_{t\wedge\tau_R}^M - X_{s\wedge\tau_R}^N + X_{s\wedge\tau_R}^M|}{(t-s)^{1+\alpha}}ds\right)^2$$
$$= \mathsf{E}(X_{t\wedge\tau_R}^N - X_{t\wedge\tau_R}^M)^2 + \mathsf{E}\left(\int_0^{t\wedge\tau_R} \frac{|X_{t\wedge\tau_R}^N - X_{t\wedge\tau_R}^M - X_s^N + X_s^M|}{(t-s)^{1+\alpha}}ds\right)^2$$
$$=: A_1^{N,M}(t) + A_2^{N,M}(t). \tag{18}$$

First, for $A_1^{N,M}(t)$ we have

$$A_1^{N,M}(t) \le 4\mathsf{E}\left(\int_0^{t\wedge\tau_R}(a(s,X_s^N)-a(s,X_s^M))ds\right)^2 + 4\mathsf{E}\left(\int_0^{t\wedge\tau_R}(b(s,X_s^N)-b(s,X_s^M))dW_s\right)^2 + 4\mathsf{E}\left(\int_0^{t\wedge\tau_R}(c(s,X_s^N)-c(s,X_s^M))dB_s^H\right)^2 + 4\mathsf{E}\left(\int_0^{t\wedge\tau_R}\left(\frac{c(s,X_s^N)}{N}-\frac{c(s,X_s^M)}{M}\right)dV_s\right)^2 =: 4(I_1+I_2+I_3+I_4).$$

Then

$$I_1 \le CTL^2\int_0^t \mathsf{E}(X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M)^2 ds,$$

$$I_2 \le CL^2\int_0^t \mathsf{E}(X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M)^2 ds,$$

$$I_4 \le CA^2T(N^{-2}+M^{-2}).$$

Now we are in position to estimate I_3 :

$$I_3 = \mathsf{E}\left(\int_0^{t\wedge\tau_R}(c(s,X_s^N)-c(s,X_s^M))dB_s^H\right)^2 \le R^2\mathsf{E}\left(\int_0^{t\wedge\tau_R}\frac{|c(s,X_s^N)-c(s,X_s^M)|}{s^\alpha}ds + \int_0^{t\wedge\tau_R}\int_0^s\frac{|c(s,X_s^N)-c(s,X_s^M)-c(u,X_u^N)+c(u,X_u^M)|}{(s-u)^{1+\alpha}}duds\right)^2$$
$$\le 2R^2\left(\mathsf{E}\left(\int_0^{t\wedge\tau_R}\frac{|c(s,X_s^N)-c(s,X_s^M)|}{s^\alpha}ds\right)^2 + \mathsf{E}\left(\int_0^{t\wedge\tau_R}\int_0^s\frac{|c(s,X_s^N)-c(s,X_s^M)-c(u,X_u^N)+c(u,X_u^M)|}{(s-u)^{1+\alpha}}duds\right)^2\right) = 2R^2(I_4+I_5).$$

Further,

$$I_4 \le CL^2T^{1-2\alpha}\mathsf{E}\int_0^{t\wedge\tau_R}(X_s^N-X_s^M)^2ds = CL^2T^{1-2\alpha}\int_0^t(A_1^{N,M}(s))^2ds.$$

Using Lemma 7.1 [1] we estimate I_5 as

$$I_5 \le \mathsf{E}\Bigg(\int_0^{t\wedge\tau_R}\int_0^s \frac{A|X_s^N - X_s^M - X_u^N + X_u^M|}{(s-u)^{1+\alpha}}duds + \int_0^{t\wedge\tau_R}\int_0^s \frac{AB|X_s^N - X_s^M|(s-u)^{\beta}}{(s-u)^{1+\alpha}}duds + \int_0^{t\wedge\tau_R}\int_0^s \frac{D|X_s^N - X_s^M|(|X_s^N - X_u^N|^{\rho} + |X_s^M - X_u^M|^{\rho})}{(s-u)^{1+\alpha}}duds\Bigg)^2$$
$$\le 3\mathsf{E}\left(\int_0^{t\wedge\tau_R}\int_0^s \frac{A|X_s^N - X_s^M - X_u^N + X_u^M|}{(s-u)^{1+\alpha}}duds\right)^2 + 3\mathsf{E}\left(\int_0^{t\wedge\tau_R}\int_0^s \frac{AB|X_s^N - X_s^M|(s-u)^{\beta}}{(s-u)^{1+\alpha}}duds\right)^2 + 3\mathsf{E}\left(\int_0^{t\wedge\tau_R}\int_0^s \frac{D|X_s^N - X_s^M|(|X_s^N - X_u^N|^{\rho} + |X_s^M - X_u^M|^{\rho})}{(s-u)^{1+\alpha}}duds\right)^2 = 3(I_6 + I_7 + I_8),$$

where

$$I_6 \le CTA^2\int_0^t \mathsf{E}\left(\int_0^{s\wedge\tau_R} \frac{|X_s^N - X_s^M - X_u^N + X_u^M|}{(s-u)^{1+\alpha}}du\right)^2 ds,$$

$$I_7 \le CTA^2\int_0^t s^{2(\beta-\alpha)}\mathsf{E}\left(|X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M|\right)^2 ds,$$

$$I_8 \le \mathsf{E}\left(\int_0^{t\wedge\tau_R}\int_0^s \frac{B|X_s^N - X_s^M|(2R(s-u)^{\rho(1/2-\delta)})}{(s-u)^{1+\alpha}}duds\right)^2 \le CTD^2R^2\int_0^t s^{\rho-2\rho\delta-2\alpha}\mathsf{E}\left(|X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M|\right)^2 ds,$$

where we choose δ in such a way that $\rho - 2\rho\delta - 2\alpha > 0$. It is possible since $\alpha < \rho - 1/2$ so $\rho - 2\alpha > 1/2 - \alpha > 0$. At last,

$$I_5 \le C\int_0^t \mathsf{E}\left(\int_0^{s\wedge\tau_R} \frac{|X_s^N - X_s^M - X_u^N + X_u^M|}{(s-u)^{1+\alpha}}du\right)^2 ds + C\int_0^t s^{2(\beta-\alpha)}\mathsf{E}\left(|X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M|\right)^2 ds + CR^2\int_0^t s^{\rho-2\rho\delta-2\alpha}\mathsf{E}\left(|X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M|\right)^2 ds,$$

and

$$A_1^{N,M}(t) \le CR^2 \int_0^t A_1^{N,M}(s)ds + CR^2 \int_0^t A_2^{N,M}(s)ds + C(N^{-2} + M^{-2}). \quad (19)$$

Return to $A_2^{N,M}(t)$. It admits the following estimate

$$\begin{aligned} A_2^{N,M}(t) \le C \Bigg(&\mathsf{E}\left(\int_0^{t\wedge\tau_R} \frac{\int_s^{t\wedge\tau_R} (a(u,X_u^N) - a(u,X_u^M))du}{(t-s)^{1+\alpha}} ds \right)^2 \\ &+ \mathsf{E}\left(\int_0^{t\wedge\tau_R} \frac{\int_s^{t\wedge\tau_R} (b(u,X_u^N) - b(u,X_u^M))dW_u}{(t-s)^{1+\alpha}} ds \right)^2 \\ &+ \mathsf{E}\left(\int_0^{t\wedge\tau_R} \frac{\int_s^{t\wedge\tau_R} (c(u,X_u^N) - c(u,X_u^M))dB_u^H}{(t-s)^{1+\alpha}} ds \right)^2 \\ &+ \mathsf{E}\left(\int_0^{t\wedge\tau_R} \frac{\int_s^{t\wedge\tau_R} (\frac{c(u,X_u^N)}{N} - \frac{c(u,X_u^M)}{M})dV_u}{(t-s)^{1+\alpha}} ds \right)^2 \Bigg) \\ &\le C(I_9 + I_{10} + I_{11} + I_{12}). \end{aligned}$$

$$\begin{aligned} I_9 &\le CT^{1-2\gamma} \mathsf{E} \int_0^{t\wedge\tau_R} \frac{(t-s)\int_0^{t\wedge\tau_R} L^2 |X_u^N - X_u^M|^2 du}{(t-s)^{2+2\alpha-2\gamma}} ds \\ &\le CT^{1-2\alpha} \int_0^t \mathsf{E}\left(X_{s\wedge\tau_R}^N - X_{s\wedge\tau_R}^M\right)^2 ds \le CT^{1-2\alpha} \int_0^t A_1^{N,M}(s)ds, \end{aligned}$$

$$\begin{aligned} I_{10} &\le CT^{1-2\gamma} \int_0^t \frac{\int_s^t \mathsf{E}|X_{u\wedge\tau_R}^N - X_{u\wedge\tau_R}^M|^2 du}{(t-s)^{2+2\alpha-2\gamma}} ds \\ &\le CT^{1-2\gamma} \int_0^t \frac{A_1^{N,M}(s)}{(t-s)^{1+2\alpha-2\gamma}} ds, \end{aligned}$$

where we choose γ in such a way that $\alpha < \gamma < 1/2$.
For I_{12} we have

$$I_{12} \le CT^{1-2\alpha}(N^{-2} + M^{-2}).$$

Now consider I_{11} :

$$I_{11} \le CR^2T^{1-2\gamma}\left(\mathsf{E}\int_0^{t\wedge\tau_R}\frac{\left(\int_s^{t\wedge\tau_R}\frac{c(u,X_u^N)-c(u,X_u^M)}{(u-s)^\alpha}du\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds\right.$$

$$\left.+\,\mathsf{E}\int_0^{t\wedge\tau_R}\frac{\left(\int_s^{t\wedge\tau_R}\int_s^u\frac{|c(u,X_u^N)-c(u,X_u^M)-c(v,X_v^N)+c(v,X_v^M)|}{(u-v)^{1+\alpha}}dvdu\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds\right)$$

$$=: CR^2T^{1-2\gamma}(I_{12}+I_{13}).$$

$$I_{12} \le C\int_0^t\frac{(t-s)\int_s^t\frac{\mathsf{E}\left(X_{u\wedge\tau_r}^N-X_{u\wedge\tau_r}^M\right)^2}{(u-s)^{2\alpha}}du}{(t-s)^{2+2\alpha-2\gamma}}$$

$$\le C\int_0^t\frac{A_1^{N,M}(s)}{(t-s)^{1+2\alpha-2\gamma}}ds.$$

$$I_{13} \le C\left(\mathsf{E}\int_0^{t\wedge\tau_R}\frac{\left(\int_s^{t\wedge\tau_R}\int_s^u\frac{L|X_u^N-X_u^M-X_v^N+X_v^M|}{(u-v)^{1+\alpha}}dvdu\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds\right.$$

$$+\,\mathsf{E}\int_0^{t\wedge\tau_R}\frac{\left(\int_s^{t\wedge\tau_R}\int_s^u\frac{D|X_u^N-X_u^M|(u-v)^\beta}{(u-v)^{1+\alpha}}dvdu\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds$$

$$\left.+\,\mathsf{E}\int_0^{t\wedge\tau_R}\frac{\left(\int_s^{t\wedge\tau_R}\int_s^u\frac{B|X_u^N-X_u^M|(|X_u^N-X_v^N|^\rho+|X_u^M-X_v^M|^\rho)}{(u-v)^{1+\alpha}}dvdu\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds\right)$$

$$=: C(I_{14}+I_{15}+I_{16}).$$

$$I_{14} \le CT^{2\gamma-2\alpha}\int_0^t\mathsf{E}\left(\int_0^{s\wedge\tau_R}\frac{|X_s^N-X_s^M-X_u^N+X_u^M|}{(s-u)^{1+\alpha}}du\right)^2ds$$

$$= CA^2T^{2\gamma-2\alpha}\int_0^t A_2^{N,M}(s)ds,$$

$$I_{15} \le C\int_0^t\frac{\mathsf{E}\left(\int_s^{t\wedge\tau_R}|X_u^N-X_u^M|(u-s)^{\beta-\alpha}\right)^2}{(t-s)^{2+2\alpha-2\gamma}}ds$$

$$\le C\int_0^t\frac{(t-s)^{1+2\beta-2\alpha}\int_s^t\mathsf{E}\left(X_{u\wedge\tau_R}^N-X_{u\wedge\tau_R}^M\right)^2du}{(t-s)^{2+2\alpha-2\gamma}}ds$$

$$\le CT^{2\beta+2\gamma-4\alpha}\int_0^t A_1^{N,M}(s)ds,$$

note that $\alpha < \beta$.

$$I_{16} \le CR^2 \mathsf{E} \int_0^{t\wedge\tau_R} \frac{\left(\int_s^{t\wedge\tau_R} \int_s^u \frac{|X_u^N - X_u^M|(u-v)^{\rho(1/2-\delta)}}{(u-v)^{1+\alpha}} dv du\right)^2}{(t-s)^{2+2\alpha-2\gamma}} ds,$$

where we chose $0 < \delta < 1/2 - \alpha/\rho$, note that $\alpha < \rho - 1/2$. Similarly to I_{15},

$$I_{16} \le CT^{2\gamma-2\alpha} \int_0^t A_1^{N,M}(s) ds.$$

Therefore we have

$$I_{13} \le CR^2 (\int_0^t A_1^{N,M}(s) ds + \int_0^t A_2^{N,M}(s) ds).$$

Hence

$$I_{11} \le CR^4 (\int_0^t \frac{A_1^{N,M}(s)}{(t-s)^{1+2\alpha-2\gamma}} ds + \int_0^t A_2^{N,M}(s) ds).$$

At last,

$$A_2^{N,M}(t) \le CR^4 \left(\int_0^t \frac{A_1^{N,M}(s)}{(t-s)^{1+2\alpha-2\gamma} ds} + \int_0^t A_2^{N,M}(s) ds \right) + C(N^{-2} + M^{-2}). \quad (20)$$

From (19) and (20) we obtain that the sum $A_1^{N,M}(t) + A_2^{N,M}(t)$ admits the same estimate as $A_2^{N,M}(t)$, i.e.

$$A_1^{N,M}(t) + A_2^{N,M}(t) \le CR^4 \left(\int_0^t \frac{A_1^{N,M}(s)}{(t-s)^{1+2\alpha-2\gamma} ds} + \int_0^t A_2^{N,M}(s) ds \right) + C(N^{-2} + M^{-2}), \quad (21)$$

and from modified Gronwall lemma [1]

$$A_1^{N,M}(t) + A_2^{N,M}(t) \le CR^4 (N^{-2} + M^{-2}) \exp\{t(CR^4)^{1/(2\gamma-2\alpha)}\}, \quad (22)$$

taking, for example, $\gamma := (1/2 + \alpha)/2$. As choosing $N, M \to 0$ see that right-hand side of (22) tends to zero whence the proof follows. □

Theorem 5. *The SDE* (1) *has the solution on interval* $[0, T]$, *and this solution is unique.*

Proof. Since the space (3) is complete, and from Theorem 4 we can define

$$X_{t\wedge\tau_R} := \lim_{N\to\infty} X^N_{t\wedge\tau_R}, \tag{23}$$

where the limit is taken in space $W_\alpha[0,T]$ (in particular, we have that the limit exists in $L_2(\Omega\times[0,T])$).Using the similar estimates an Theorem 4, we can prove that $X_{t\wedge\tau_R}$ is the unique solution of the original equation (1) on interval $[0,\tau_R]$.

From definition (17) of τ_R we have $\tau_{R_1} \le \tau_{R_2}$ for $R_1 \le R_2$. So $X_{\tau_{R_1}}$ and $X_{\tau_{R_2}}$ coincide a.s. on interval $[0,\tau_{R_1}]$. Where $R\to\infty$ we obtain the existence and uniqueness of solution of SDE (1) on interval $[0,T]$. □

3. Conclusion

So, we proved the existence and uniqueness of solution of stochastic differential equation driven by standard and fractional Brownian motions (1).

Bibliography

1. Nualart, D., Răşcanu *Differential equations driven by fractional Brownian motion,* Collect. Math., **53**, (2000), 55–81.
2. Cheridito, P. *Regularizing fractional Brownian motion with a view towards stock price modeling.* PhD thesis, Zurich (2001).
3. Mishura, Yu., Posashkov, S., *Existence and uniqueness of solution of stochastic differential equation driven by fractional Brownian motion with stabilizing term* Theory Prob. Math. Stat, **76**, (2007).
4. Zähle, M. *On the link between fractional and stochastic calculcus.* In: Grauel, H., Gundlach, M. (eds.) Stochastic Dynamics, Springer, (1999), 305–325.
5. Zähle, M. *Integration with respect to fractal functions and stochastic calculus. I.* Probab. Theory Rel. Fields, **111**,(1998), 333–374.
6. Zähle, M. *Integraton with respect to fractal functions and stochastic calculus. II.* Math. Nachr., **225**,(2001), 145–183.
7. Krvavych, Yu., Mishura, Yu., *Exponential formula and Girsanov theorem for mixed semilinear stochastic differential equatios*, in: Mathematical Finance (Trends in Mathematics), Basel, Birkhäuser, (2001), 230–39.
8. Hitsuda, M. *Representation of Gaussian processes equivalent to Wiener process*, Osaka Journal of Mathematics, **5**, (1968), 299–312.

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: myus@univ.kiev.ua

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: corlagon@univ.kiev.ua