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ON ACCURACY OF SIMULATION OF GAUSSIAN STATIONARY PROCESSES IN $L_2([0,T])$

A theorem about simulation of a Gaussian stochastic process with given accuracy and reliability in $L_2([0,T])$ using wavelets has been proved.

1. Introduction

Problems of expansion of random processes in series over uncorrelated random variables using wavelets were considered in [3], [4], [5] and other papers. There has been proved in [3] a theorem about accuracy and reliability of such expansions in $L_p([0,T])$ for Gaussian wide-sense stationary processes. We'll give a refinement of this theorem for $L_2([0,T])$. Since Gaussian processes are widely used in financial and actuarial mathematics results of the article may be used in these areas.

First of all we need to list some necessary facts.

Let $\varphi \in L_2(R)$ be such a function that the following assumptions hold:

i) $\sum_{k\in Z} |\hat{\varphi}(y+2\pi k)|^2 = 1$ almost everywhere, where $\hat{\varphi}(y)$ is the Fourier transform of φ,

$$\hat{\varphi}(y) = \int_R \exp\{-iyx\}\varphi(x)dx;$$

ii) There exists a function $m_0 \in L_2([0,2\pi])$ such that $m_0(x)$ has period 2π and almost everywhere

$$\hat{\varphi}(y) = m_0\left(\frac{y}{2}\right)\hat{\varphi}\left(\frac{y}{2}\right);$$

iii) $\hat{\varphi}(0) \neq 0$ and the function $\hat{\varphi}(y)$ is continuos at 0.

Function $\varphi(x)$ is called f-wavelet. Let $\psi(x)$ be the inverse Fourier transform of the function

$$\hat{\psi}(y) = \overline{m_0\left(\frac{y}{2}+\pi\right)}\exp\Big\{-i\frac{y}{2}\Big\}\hat{\varphi}\left(\frac{y}{2}\right).$$

Function $\psi(x)$ is called m-wavelet. Let $\varphi_{jk}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$, $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, $k \in Z$, $j = 0,1,2,\ldots,$ It is known that the family of functions $\{\varphi_{0k}, \psi_{jk}, j = 0,1,2,\ldots, k \in Z\}$ is an orthonormal basis in $L_2(R)$ (see, for

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example, the book [1]). Such a basis is called wavelet basis. Any function $f \in L_2(R)$ can be represented in the form

$$f(x) = \sum_{k\in Z} \alpha_{0k}\varphi_{0k}(x) + \sum_{j=0}^{\infty}\sum_{k\in Z} \beta_{jk}\psi_{jk}(x), \tag{1}$$

where $\alpha_{0k} = \int_R f(x)\overline{\varphi_{0k}(x)}dx$, $\beta_{jk} = \int_R f(x)\overline{\psi_{jk}(x)}dx$,

$$\sum_{k\in Z} |\alpha_{0k}|^2 + \sum_{j=0}^{\infty}\sum_{k\in Z} |\beta_{jk}|^2 < \infty.$$

That is, series (1) converges in the norm of the space $L_2(R)$. Representation (1) is called wavelet representation.

Let $X = \{X(t), t \in R\}$ be a centered wide-sense stationary random process (from this moment we will refer to wide-sense stationary processes simply as stationary processes), $X(t,\omega) \in L_2(\Omega, F, P)$ for all $t \in R$ (where (Ω, F, P) is the standard probability space to which belong all random variables $X(t,\omega)$), $R(\tau) = EX(t+\tau)\overline{X(t)}$.

There has been proved the following theorem in [3].

Theorem 1. *Let $X = \{X(t), t \in R\}$ be centered stationary random process, $R(\tau) = EX(t+\tau)\overline{X(t)}$. Suppose that $R(\tau)$ is a continuous function and process $X(t)$ has spectral density, i.e. $R(\tau) = \int_R \exp\{-i\tau\lambda\} f(\lambda)d\lambda$, where f is real-valued, $f(\lambda) \geq 0$, $\int_{-\infty}^{\infty} f(\lambda)d\lambda = R(0) < \infty$. Let $\{\varphi_{0k}(x), \psi_{jk}(x), k \in Z, j = 0, 1, 2, \ldots\}$ be a wavelet basis. Then*

$$X(t) = \sum_{k\in Z} \xi_{0k}\alpha_{0k}(t) + \sum_{j=0}^{\infty}\sum_{k\in Z} \eta_{jk}\beta_{jk}(t), \tag{2}$$

where

$$\alpha_{0k}(t) = \frac{1}{\sqrt{2\pi}}\int_R (f(y))^{1/2} \exp\{-iy(t-k)\}\overline{\hat{\varphi}(y)}dy, \tag{3}$$

$$\beta_{jk}(t) = \frac{1}{\sqrt{2\pi}2^{j/2}}\int_R (f(y))^{1/2} \exp\left\{-iy\left(t-\frac{k}{2^j}\right)\right\}\overline{\hat{\psi}\left(\frac{y}{2^j}\right)}dy, \tag{4}$$

ξ_{0k} and η_{jk} are centered random variables such that $E\xi_{0k}\overline{\xi_{0l}} = \delta_{kl}$, $E\eta_{mk}\overline{\eta_{nl}} = \delta_{mn}\delta_{kl}$, $E\xi_{0k}\overline{\eta_{nl}} = 0$, δ_{kl} is Kronecker delta and series (2) converges in mean square.

The expansion (2) has been used in [3] for modeling of Gaussian stochastic processes and obtaining inequality for given accuracy and reliability of appropriate model in $L_p([0,T])$. We'll give refinement of these theorems for $L_2([0,T])$.

2. Simulation of stationary Gaussian processes.

If we have a Gaussian process $X(t)$ which satisfies conditions of theorem 1, then we can consider as a model of $X(t)$ a process

$$\hat{X}(t)=\sum_{k=-(N_0-1)}^{N_0-1}\xi_{0k}\alpha_{0k}(t)+\sum_{j=0}^{N-1}\sum_{k=-(M_j-1)}^{M_j-1}\eta_{jk}\beta_{jk}(t),$$

where ξ_{0k}, η_{jk} are independent random variables with distribution $N(0,1)$, $\alpha_{0k}(t)$ and $\beta_{jk}(t)$ are calculated using formulae (3) and (4), $N_0>1$, $N>1$, $M_j>1$ $(j=0,\dots,N-1)$.

Definition. *Model $\hat{X}(t)$ approximates process $X(t)$ with given reliability $1-\delta$, $0<\delta<1$, and accuracy $\varepsilon>0$ in $L_p([0,T])$ if*

$$P\Big\{\Big(\int_0^T|X(t)-\hat{X}(t)|^p dt\Big)^{1/p}>\varepsilon\Big\}\le\delta.$$

There has been proved the following theorem in [3].

Theorem 2. *Stochastic process $\hat{X}(t)$ approximates process $X(t)$ with reliability $1-\delta$ and accuracy ε in $L_p([0,T])$ ($p\ge 1, 0<\delta<\min\{1,2e^{-p/2}\}$),*

$$P\Big\{\Big(\int_0^T|X(t)-\hat{X}(t)|^p dt\Big)^{1/p}>\varepsilon\Big\}\le\delta \tag{5}$$

if

$$\sup_{t\in[0,T]}E|X(t)-\hat{X}(t)|^2=\sup_{t\in[0,T]}\Big(\sum_{k:|k|\ge N_0}|\alpha_{0k}(t)|^2+\sum_{j=0}^{N-1}\sum_{k:|k|\ge M_j}|\beta_{jk}(t)|^2+$$

$$\sum_{j=N}^{\infty}\sum_{k\in Z}|\beta_{jk}(t)|^2\Big)\le\frac{\varepsilon^2}{2T^{2/p}\ln\frac{2}{\delta}}.$$

This result can be made more exact for $p=2$. We'll give first some auxiliary facts.

There has been proved the following assertion in [2], which we give here in simplified form.

Theorem 3. *Let $\{\mathbf{T},\mathbf{U},\mu\}$ be a measurable space. Consider a random series*

$$S(t)=\sum_{k=1}^{\infty}\xi_k f_k(t),\ \ t\in\mathbf{T}, \tag{6}$$

where $\xi=\{\xi_k,k=1,2,\dots\}$ is a family of Gaussian random variables and $f=\{f_k(t),k=1,2,\dots\}$ is a family of real-valued $L_2(\mathbf{T})$ functions. Let the *random variables in (6) be either uncorrelated with $E\xi_k^2 = \sigma_k^2$ or the system of functions $f_k(t)$ be orthogonal, that is,*

$$\int_{\mathbf{T}} f_k(t) f_l(t) d\mu(t) = \delta_{kl} a_k^2,$$

where δ_{kl} is the Kronecker symbol. If the series

$$\sum_{k=1}^{\infty} \sigma_k^2 a_k^2 < \infty$$

converges, then the series (6) is mean square convergent in $L_2(\mathbf{T})$ and for all $x > \sqrt{A_n}$ and n=1,2, ... we have

$$P\left\{ \left\| \sum_{k=n}^{\infty} \xi_k f_k(t) \right\|_{L_2(\mathbf{T})} > x \right\} \le e^{1/2} \frac{x}{\sqrt{A_n}} \exp\left\{ -\frac{x^2}{2A_n} \right\}, \tag{7}$$

where $A_n = \sum_{k=n}^{\infty} \sigma_k^2 a_k^2$.

The following statement about bounds for coefficients $\alpha_{0k}(t)$ and $\beta_{jk}(t)$ was proved in [3].

Lemma 1. *Let $R(\tau)$ be a covariance function which satisfies conditions of theorem 1, $R(\tau) = \int_R \exp\{-i\tau\lambda\} f(\lambda) d\lambda$. Let $\hat{\varphi}(y)$ be the Fourier transform of a f-wavelet $\varphi(x)$. Let $\hat{\varphi}(y)$ be a continuous function and assertions i) – iii) hold true for all $y \in R$. Let $\hat{\psi}(y)$ be the Fourier transform of m-wavelet $\psi(x)$ corresponding to $\varphi(x)$. Let $g(y) = \sqrt{f(y)}$ and there exist $g'(y)$, $\hat{\psi}'(y)$, $\hat{\varphi}'(y)$; $|\hat{\psi}(y)| < C_1$, $|\hat{\psi}'(y)| < C_2$, $|\hat{\varphi}(y)|$ is bounded,*

$$\int_R g(y) dy < \infty, \qquad \int_R |g'(y)||y| dy < \infty, \qquad \int_R g(y)|y| dy < \infty,$$

$$\int_R |g'(y)||\hat{\varphi}(y)| dy < \infty, \qquad \int_R g(y)|\hat{\varphi}'(y)| dy < \infty,$$

$\alpha_{0k}(t)$ and $\beta_{jk}(t)$ are given in (3) and (4). If $k \neq 0$ then for all $t \in R$

$$|\beta_{jk}(t)| \le \frac{A + B|t|}{|k| 2^{j/2}},$$

where

$$A = \frac{C_2}{\sqrt{2\pi}} \int_R (|g'(y)||y| + g(y)) dy, \quad B = \frac{C_2}{\sqrt{2\pi}} \int_R g(y)|y| dy;$$

$$|\alpha_{0k}(t)| \le \frac{A_1 + B_1|t|}{|k|},$$

where

$$A_1 = \frac{1}{\sqrt{2\pi}} \int_R (|g'(y)||\hat{\varphi}(y)| + g(y)|\hat{\varphi}'(y)|) dy,$$

$$B_1 = \frac{1}{\sqrt{2\pi}} \int_R g(y)|y||\hat{\varphi}(y)| dy.$$

For all $t \in R$ *and* $j = 0, 1, 2, \ldots$

$$|\beta_{j0}(t)| \le \frac{C_2}{\sqrt{2\pi}2^{3j/2}} \int_R g(y)|y|dy, \; |\alpha_{00}(t)| \le \frac{1}{\sqrt{2\pi}} \int_R g(y)|\hat{\varphi}(y)|dy.$$

Now we can formulate the main result of our paper.

Theorem 4. *Let Gaussian process X(t) satisfy restrictions of theorem 1, f-wavelet* φ *and m-wavelet* ψ *satisfy conditions of lemma 1. Model* $\hat{X}(t)$ *approximates process* $X(t)$ *with accuracy* ε *and reliability* $1-\delta$ $(0 \le \delta \le 1)$ *in* $L_2([0,T])$,

$$P\Big\{ \Big(\int_0^T |X(t) - \hat{X}(t)|^2 dt \Big)^{1/2} > \varepsilon \Big\} \le \delta, \tag{8}$$

if $A_n < \frac{\varepsilon^2}{x_\delta^2}$, *where* x_δ $(x_\delta \ge 1)$ *is a root of equation* $e^{1/2} x e^{-x^2/2} = \delta$,

$$A_n = \sum_{k:|k|\ge N_0} \int_0^T |\alpha_{0k}(t)|^2 dt +$$

$$+ \sum_{j=0}^{N-1} \sum_{k:|k|\ge M_j} \int_0^T |\beta_{jk}(t)|^2 dt + \sum_{j=N}^{\infty} \sum_{k \in Z} \int_0^T |\beta_{jk}(t)|^2 dt,$$

$\alpha_{0k}(t)$ *and* $\beta_{jk}(t)$ *are defined by formulae (3) and (4).*

Proof. Let's consider stochastic process

$$\tilde{X}(t) = \sum_{k \in Z} \xi^*_{0k} \alpha_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k \in Z} \eta^*_{jk} \beta_{jk}(t), \tag{9}$$

where ξ^*_{0k}, η^*_{jl} $(j = 0, 1, 2, \ldots; k, l \in Z)$ are i.i.d. random variables with distribution $N(0,1)$. It's easy to see that correlation functions of processes $X(t)$ and $\tilde{X}(t)$ are the same. Therefore we may consider process $\tilde{X}(t)$ instead of process $X(t)$. If we apply theorem 3 to series (9) we obtain inequality (8).

Corollary. *Model* $\hat{X}(t)$ *approximates process* $X(t)$ *with accuracy* ε *and reliability* $1-\delta$ *in* $L_2([0,T])$ *if the following inequalities are satisfied:*

$$N_0 > 1 + \frac{6D_2}{\varepsilon_1}, \; N > \log_2 \left(\frac{3}{\varepsilon_1} \left(8D_1 + \frac{8D^2 T}{7} \right) \right),$$

$$M_j > \frac{12 D_1}{\varepsilon_1} \left(1 - \frac{1}{2^N} \right) + 1,$$

$(j = 0, 1, \ldots, N-1)$, *where* $\varepsilon_1 = \frac{\varepsilon^2}{x_\delta^2}$,

$$D = \frac{C_2}{\sqrt{2\pi}} \int_R g(y)|y|\, dy,$$

$D_1 = A^2 T + ABT^2 + \frac{1}{3} B^2 T^3$, $D_2 = A_1^2 T + A_1 B_1 T^2 + \frac{1}{3} B_1^2 T^3$; $g(y), A, B,$ A_1, B_1 C_2 *are defined in lemma 1,* x_δ *is defined in theorem 4.*

Proof. If we apply lemma 1 it's easy to see that under conditions of the corollary the following inequalities are true:

$$A_n < \frac{2D_2}{N_0 - 1} + \frac{4D_1}{2^{N-1}} + D^2 T \frac{1}{7 \cdot 8^{N-1}} + D_1 \sum_{j=0}^{N-1} \frac{1}{2^{j-1}(M_j - 1)},$$

$$\frac{2D_2}{N_0 - 1} < \frac{\varepsilon_1}{3}, \quad \frac{4D_1}{2^{N-1}} + D^2 T \frac{1}{7 \cdot 8^{N-1}} < \frac{\varepsilon_1}{3},$$

$$D_1 \sum_{j=0}^{N-1} \frac{1}{2^{j-1}(M_j - 1)} < \frac{\varepsilon_1}{3}.$$

Statement of the corollary immediately follows from these inequalities and theorem 4. □

3. Conclusions

There has been obtained a theorem about accuracy and reliability of simulation in $L_2([0,T])$ for a certain class of stationary Gaussian processes. This theorem is more exact than previous result for $L_p([0,T])$ when $p = 2$.

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References

1. W.Härdle, G. Kerkacharian, D. Picard, A. Tsybakov, *Wavelets, Approximation and Statistical Applications*, Springer, New York, (1998).
2. Yu.V. Kozachenko, A.O. Pashko, *Accuracy of simulation of stochastic processes in norms of Orlicz spaces. I*, Theory Probab. Math. Statist., **58**, (1999), 51–92.
3. Yu. Kozachenko, E. Turchyn, *On a generalization of Walter-Zhang's wavelet-based KL-like expansion for stationary random processes*, submitted to Applied and Computational Harmonic Analysis.
4. G. Walter, J. Zhang, *A wavelet-based KL-like expansion for wide-sense stationary random processes*, IEEE Trans. Signal Process., **42**, (1994), no.7, 1737–1745.
5. G. Walter, J. Zhang, *Wavelets based on band-limited processes with a KL-type property*, Proc. SPIE Conf. Visual Inform. Processing II (Orlando, FL), Apr. 1993, 336–343.

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