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ANDREI N. LEPEYEV

UNIQUENESS IN LAW OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL INCLUSIONS

The paper deals with one-dimensional homogeneous stochastic differential inclusions without drift with Borel measurable mapping at the right side. We give the conditions for uniqueness in law and existence of unique in law weak solutions of the inclusions with locally unbounded right sides.

1. Introduction

The following stochastic differential inclusion (SDI) is considered

$$dX_t \in B(X_t)\, dW_t, \quad t \geq 0, \tag{1}$$

where $B : \mathbb{R} \to comp(\mathbb{R})$ - multi-valued Borel measurable mapping, $comp(\mathbb{R})$ - the set of all non-empty compact subsets of $\mathbb{R}$ with Hausdorff metric $\rho(A, B) = \max(\beta(A, B), \beta(B, A)), \forall A, B \in comp(\mathbb{R})$, where $\beta(A, B) = \sup_{x \in A}(\inf_{y \in B} |x - y|)$ - excess of A over B, W - one-dimensional Wiener process.

One can suspect the uniqueness concept for the SDI solutions to be ambiguous, because the definition of stochastic integral of multi-valued mappings implies the set of possibly different processes (see [2]). But despite that, the uniqueness of SDI solutions is very important due to the fact that availability of uniqueness helps in description of the SDI solution set.

Inclusion (1) was investigated in paper [7]. The necessary and sufficient conditions were given for the existence of weak and explicit weak solutions of the SDI, including the case of non-trivial solutions. This paper is the natural continuation of the investigation.

The aim of the paper is to give the conditions for uniqueness in law and existence of unique in law weak solutions of the inclusions with locally unbounded right sides. For this purpose, we use a selectionwise approach

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that requires introduction of several intermediate definitions of uniqueness in law in terms of their selections. The main results are necessary, sufficient, and necessary and sufficient conditions for existence of unique in law explicit weak solutions. On the one hand, the results express the way of classifying the solutions and, on the other hand, given some additional conditions, they guarantee the existence of unique in law weak solutions in the general context.

In addition to notions from paper [7], the article uses the definitions and results obtained for stochastic differential equations by H.-J. Engelbert and W. Schmidt (see [4] and [5]). We also use the methods they have developed.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. As usually, assume that filtration $\mathbb{F}$ satisfies the natural conditions, i.e. it is right-continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$ - zero subsets of $\mathcal{F}$. If stochastic process $(X_t)_{t\geq 0}$ is $\mathbb{F}$-adapted, we write $(X, \mathbb{F})$.

The first part of the section is devoted to notions from the theory, developed for investigation of stochastic differential equations (SDEs). The notions are also important for inclusions due to the fact that an inclusion is a generalization of an equation. For instance, the considered SDI (1) is the natural generalization of the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)\, dW_s, \quad t \geq 0, \tag{2}$$

where diffusion coefficient $b : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function and W is a Wiener process.

Measurable process $(X, \mathbb{F})$, defined on probability space $(\Omega, \mathcal{F}, \mathbf{P})$, possesses the *representation property*, if any martingale $(Y, \mathbb{F})$ can be represented in the form of stochastic Ito integral driven by $(X, \mathbb{F})$

$$Y_t = Y_0 + \int_0^t f(s) dX_s, \ \forall t \geq 0,$$

where $(f, \mathbb{F})$ – some predictable process.

Let X be a weak solution of SDE (2) with Borel measurable diffusion coefficient v, i.e. there exist probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$ and Wiener process $(W^v, \mathbb{F})$ such that $(X, \mathbb{F})$ is measurable and (2) is valid $\mathbf{P}$-a.s. for all $t \geq 0$. The quadratic variation can be defined by $\langle X \rangle_t = A_t^v = \int_0^t v^2(X_s) ds$, $\forall t \geq 0$ and the process reverse to the quadratic variation can be defined by $T_t^v = \inf\{s \geq 0 : A_s^v > t\}$, $\forall t \geq 0$.

Proposition 1. [3, Theorem 2, Proposition 7] *If $A_\infty^v = \lim_{t\to+\infty} A_t^v$ is $\mathbb{F}^{W^v}$-predictable stopping time (where $\mathbb{F}^{W^v}$ is the filtration generated by W^v and completed in the natural way) and for all fixed $t \geq 0$ variable value T_t^v*

is $\mathcal{F}_\infty^{W^v}$-measurable (where sigma-algebra $\mathcal{F}_\infty^{W^v} = \cup_{t\geq 0}\mathcal{F}_t^{W^v}$), then $(X, \mathbb{F}^X)$ possesses the representation property.

The next result guarantees the availability of representation property for certain processes. We will use sets $N_v = \{x \in \mathbb{R} | v(x) = 0\}$ and $M_v = \{x \in \mathbb{R} | \int_{U(x,C)} v^{-2}(y)\, dy = \infty, \forall C > 0\}$, where $U(x, C)$ is an open ball with the center in x and radius C. The sets can be defined for any Borel measurable function $v : \mathbb{R} \to \mathbb{R}$.

Proposition 2. [4, Lemma 2] *Let $N_v \subseteq M_v$. Then:*

1) $T_t^v = \int_0^t v^{-2}(W_s^v)ds, \forall t < A_\infty^v, \mathbf{P}$-a.s.;

2) A_∞^v is $\mathbb{F}^{W^v}$-predictable stopping time.

The representation property ensures some important properties of the martingale distributions. An *extremal point* of some convex set of measures is measure P such that if there exist measures Q and S $(Q \neq S)$ from this set of measures and $P = \lambda Q + (1 - \lambda)S, \lambda \in [0, 1]$, then λ can just have values of either 0 or 1.

Proposition 3. [3, p.326] *The next properties are equivalent:*

i) process $(X, \mathbb{F})$ on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ possesses the representation property for continuous local martingales;

ii) measure $\mathbf{P}$ is an extremal point of the convex set of such probability measures on $\mathcal{F}_\infty$ that $(X, \mathbb{F})$ is a continuous local martingale with respect to them.

Let $(X, \mathbb{F})$ be a continuous local martingale, the *local time* of the process is function L^X from $[0, +\infty) \times \mathbb{R} \times \Omega$ to $\mathbb{R}$ such that, for any non-negative Borel function g and all $t \geq 0$, $\mathbf{P}$-a.s.

$$\int_0^t g(X_s)d\langle X\rangle_s = \int_{\mathbb{R}} g(y)L^X(t, y)dy. \tag{3}$$

The local time of Wiener process possesses some important properties, which describe the process itself and the integrals driven by the process. The next property is used in the paper.

Proposition 4. [5, Lemma 2.24] *For local time of Wiener process, $L^W(t, x) > 0$ holds for all $t \geq 0$, $l \times \mathbf{P}$-a.e..*

The existence conditions of the paper will guarantee the solutions existence *for every initial distribution*, that means: for every probabilistic measure $\bar{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ there exists solution X with initial value X_0, such that the following holds:

$$\mathbf{P}(\{X_0 \in B\}) = \bar{P}(B), \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Stochastic process $(X, \mathbb{F})$, defined on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, is called a *weak solution* of SDI (1), if there exist

Wiener process $(\overline{W}, \mathbb{F})$ with $\overline{W}_0 = 0$ and measurable process $(u, \mathbb{F})$, such that $u(t,\omega) \in B(X(t,\omega))$ $l_+ \times \mathbf{P}$-a.e. and the following holds $\mathbf{P}$- a.s. $\forall t \geq 0$

$$X_t = X_0 + \int_0^t u(s)\, d\overline{W}_s.$$

Stochastic process $(X, \mathbb{F})$, defined on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, is called an *explicit weak solution* of SDI (1), if there exist Wiener process $(\overline{W}, \mathbb{F})$ with $\overline{W}_0 = 0$ and Borel measurable function $v : \mathbb{R} \to \mathbb{R}$, such that $v(x) \in B(x), \forall x \in \mathbb{R}$ and the following holds $\mathbf{P}$- a.s. $\forall t \geq 0$

$$X_t = X_0 + \int_0^t v(X_s)\, d\overline{W}_s.$$

The definitions were considered more in detail in paper [7]. Using excess β of elements from $comp(\mathbb{R})$, the paper also introduced some sets and selections, which will be used in this article as well. We call function

$$b_{int}(x) = \begin{cases} \beta(0, B(x)), & \beta(0, B(x)) \in B(x); \\ -\beta(0, B(x)), & \beta(0, B(x)) \notin B(x); \end{cases} \tag{4}$$

the *internal characteristic selection* of mapping B and function

$$b_{ext}(x) = \begin{cases} \beta(B(x), 0), & \beta(B(x), 0) \in B(x); \\ -\beta(B(x), 0), & \beta(B(x), 0) \notin B(x); \end{cases} \tag{5}$$

the *external characteristic selection* of mapping B. Additionally, paper [7] introduced sets

$$\underline{M}_B = \left\{ x \in \mathbb{R} \,\middle|\, \int_{U(x,C)} \beta(0, B(y))^{-2}\, dy = \infty, \forall C > 0 \right\},$$

$$\overline{M}_B = \left\{ x \in \mathbb{R} \,\middle|\, \int_{U(x,C)} \beta(B(y), 0)^{-2}\, dy = \infty, \forall C > 0 \right\},$$

$$\underline{N}_B = \{x \in \mathbb{R} | \{0\} \in B(x)\},$$

$$\bar{N}_B = \{x \in \mathbb{R} | B(x) = \{0\}\}.$$

Using the sets we can define the *optimal characteristic selection* of mapping B as function

$$b_{opt}(x) = \begin{cases} 0, & 0 \in B(x) \text{ and } x \in \overline{M}_B; \\ b_{ext}(x), & \text{otherwise}. \end{cases} \tag{6}$$

This selection is an improvement of that one introduced in paper [7]. The following relations are valid for the selection: $N_{b_{opt}} = \bar{N}_B \cup (\overline{M}_B \cap \underline{N}_B) \subseteq \underline{N}_B$; $M_{b_{opt}} = Cl(\underline{N}_B \cap \overline{M}_B) \cup \overline{M}_B = \overline{M}_B$, since $\overline{M}_B$ is closed and $Cl(\underline{N}_B \cap \overline{M}_B) \subseteq Cl(\overline{M}_B)$; $N_{b_{opt}} \setminus M_{b_{opt}} = \bar{N}_B \setminus \overline{M}_B$; $M_{b_{opt}} \setminus N_{b_{opt}} = \overline{M}_B \setminus (\bar{N}_B \cup (\overline{M}_B \cap \underline{N}_B)) =$

$\bar{M}_B \setminus \underline{N}_B$, where Cl stands for the closure of a set. All the results from paper [7] are valid for this new selection, due to the relations above.

Proposition 5. [7, Theorem 1]*Stochastic differential inclusion (1) has weak and explicit weak solutions for every initial distribution if and only if the following holds*

$$\bar{M}_B \subseteq \underline{N}_B. \tag{7}$$

The proof of the result contains the main way of constructing a solution of inclusion (1), which basically can be described in five steps:

1) Take selection $v : \mathbb{R} \to \mathbb{R}$ equal to b_{opt} and choose a Wiener process $\tilde{W}$ with the needed initial distribution.

2) Using the Wiener process the following processes can be introduces for a given selection v:

$$T_t^v = \int_0^{t+} v^{-2}(\tilde{W}_s)ds, \quad A_t^v = \inf\{s \geq 0 : T_s^v > t\}, \quad \forall t \geq 0. \tag{8}$$

3) Process X can be introduced as a composition of the Wiener process and process A.

4) Having introduced the set of the first entry of the Wiener process to M_v

$$U(M_v) = \inf\{s \geq 0 : \tilde{W}_s \in M_v\} \tag{9}$$

and using condition (7), one can show the existence of quadratic variation of process X.

5) From Doob Theorem (see [6, Theorem II.7.1′]), the existence of the quadratic variation guaranties the existence of a filtrated probability space with a Wiener process where process X satisfies the conditions of the weak solution definition.

Proposition 6. [7, Corollary 2] *If stochastic differential inclusion (1) has weak solutions for every initial distribution, then the inclusion has explicit weak solutions with respect to selection b_{opt} for every initial distribution.*

Proposition 7. [7, Theorem 3] *Stochastic differential inclusion (1) has explicit weak solutions with respect to every Borel measurable selection for every initial distribution if and only if the following holds*

$$\underline{M}_B \subseteq \bar{N}_B. \tag{10}$$

Multi-valued mapping $B : \mathbb{R} \to \mathcal{P}(\mathbb{R} \cup \{\pm\infty\})$ is *locally integrable in the wide sense (locally integrable in the narrow sense)*, if its every Borel measurable selection is locally integrable (there exists its Borel measurable locally integrable selection). Using the notions, the existence conditions for non-trivial weak and explicit weak solutions were also given in paper [7].

Proposition 8. [7, Theorem 4] *Stochastic differential inclusion (1) has weak and explicit weak non-trivial solutions for every initial distribution if and only if mapping B^{-2} is locally integrable over $\mathbb{R}$ in the narrow sense.*

Proposition 9. [7, Theorem 5] *Stochastic differential inclusion (1) has explicit weak non-trivial solutions with respect to every Borel measurable selection for every initial distribution if and only if mapping B^{-2} is locally integrable over $\mathbb{R}$ in the wide sense.*

Let us go on to uniqueness concepts. In correspondence with the SDE theory, we say that the *uniqueness in law* holds for SDI (1) if any two solutions $(X^1, \mathbb{F}^1)$ and $(X^2, \mathbb{F}^2)$ with coinciding initial distributions possess the same image law on the space of continuous functions over $\mathbb{R}$.

Definition 1. Additionally, we say that the *uniqueness in law in the class of explicit solutions* holds for SDI (1) if any two explicit solutions $(X^1, \mathbb{F}^1)$ and $(X^2, \mathbb{F}^2)$ with coinciding initial distributions possess the same image law on the space of continuous functions over $\mathbb{R}$.

Separating the solutions according to their selections, we can introduce the following definitions.

Definition 2. We say that the *uniqueness in law with respect to selection v* holds for SDI (1) if any two explicit solutions $(X^1, \mathbb{F}^1)$ and $(X^2, \mathbb{F}^2)$ with respect to explicit selection v with coinciding initial distributions possess the same image law on the space of continuous functions over $\mathbb{R}$.

Definition 3. We say that the *selectionwise uniqueness in law* holds for SDI (1) if uniqueness in law holds for every explicit Borel measurable selection of the SDI right side.

Remarks 1. The notions of uniqueness with respect to a selection and the selectionwise uniqueness does not make any sense for non-explicit solutions, due to the fact that every non-explicit selection corresponds to one solution.

2. Uniqueness in law implies uniqueness in law in the class of explicit solutions, which, in its turn, implies the selectionwise uniqueness in law.

3. If the right side of SDI (1) is single-valued then the definitions of the uniqueness in law coincide and equal the definition of uniqueness in law of SDE solutions.

The following result is crucial for the following calculations.

Lemma 1. *If there exists set $V \subset \mathbb{R}$ such that $l\{V\} > 0$, then there exists point $x_0 \in \mathbb{R}$ such that for every open ball $U(x_0, C)$ inequality $l\{V \cap U(x_0, C)\} > 0$ holds.*

Proof. Sigma-additivity of Lebesgue measure implies the existence of $z \in \mathbf{Z}$ such that $V_z = V \cap [z, z+1]$ has non-zero measure. We will prove that $\exists x_0 \in [z, z+1]$ such that $\forall C \in (0,1) : l\{U(x_0, C) \cap V\} > 0$. Using the rule of contraries, assume that, for every point $x \in [z, z+1]$, there exists radius C_x such that $l\{U(x, C_x) \cap V\} = 0$. The set of open balls $U(x, C_x)$ is an open cover of $[z, z+1]$. The compactness implies the existence of finite subcover $\{m \in \overline{1,n} | U(x_m, C_{x_m})\}$ of $[z, z+1]$. We obtained that $l\{\cup_{m\in\overline{1,n}} U(x_m, C_{x_m}) \cap V\} = 0$, but, on the other hand, $l\{\cup_{m\in\overline{1,n}} U(x_m, C_{x_m}) \cap V\} \geq l\{[z, z+1] \cap V\} > 0$. Lemma is proved.

3. Uniqueness Theorems

Let us start with the seletionwise uniqueness in law.

Theorem 1. *Stochastic differential inclusion (1) has selectionwise unique in law explicit weak solutions with respect to every Borel measurable selection for every initial distribution if and only if (10) holds and*

$$\underline{N}_B \subseteq \overline{M}_B. \tag{11}$$

Proof. Necessity. Let unique in law explicit weak solutions exist with respect to every selection and for every initial distribution. Condition (10) holds from Proposition 7. The next step is to prove relation (11). Using the rule of contraries, assume the existence of $x_0 \in (\underline{N}_B \setminus \underline{M}_B)$. From the statement of the theorem, there exists weak solution X of SDI (1), defined on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\mathbb{F}$, with respect to selection $v = b_{int}$ and initial value $X_0 = x_0$. Moreover, since $\underline{M}_B \subseteq \bar{N}_B \subseteq \underline{N}_B$, then, for this selection, we can construct the solution in the way described by Proposition 5, where original Wiener process $\tilde{W}$ can be taken with initial value x_0. In one's turn, quadratic variation $\langle X \rangle$ $\mathbf{P}$-a.s. coincides with process A^v defined in (8). Using the continuity of Wiener process trajectories, one can conclude that the corresponding set $U(M_v)$, defined in (9), is separated from zero, but $U(M_v) = A^v_\infty$ $\mathbf{P}$-a.s. (the last equality follows from Lemma 1 in [4]) and, hence, $A^v_\infty > 0$ $\mathbf{P}$-a.s., that implies non-triviality of $(X, \mathbb{F})$. On the other hand, since $v(x_0) = b_{int}(x_0) = 0$ $\mathbf{P}$-a.s., then $\bar{X}_t = x_0$ $\forall t \geq 0$ is a trivial solution of SDI (1) with respect to selection v, hence the condition of selectionwise uniqueness in law fails. The same proof can be conducted for selection $v = b_{ext}$, hence, $\bar{N}_B \subseteq \overline{M}_B$ and $\underline{N}_B \subseteq \underline{M}_B \subseteq \bar{N}_B \subseteq \overline{M}_B$.

Sufficiency. From Proposition 7, condition (10) guarantees the existence of explicit weak solutions with respect to every Borel measurable selection for every initial distribution. To finish the proof we need to show, that condition (11) implies selectionwise uniqueness in law. Let $(X, \mathbb{F})$ be an arbitrary solution of SDI (1) with respect to some explicit selection v. Similarly to the necessity proof of Theorem 1 in [7] (Proposition 5) we can introduce process A^v, his inverse T^v and Wiener process $(W^v, \mathbb{F}^v)$ stopped at A^v_∞, where $(W^v_t)_{t\geq 0} = (X_{T^v_t})_{t\geq 0}$, and the corresponding filtration $\mathbb{F}^v = (\mathcal{F}^v_t)_{t\geq 0} = (\mathcal{F}_{T^v_t})_{t\geq 0}$.

Conditions of Proposition 2 are valid for our case, due to the relation $N_v \subseteq \underline{N}_B \subseteq \overline{M}_B \subseteq M_v$ and, hence, process T^v is $\mathbb{F}^{W^v}$-adapted and A^v_∞ is $\mathbb{F}^{W^v}$-predictable stopping time. From Proposition 1, we can conclude that

$(X, \mathbb{F}^X)$ possesses the representation property for continuous local martingale. That is why, any solution of inclusion (1) with respect to the same explicit selection v possesses the representation property.

Now let $(X, \mathbb{F})$ and $(X', \mathbb{F}')$ be two weak solutions on probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\Omega', \mathcal{F}', \mathbf{P}')$ with the same initial distribution and with respect to the same selection v. From Proposition IV.2.1 in [6], the existence of solutions means the existence of the corresponding distributions Q and Q' on measurable space $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})))$. To prove the uniqueness in law, we should prove the coincidence of the distributions. Using the coordinate method, let us introduce process

$$Z_t(w) = w(t), \quad \forall t \geq 0, w \in C(\mathbb{R}_+, \mathbb{R}).$$

Measure $\bar{\mathbf{P}}$ can be defined on $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})))$ in the following way

$$\bar{\mathbf{P}}(C) = \lambda Q(C) + (1-\lambda)Q'(C), \quad \forall C \in \mathcal{B}(C(\mathbb{R}_+, \mathbb{R}))$$

for an arbitrary, but fixed $\lambda \in [0, 1]$. Proposition IV.2.1 in [6] implies, that process Z is a local continuous martingale with respect to filtration $\mathbb{F}^Z$ and measures Q, Q' and $\bar{\mathbf{P}}$, and $\bar{\mathbf{P}}$-a.s.

$$\langle Z \rangle_t = \int_0^t v^2(Z_s) ds.$$

Using the Doob Theorem (see [6, Theorem II.7.1′]) one can conclude, that there exists Wiener process W such that Z is the solution of equation (2) with diffusion coefficient v. Hence, process Z with distribution $\bar{\mathbf{P}}$ possesses the representation property and, from Proposition 3, it is an extremal point of the convex set of probability measures on $\mathcal{F}_\infty^Z$, for which process $(Z, \mathbb{F}^Z)$ is a local martingale. Since $(Z, \mathbb{F}^Z)$ is also a local martingale for measures Q and Q', and λ is an arbitrary, then $Q = \bar{\mathbf{P}} = Q'$. The proof is completed.

Corollary 1. *The selectiowise uniqueness in law holds for SDI (1), if it satisfies condition (11).*

Using Proposition 9, we can deduce the following statement.

Corollary 2. *Stochastic differential inclusion (1) has selectionwise unique in law non-trivial explicit weak solutions with respect to every Borel measurable selection for every initial distribution if and only if*

$$\underline{N}_B = \underline{M}_B = \emptyset. \tag{12}$$

Corollary 3. *Stochastic differential inclusion (1) does not have trivial solutions if it satisfies condition (12).*

Now we will consider the question of the uniqueness in law with respect to a specific selection.

Theorem 2. *If (7) holds and*

$$\bar{N}_B \subseteq \overline{M}_B, \tag{13}$$

then there exists explicit selection v such that stochastic differential inclusion (1) has unique in law with respect to the selection explicit weak solutions for every initial distribution.

Proof. Let explicit weak solutions of inclusion (1) exist, then Proposition 5 implies the existence of explicit weak solutions with respect to selection $v = b_{opt}$ for every initial distribution. The proof of the uniqueness repeats the steps of the Theorem 1 proof, using Proposition 2 and the fact that $N_v = \bar{N}_B \cup (\overline{M}_B \cap \underline{N}_B) \subseteq \overline{M}_B = M_v$. The proof is completed.

Corollary 4. *If conditions (7) and (13) are satisfied, then stochastic differential inclusion (1) has unique in law with respect to selection b_{opt} explicit weak solutions for every initial distribution.*

Corollary 5. *If condition (13) is satisfied, then there exists explicit selection v such that the uniqueness in law with respect to selection v holds for SDI (1).*

Using Proposition 8, we can deduce the following statement.

Corollary 6. *If condition (13) is satisfied and B^{-2} is locally integrable over $\mathbb{R}$ in the narrow sense, then there exists explicit selection v such that stochastic differential inclusion (1) has unique in law with respect to the selection non-trivial explicit weak solutions for every initial distribution.*

Theorem 3. *If there exists explicit selection v such that stochastic differential inclusion (1) has unique in law with respect to the selection explicit weak solutions for every initial distribution, then conditions (7) and (11) are satisfied.*

Proof. Basically, we should follow the proof of necessity in Theorem 1. Statement $\overline{M}_B \subseteq \underline{N}_B$ holds from Proposition 5. In order to prove statement (11), we use the rule of contraries. Assume the existence of point $x_0 \in (\underline{N}_B \setminus \overline{M}_B)$. Let us take an arbitrary Wiener process $(\tilde{W}, \tilde{\mathbb{F}})$ with initial value $\tilde{W}_0 = x_0$. Analogously to Proposition 5, we can construct solution $(X, \mathbb{F})$ with respect to selection $v = b_{opt}$ and initial distribution $X_0 = x_0$ and the quadratic variation $\langle X \rangle$ will coincide with A^v $\mathbf{P}$-a.s.. Set $U(M_v)$ for the constructed Wiener process is separated from zero, since the trajectories of Wiener process are continuous, set $M_v = \overline{M}_B$ is closed and does not contain x_0. Additionally, $U(M_v) = A^v_\infty$ **P**-a.s. and hence $A^v_\infty > 0$ **P**-a.s., that implies non-triviality of $(X, \mathbb{F})$. On the other hand, since $v(x_0) = b_{opt}(x_0) = 0$ **P**-a.s., then $\bar{X}_t = x_0, \forall t \geq 0$ is a trivial solution of inclusion (1) with respect to selection v, hence, the condition of Definition 2 fails. Thus, relation (11) holds. The proof is completed.

Corollary 7. *If there exists selection v such that stochastic differential inclusion (1) has unique in law with respect to the selection non-trivial explicit weak solutions for every initial distribution, then condition (11) is satisfied and B^{-2} is locally integrable over $\mathbb{R}$ in the narrow sense.*

Theorem 4. *For every initial distribution, stochastic differential inclusion (1) has unique in law non-trivial explicit weak solution if and only if (12) holds and mapping B is almost everywhere single-valued, i.e.*

$$B(x) = b_{ext}(x) = b_{int}(x), \quad l - a.e. \tag{14}$$

Proof. Sufficiency. From Corollary 2 condition (12) ensures the existence of non-trivial explicit weak solutions with respect to every Borel measurable selection for every initial distribution as well as the selectionwise uniqueness in law of the solutions. Let us prove that condition (14) implies coincidence of the finite dimensional distributions of the solutions. Selectionwise uniqueness in law means, that finite dimensional distributions of every explicit solution with respect to selection v coincide with finite dimensional distribution of the explicit solution constructed in the way of Proposition 1 for this selection v. Let us take any almost everywhere equal selections v and v'. Statement $N_v = M_v = N_{v'} = M_{v'} = \emptyset$ implies that the corresponding processes T^v_t and $T^{v'}_t$ are continuous for all t (see proof of Theorem 1 in [7]) and **P**-a.s. identical. That is why, for quadratic variations of processes $X_t = W_{A^v_t}$ and $X'_t = W_{A^{v'}_t}$ with respect to selections v and v' for all $t \geq 0$, almost sure holds

$$\langle X \rangle_t = A^v_t = A^{v'}_t = \langle X' \rangle_t \ .$$

The accomplished equality also holds almost sure with the measure of, possibly, extended probability space, where there exists Wiener process $\bar{W}$ and the conditions of the weak solution definitions are satisfied (see [6, Theorem II.7.1′]). Hence, the finite dimensional distributions of the constructed solutions coincide.

Necessity. Condition (12) holds from Corollary 2. At the same time, the condition guarantees the correctness of the construction of solutions with respect to the selections in the way of Proposition 5. To prove condition (14), we will use the rule of contraries. Let condition (14) fail, hence, selections b_{int} and b_{ext} differ on some set V such that $l\{V\} > 0$. Hence, from Lemma 1, there exists point x_0 such that, for every open ball $U(x_0, C)$,

$l\{V \cap U(x_0, C)\} > 0$. Using any Wiener process W with initial value x_0, we can construct processes $T^{b_{int}}$ and $T^{b_{ext}}$. For all $t \geq 0$ **P**-a.s.

$$T_t^{b_{int}} = \int_0^t b_{int}^{-2}(W_s)ds = \int_{\mathbb{R}} b_{int}^{-2}(y)L^W(t,y)dy,$$

where the last equality follows from the definition of the local time (3). Similar, for all $t \geq 0$ **P**-a.s.

$$T_t^{b_{ext}} = \int_{\mathbb{R}} b_{ext}^{-2}(y)L^W(t,y)dy,$$

that implies, for all $t \geq 0$ **P**-a.s.

$$T_t^{b_{int}} - T_t^{b_{ext}} = \int_{\mathbb{R}} (b_{int}^{-2}(y) - b_{ext}^{-2}(y))L^W(t,y)dy > 0,$$

where the last inequality follows from Proposition 4.

Using the continuity of processes $T^{b_{int}}$ and $T^{b_{ext}}$ for quadratic variations of solutions $X_t = W_{A_t^{b_{int}}}$ and $X'_t = W_{A_t^{b_{ext}}}$ with respect to selections b_{int} and b_{ext} for all $t \geq 0$ almost sure holds

$$\langle X \rangle_t = A_t^{b_{int}} < A_t^{b_{ext}} = \langle X' \rangle_t.$$

The inequality also holds almost sure with the measure of, possibly, extended probability space, where there exists Wiener process $\bar{W}$ and the conditions of the definitions of explicit weak solution are satisfied (see [6,Theorem II.7.1′]). The proof is completed.

Remark 4. In order to state the uniqueness in law for all weak solutions (not just in the class of explicit solutions), one have to additionally impose a condition, which can guarantee the existence of an explicit selection corresponding to the selection of the weak solution (as an example, Theorem 9.5.3 of [1] can be used). The condition means that any weak solution can be expressed in the form of an explicit weak solution with an appropriate explicit selection. Hence, given the condition, the uniqueness in law coincides with the uniqueness in law in the class of explicit solutions.

4. Examples

Example 1. Let us consider stochastic differential inclusion (1) with right side

$$B(x) = \begin{cases} |x|, & x \in (-\infty, 1) \cup (1, +\infty); \\ \{0, 1\}, & x = 1. \end{cases}$$

For this equation, $\overline{M}_B = \underline{M}_B = \overline{N}_B = \{0\}$ and $\underline{N}_B = \{0, 1\}$. Hence, from Theorem 2, there exists a solution with respect to selection $b_{opt}(x) = |x|$

and the solution is unique in low for the selection. But condition (11) of selectionwise uniqueness is not satisfied and it can be verified, that SDI (1), for initial value $X_0 = 1$, has two solutions (trivial and non-trivial) with respect to selection $b_{int}(x) = \begin{cases} |x|, & x \in (-\infty, 1) \cup (1, +\infty); \\ 0, & x = 1. \end{cases}$

Example 2. Stochastic differential inclusion (1) with right side

$$B(x) = \begin{cases} Arcth(x), & x \in (-\infty, -1) \cup (1, +\infty); \\ Arth(x), & x \in (-1, +1); \\ [1, 2], & x \in \{-1, 1\}; \end{cases}$$

has selectionwise unique in law explicit weak solutions with respect to every Borel measurable selection for every initial distribution, due to $\bar{M}_B = \underline{M}_B = \bar{N}_B = \underline{N}_B = \{0\}$.

Example 3. For stochastic differential inclusion (1) with right side

$$B(x) = \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\}; \\ [1, 2], & x = 0; \end{cases}$$

$\bar{M}_B = \underline{M}_B = \bar{N}_B = \underline{N}_B = \emptyset$ holds and the right side is single-valued almost everywhere on $\mathbb{R}$, except one point. Hence, for every initial distribution, this inclusion has a non-trivial explicit weak solution unique in law in the class of explicit solutions.

Bibliography

1. Aubin J.-P., Frankowska H., *Set Valued Analysis,* Boston. Birkhauser, (1990).
2. Auman R.J., *Integrals of set-valued functions.* Jornal of Math. Anal. and Appl. (1965). Vol. 12. 1–12.
3. Engelbert H.-J., Hess J., *Stochastic Integrals of Continuous Local Martingales, I, II.* Math. Nachr. (1980),(1981); Vol. 97,100; 325–343,249–269.
4. Engelbert H.-J., Schmidt W., *On one-dimensional stochastic differential equations with generalized drift.* Lect. Notes Control Inf. Sci. Springer-Verlag, Berlin, (1985), Vol. 69, 143–155.
5. Engelbert H.-J., Schmidt W., *Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations, I, II, III.* Math. Nachr., (1989), (1989), (1991); Vol. 143,144,151; 167–184,241–281,149–197.
6. Ikeda N., Watanabe S., *Stochastic differential equations and diffusion processes,* North-Holland Pub. (1981).
7. Lepeyev A.N., *On Stochastic Differential Inclusions with Unbounded Right Sides,* Theory of Stochastic Processes, Kyiv, (2006), Vol. 12(28), No.1-2, 94-105.

Department of Functional Analysis, Belarusian State University, pr. Nezavisimosti 4a, 220030, Minsk, Belarus

E-mail address: ALepeev@iba.by