Theory of Stochastic Processes
Vol. 12 (28), no. 3–4, 2006, pp. 63–74

ALEXANDER KUKUSH AND MARIA POLEKHA

A GOOGNESS OF-FIT-TEST FOR A MULTIVARIATE ERRORS-IN-VARIABLES MODEL

A multivariate errors-in-variables model $AX \approx B$ is considered, where the data matrices A and B are observed with errors, and a matrix parameter X is to be estimated. A goodness-of-fit test which is based on the moment estimator is constructed. The proposed test is asymptotically chi-squared under null hypothesis. The power of the test is discussed.

1. Introduction

Errors-in-variables (EIV) models are rather important in practical applications. It is reasonable to develop appropriate goodness–of–fit test for such models.

Consistent estimators for a multivariate errors-in-variables model under various conditions are presented in [1 – 3]. A goodness–of–fit test is constructed in [4] for a linear structural EIV model, where the distribution of the latent variable and the error distributions are normal. A polynomial EIV model is considered in [5], without the normality assumption. Present paper modifies the results of [5] for a multivariate errors-in-variables model.

We use the following notations: $\|A\|$ is Frobenius norm of a matrix A, I_p is the unit matrix of size p. The symbols $\mathbf{E}$, $\mathbf{D}$, and $\mathbf{cov}$ denote the expectation of a random matrix, the variance of random variable, and the variance - covariance matrix of a random vector, respectively. $O_p(1)$ denotes a sequence of stochastically bounded random variables, and $o_p(1)$ is a sequence of random variables that converges to 0 in probability. All the vectors in the paper are column vectors.

The paper is organized as follows. In Section 2 we introduce the model and construct an estimator. In Section 3 we present a goodness–of–fit test and show that it is asymptotically chi-squared with p degrees of freedom under null hypothesis. We introduce a local alternative and investigate the power of the test in Section 4. Section 5 concludes, and the proofs of the results are presented in Appendix.

2. The model and the estimator

Consider the model of observations:

$$A^0X = B^0,\ A = A^0 + \tilde{A},\ B = B^0 + \tilde{B}, \tag{1}$$

2000 *Mathematics Subject Classification.* Primary 62H15.
Key words and phrases. Multivariate errors-in-variables model, goodness-of-fit test, moment estimator, asymptotically chi-squared, power of the test.

where $A^0 \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$, $B^0 \in \mathbb{R}^{m\times p}$. Here the data matrices A, B are observed, and A^0, B^0 are unknown nonrandom matrices, and $\tilde{A}$, $\tilde{B}$ are matrices of random errors.

Let $A^T = [a_1 ... a_m]$, $B^T = [b_1 ... b_m]$, and we use similar notations for the rows of A^0, B^0, $\tilde{A}$, $\tilde{B}$.

Rewrite the model (1) as a multivariate lineal model:

$$X^T a_i^0 = b_i^0, \ i = \overline{1,m}; \tag{2}$$

$$b_i = b_i^0 + \tilde{b}_i, \ a_i = a_i^0 + \tilde{a}_i, \ i = \overline{1,m}.$$

We assume the following conditions:

a) the sequences of errors vectors $\{\tilde{a}_i, i \geq 1\}$ and $\{\tilde{b}_i, i \geq 1\}$ are two IID centered sequences of random errors, independent of each other,

b) for all i, $\tilde{a} \overset{d}{=} \tilde{a}_i$, $\tilde{b} \overset{d}{=} \tilde{b}_i$ and $\mathbf{E}\tilde{a} = 0$, $\mathbf{E}\tilde{b} = 0$;

c) $\mathbf{cov}\tilde{a} =: S_{\tilde{a}}$ is known and $\mathbf{cov}\tilde{b} =: S_{\tilde{b}}$ is unknown.

The adjusted least squares (ALS) estimator of matrix parameter X is

$$\hat{X} := (A^T A - \mathbf{E}\tilde{A}^T\tilde{A})^{-1} A^T B = (\sum_{i=1}^m a_i a_i^T - \mathbf{E}\tilde{a}_i\tilde{a}_i{}^T)^{-1} \sum_{i=1}^m a_i b_i^T =$$

$$= (\overline{aa^T - \mathbf{E}\tilde{a}\tilde{a}^T})^{-1}\overline{ab^T},$$

$$\hat{X} = \bar{H}^{-1}\overline{ab^T}, \tag{3}$$

where $H := aa^T - \mathbf{E}\tilde{a}\tilde{a}^T$. Hereafter the bars denote averages, e.g., $\overline{ab^T} = \sum\limits_{i=1}^m a_i b_i^T / m$.

Lemma 1[6]. *Assume that the following conditions are satisfied.*

(i) $\mathbf{E}\|\tilde{a}\|^4 < \infty$, $\mathbf{E}\|\tilde{b}\|^4 < \infty$.

(ii) There exists $V := \lim\limits_{m\to\infty} \overline{a^0 a^{0T}}$ *and* V *is positive definite.*

Then $\bar{H}$ *is nonsingular with probability tending to 1, and*

$$\hat{X} \xrightarrow{P} X \ \ as \ \ m \to \infty, \tag{4}$$

$$\hat{S}_{\tilde{b}} := \overline{bb^T} - \overline{ba^T}\hat{X} \xrightarrow{P} S_{\tilde{b}} \ \ as \ \ m \to \infty. \tag{5}$$

The estimator of $\hat{X}$ is well-defined for $m \geq m_0(\omega)$ a.s., under the conditions of Lemma 1. If the matrix $\bar{H} = \bar{H}(m,\omega)$ is singular, then the estimator is $\hat{X} = \bar{H}^{\dagger}\overline{ab^T}$, where $\bar{H}^{\dagger}$ is pseudoinverse matrix.

3. Construction of the test

For the response vector b and the corresponding latent vector a^0 we consider the following hypotheses

$\mathbf{H_0}$: there exists a matrix $X \in \mathbb{R}^{n\times p}$, for which the equality holds true:

$$\mathbf{E}(b - X^T a^0) = 0, \tag{6}$$

and $\mathbf{H_1}$: for all matrices $X \in \mathbb{R}^{n\times p}$,

$$\mathbf{E}(b - X^T a^0) \text{ is not identically equal to } 0. \tag{7}$$

We want to construct a test statistic for the null hypothesis using the observations a_i and b_i, $i = 1, 2, ..., n$.

Let $w(a^0)$ be a scalar weight function. Then under null hypothesis we have equality $\mathbf{E}[(b-X^T a^0)w(a^0)] = 0$. We will construct a vector polynomial $s(a)$, such that under $\mathbf{H_0}$ the following relation is true:

$$\mathbf{E}[(b - X^T s(a))w(a)] = 0. \tag{8}$$

Such a construction is possible if one chooses $w(a)$ as follows: $w(a) = e^{\lambda^T a}$, $a \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)^T$ is fixed, $\lambda_k \neq 0$, $k = \overline{1,n}$. We fix such a λ and assume that the corresponding exponential moment of $\tilde{a}$ exists and satisfies the condition:

(iii) $\mathbf{E}[(1 + \|\tilde{a}\|)e^{\lambda^T \tilde{a}}] < \infty$.

For the chosen weight function, relation (8) holds if for every a^0 one has:

$$a^0 \cdot \mathbf{E}(e^{\lambda^T \tilde{a}}) = \mathbf{E}(s(a^0 + \tilde{a})e^{\lambda^T \tilde{a}}).$$

Then (8) holds for $s(a) = a - \frac{\mathbf{E}(\tilde{a}e^{\lambda^T \tilde{a}})}{\mathbf{E}(e^{\lambda^T \tilde{a}})}$. Denote $\mu_0 = \mathbf{E}(e^{\lambda^T \tilde{a}})$ and $\mu_1 = \mathbf{E}(\tilde{a}e^{\lambda^T \tilde{a}})$, then $s(a) = a - \frac{\mu_1}{\mu_0}$.

Define a statistic of the score type

$$T_m^0 = \frac{1}{m}\sum_{i=1}^{m}(b_i - \hat{X}^T s(a_i))e^{\lambda^T a_i} = \overline{(b - \hat{X}^T s(a))e^{\lambda^T a}}. \tag{9}$$

We introduce further assumptions to derive an asymptotic expansion of $\sqrt{m} \cdot T_m^0$.

(iv) $\mathbf{E}[(1 + \|\tilde{a}\|^2)e^{2\lambda^T \tilde{a}}] < \infty$.

This condition is stronger than (iii). For arbitrary function $f(a^0)$, we denote $M(f(a^0)) = \lim\limits_{m\to\infty} \overline{f(a^0)}$, provided the limit exists and finite; a_j^0 is jth component of the vector a^0.

(v) $\exists M((a^0(j))^l (a^0(k))^r e^{\lambda^T a^0})$, for all $l, r \geq 0$, $l + r \leq 2$, $j, k = \overline{1,n}$.

(vi) $\|\overline{a^0 a^{0T}} - V\| = o(m^{-1/4})$, as $m \to \infty$.

Lemma 2. *Assume (i), (ii), and (iv) to (vi). Then*

$$\sqrt{m} \cdot T_m^0 = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\tilde{b}_i(e^{\lambda^T a_i} - a_i^T f) + X^T \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\eta_i + o_p(1), \tag{10}$$

where $\eta_i := (a_i^0 - s(a_i))e^{\lambda^T a_i} + (H_i - a_i^0 a_i^T)f$ *are independent random vectors with expectation 0,* $H_i = a_i a_i^T - \mathbf{E}\tilde{a}_i\tilde{a}_i^T$, $f := V^{-1}M(a^0 e^{\lambda^T a^0})\mu_0$, *and matrix* V *comes from (ii).*

We need some more assumptions in order to apply the central limit theorem in the Lyapunov form to the statistic $\sqrt{m} \cdot T_m^0$.

(vii) $\exists \delta > 0 : \mathbf{E}[(1 + \|\tilde{a}\|^{2+\delta})e^{(2+\delta)\lambda^T \tilde{a}}] < \infty$, and $\mathbf{E}\|\tilde{b}\|^{2+\delta} < \infty$.

(viii) There exist $M((a^0(j))^l(a^0(k))^r e^{\lambda^T a^0})$, for all $l, r \geq 0,\ l+r \leq 3$, and $M((a^0(j))^l(a^0(k))^r e^{2\lambda^T a^0})$, for all $l, r \geq 0,\ l+r \leq 2;\ j,k = \overline{1,n}$.

(ix) $\exists \delta > 0 : \overline{\|a^0\|^{4+\delta}} + \overline{e^{(2+\delta)\lambda^T a^0}} + \overline{\|a^0\|^{2+\delta} e^{(2+\delta)\lambda^T a^0}} \leq const.$

(x) $\overline{e^{2\lambda^T a^0} \cdot \|a^0\|^4} = o(m)$, as $m \to \infty$.

Condition (vii) absorbs conditions (iii) and (iv), and conditions (viii) absorbs condition (v). Condition (ix) means that the higher empirical moments are bounded.

Lemma 3. *Assume (ii), and (vii) to (ix). Then* $\sqrt{m} \cdot T_m^0 \xrightarrow{d} N(0, \Sigma_T)$, *where* $\Sigma_T := S_{\tilde{b}} \cdot M[\mathbf{E}(e^{\lambda^T a} - a^T f)^2] + X^T [I_n, f^T \otimes I_n] \cdot M(U) \cdot [I_n, f^T \otimes I_n]^T X$,

$$M(U) := \lim_{m\to\infty} \overline{\mathbf{cov}(Z(a))},\ Z(a_i) := \begin{bmatrix} (a_i^0 - s(a_i))e^{\lambda^T a_i} \\ vec(H_i) - vec({a_i}^0 {a_i}^T) \end{bmatrix},\ i = \overline{1,m},$$

the symbol $\otimes$ is Kronecker product, and vector f comes from Lemma 2.

Under the conditions of Lemma 3 and condition (x), a consistent estimator $\hat{\Sigma}$ of Σ_T is constructed,

$\hat{\Sigma}_T := \hat{S}_{\tilde{b}} \cdot \overline{(e^{\lambda^T a} - a^T f)^2} +$

$$+\hat{X}^T [I_n, f^T \otimes I_n] \cdot \widehat{\mathbf{cov}} \begin{bmatrix} (a^0 - s(a))e^{\lambda^T a} \\ vec(H - a^0 a^T) \end{bmatrix} \cdot [I_n, f^T \otimes I_n]^T \hat{X}, \tag{11}$$

where $\hat{f}$, $\widehat{\mathbf{cov}}$ are approximations described below.

A. Since $\bar{H} \xrightarrow{P} V$ and $\overline{s(a)e^{\lambda^T a}} \xrightarrow{P} M(a^0 e^{\lambda^T a^0})\mu_0$ as $m \to \infty$, we get the estimator $\hat{f} = \bar{H}^{-1}\overline{s(a)e^{\lambda^T a}}$.

B.

$$M\left(\mathbf{cov}\begin{bmatrix} (a^0 - s(a))e^{\lambda^T a} \\ vec(H - a^0 a^T) \end{bmatrix}\right) = M\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix}.$$

We want to construct $\hat{\Sigma}_{ij}$ for $M(\Sigma_{ij}),\ i,j = \overline{1,2}$, based on observations $a_i,\ i = \overline{1,m}$. We need the following auxiliary statement.

Lemma 4. *Let $k \geq 0$, and $p(a^0)$ be a polynomial of degree k, and $\{a_i^0, i \geq 1,\}$ be a sequence of nonrandom vectors in $\mathbb{R}^n$, satisfying the condition*

(xi) $\overline{(1 + \|a^0\|^{2k})e^{2\lambda^T a^0}} = o(m)$, *as* $m \to \infty$.

Let $a_i = a_i^0 + \tilde{a}_i,\ i \geq 1$, and vectors $\tilde{a}_i$ satisfy the conditions a) and b), and the following condition

(xii) $\mathbf{E}[(1 + \|\tilde{a}\|^{2k})e^{2\lambda^T \tilde{a}}] < \infty$.

Assume also that the limit $M(p(a^0)e^{\lambda^T a^0}) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m p(a_i^0)e^{\lambda^T a_i^0}$ *exists and is finite.*

Then there exists a polynomial $p_1(a)$ of degree k, $a \in \mathbb{R}^n$, such that

$$\frac{1}{m}\sum_{i=1}^m p_1(a_i)e^{\lambda^T a_i} \xrightarrow{P} M(p(a^0)e^{\lambda^T a^0}),\ \textit{as } m \to \infty. \tag{12}$$

Consider the matrix $\Sigma_{11} = a^0 {a^0}^T e^{2\lambda^T a^0}\mathbf{E}e^{2\lambda^T \tilde{a}} - \mathbf{E}s(a)e^{2\lambda^T a}{a^0}^T -$
$-a^0\mathbf{E}s(a)^T e^{2\lambda^T a} + \mathbf{E}s(a)s(a)^T e^{2\lambda^T a} =: U_1 - U_2 - U_2^T + U_3$. Next,

$\mathbf{E}(aa^T e^{2\lambda^T a}) = a^0 a^{0T} e^{2\lambda^T a^0} m_1 + a^0 e^{2\lambda^T a^0} m_2 + a^{0T} e^{2\lambda^T a^0} m_2^T + e^{2\lambda^T a^0} m_3$,
$\mathbf{E}(a e^{2\lambda^T a}) = a^0 e^{2\lambda^T a^0} m_1 + e^{2\lambda^T a^0} m_2$,
$\mathbf{E}(a^T e^{2\lambda^T a}) = a^{0T} e^{2\lambda^T a^0} m_1 + e^{2\lambda^T a^0} m_2^T$, $\mathbf{E}(e^{2\lambda^T a}) = e^{2\lambda^T a^0} m_1$, where
$m_1 = \mathbf{E} e^{2\lambda^T \tilde{a}}$, $m_2 = \mathbf{E}\tilde{a} e^{2\lambda^T \tilde{a}}$, $m_3 = \mathbf{E}\tilde{a}\tilde{a}^T e^{2\lambda^T \tilde{a}}$. Then by Lemma 4, the estimator of U_1 equals

$$\hat{U}_1 = \overline{aa^T e^{2\lambda^T a}} - \overline{a e^{2\lambda^T a}} \cdot \frac{m_2^T}{m_1} - \overline{a^T e^{2\lambda^T a}} \cdot \frac{m_2}{m_1} - \overline{e^{2\lambda^T \tilde{a}}}(\frac{m_3}{m_1} - \frac{2m_2^T m_2}{m_1^2}).$$

Again from the previous expression and the following identity

$$\mathbf{E}(s(a) a^T e^{2\lambda^T a}) = \mathbf{E}(s(a) e^{2\lambda^T a})(a^0)^T + a^0 e^{2\lambda^T a^0} m_2 + e^{2\lambda^T a^0}(m_3 - \mu_1/\mu_0 \cdot m_2^T),$$

we get an approximation:

$$\hat{U}_2 = \overline{s(a) a^T e^{2\lambda^T a}} - \overline{a e^{2\lambda^T a}} \cdot \frac{m_2^T}{m_1} - \overline{e^{2\lambda^T \tilde{a}}}(\frac{m_3 - \mu_1/\mu_0 \cdot m_2^T}{m_1} - \frac{m_2^T m_2}{m_1^2}).$$

The next approximation is $\hat{U}_3 = \overline{s(a) s(a)^T e^{2\lambda^T a}}$. Finally,

$$\hat{\Sigma}_{11} =: \hat{U}_1 - \hat{U}_2 - \hat{U}_2^T + \hat{U}_3.$$

In a similar way one can construct other approximations $\hat{\Sigma}_{ij}$ and obtain the approximation (11).

Then the test statistic defined as follows: $T_m^2 = m \cdot \|\hat{\Sigma}_T^{-1/2} T_m^0\|^2$. Since $\hat{\Sigma}_T$ is the consistent estimator of Σ_T, we obtain by Lemma 3 the following theorem.

Theorem 1. *Suppose that the conditions of lemma 3 and condition (x) are satisfied. Assume as well that at least one of the following two conditions is satisfied:*

(xiii) $M[\mathbf{E}(e^{\lambda^T a} - a^T f)^2] > 0$, *and* $S_{\tilde{b}}$ *is positive definite;*

(xvi) $n \geq p$, $rank X = p$, *and the matrix*

$$M(U) := M\left(\mathbf{cov}\begin{bmatrix}(a^0 - s(a)) e^{\lambda^T a} \\ vec(H - a^0 a^T)\end{bmatrix}\right) \quad \textit{is nonsingular.}$$

Then $T_m^2 \xrightarrow{d} \chi_p^2$, *under hypothesis* $\mathbf{H_0}$.

Let $\alpha > 0$ and $\chi_{p\alpha}^2$ be corresponding quantile of the χ_p^2 distribution, i.e., $P\{\chi_p^2 > \chi_{p\alpha}^2\} = \alpha$. Based on Theorem 1, we construct the following goodness-of -fit test with asymptotic confidence probability $1 - \alpha$. If $T_m^2 \leq \chi_{p\alpha}^2$ then we accept the hypothesis $\mathbf{H_0}$; if $T_m^2 > \chi_{p\alpha}^2$ then we reject the null hypothesis.

4. The power properties of the test

Consider the following sequences of models:

$$\mathbf{H_{1,m}}: \; b_i = X^T a_i^0 + \frac{g(a_i^0)}{\sqrt{m}} + \tilde{b}_i, \; a_i = a_i^0 + \tilde{a}_i, \; i = \overline{1, m}, \tag{13}$$

where $g : \mathbb{R}^n \to \mathbb{R}^p$ is a nonlinear vector function which satisfies the conditions:

(xv) $\exists\ M(g(a^0)e^{\lambda^T a^0})$ and $\exists\ M(g(a^0){a^0}^T)$;

(xvi) $\overline{\|g(a^0)\|^2 \cdot (1 + \|a^0\|^2 + e^{2\lambda^T a^0})} = o(m)$, as $m \to \infty$.

Then under $\mathbf{H_{1,m}}$ we have: $\frac{1}{\sqrt{m}} \sum_{i=1}^{m} (b_i - \hat{X}^T s(a_i)) e^{\lambda^T a_i} \xrightarrow{d} N(C, \Sigma_T)$, where a vector C is found bellow.

Now, we define a noncentral chi-squared distribution $\chi_p^2(\tau)$ with p degrees of freedom, and noncentrality parameter τ.

Definition.

For $p \geq 1$ and $\tau \geq 0$, let $\chi_p^2(\tau) \stackrel{d}{=} \|N(\tau e, I_p)\|^2$, where $e \in \mathbb{R}^p$, $\|e\| = 1$, or equivalently $\chi_p^2(\tau) \stackrel{d}{=} (\gamma_1 + \tau)^2 + \sum_{i=2}^{p} \gamma_i^2$, where $\{\gamma_i\}$ are independent standard normal variables.

Theorem 2. *Suppose that all the conditions of Theorem 1 and conditions (xv), (xvi) are satisfied. Then, under* $\mathbf{H_{1,m}}$, $T_m^2 \xrightarrow{d} \chi_p^2(\|\Sigma_T^{-1/2} C\|)$,

$$\text{where } C := \mu_0 \cdot M(g(a^0)e^{\lambda^T a^0}) - M(g(a^0){a^0}^T) V^{-1} M(a^0 e^{\lambda^T a^0}). \tag{14}$$

Here $\chi_p^2(\|\Sigma_T^{-1/2} C\|)$ *is noncentral chi-squared random variable with p degrees of freedom and noncentrality parameter* $\|\Sigma_T^{-1/2} C\|$.

From Theorem 2 we can find the asymptotic power of the test under local alternative (13). It is easy to see that the asymptotic power of the test is increasing function of $\|\Sigma_T^{-1/2} C\|$. In other words, the larger $\|\Sigma_T^{-1/2} C\|$, the more powerful test we will have.

Since in present paper the vector λ is arbitrary chosen and the function g is unknown, it is reasonable to consider the next two problems.

1) We assume that the weight function $w(a) = e^{\lambda^T a}$ is fixed. We discuss for which g the power is the largest. For simplicity we suppose that $\{a_i^0,\ i \geq 1\}$ are IID random vectors, independent of $\{\tilde{a}_i$, and $\tilde{b}_i,\ i \geq 1\}$, and $a^0 \stackrel{d}{=} a_i^0$. Then $\|\Sigma_T^{-1/2} C\| = \mu_0 \cdot \|\Sigma_T^{-1/2} [\mathbf{E}(g(a^0)e^{\lambda^T a^0}) -$

$$-\mathbf{E}(g(a^0){a^0}^T)\mathbf{E}(a^0 {a^0}^T)^{-1}\mathbf{E}(a^0 e^{\lambda^T a^0})]\| = \mu_0 \|\mathbf{E}(\Sigma_T^{-1/2} g(a^0) h_\lambda(a^0))\|.$$

Here h_λ is defined from the expansion: $e^{\lambda^T a^0} = z^T a^0 + h_\lambda(a^0)$, $z \in \mathbb{R}^n$ and $\mathbf{E} h_\lambda(a^0)(v^T a^0) = 0$, for all $v \in \mathbb{R}^n$. The ratio $\|\Sigma_T^{-1/2} C\|^2 / \|\Sigma_T^{-1/2} g(a^0)\|_{L_2}^2$ is maximal, if $g(a^0) = h_\lambda(a^0) w$, for certain nonrandom $w \in \mathbb{R}^p$, $w \neq 0$. We have $h_\lambda(a^0) = e^{\lambda^T a^0} - \mathbf{E}(a^0 {a^0}^T)^{-1/2} \mathbf{E}(e^{\lambda^T a^0} {a^0}^T) a^0 \mathbf{E}(a^0 {a^0}^T)^{-1/2}$, and its consistent estimator is

$$\hat{h}_\lambda(a^0) = e^{\lambda^T a^0} - \bar{H}^{-1/2} \cdot \frac{1}{\mu_0} (\overline{e^{\lambda^T a} s(a)^T} - \overline{e^{\lambda^T a}} (\mathbf{E} \tilde{a}^T e^{\lambda^T \tilde{a}} - \mathbf{E} \tilde{a} e^{\lambda^T \tilde{a}})) a^0 \bar{H}^{-1/2}.$$

The function $\hat{h}_\lambda(a^0) w$, $w \neq 0$, is asymptotically optimal choice of the function g for a local alternative (13), when the weight function w is fixed.

2) Now, we consider the second problem. Let the function g be fixed and we want to choose optimally the weight function $w(a^0) = e^{\lambda^T a^0}$. We need to maximize the function $\|\Sigma_T^{-1/2} C(\lambda)\|^2$ for $\lambda_i \in \mathbb{R}^n \backslash \{\bar{0}\}$, $i = \overline{1,m}$. Here the vector function $C = C(\lambda)$ is given in (14) provided all corresponding moments of random vectors $\{\tilde{a}_i,\ a_i^0,\ i \geq 1\}$ are exist. This is a nonlinear problem, and it can be solved numerically. Of course, one has to incorporate the approximations for $\|\Sigma_T^{-1/2} C(\lambda)\|^2$ constructed by data.

5. Conclusion

We constructed a goodness-of-fit test for a multivariate errors-in-variables model if the covariance structure of errors $\tilde{b}$ is unknown, and the exponential moments and the covariance structure of errors $\tilde{a}$ are known. Using an exponential weight function, we obtained an asymptotically chi-squared statistic under null hypothesis. A local alternative hypothesis is introduced, under which the test has a noncentral chi-squared asymptotic distribution. We discussed for what local alternatives the power of the test is the largest.

Appendix

Proof of Lemma 1. First we prove (4).

With probability tending to 1, as $m \to \infty$, we have $\bar{H}\hat{X} = \overline{ab^T}$. Hence

$$(15)\quad (\overline{a^0 a^{0T}})^{-1}(\overline{a^0 a^{0T}} + \overline{\tilde{a} a^{0T}} + \overline{a^0 \tilde{a}^T} + \overline{\tilde{a}\tilde{a}^T - \mathbf{E}\tilde{a}\tilde{a}^T})\hat{X} = \\ = (\overline{a^0 a^{0T}})^{-1}(\overline{a^0 a^{0T}} X + \overline{\tilde{a} a^{0T}} X + \overline{a^0 \tilde{b}^T} + \overline{\tilde{a}\tilde{b}^T}),$$

or $V_m^{-1}\bar{H}\hat{X} = V_m^{-1}\overline{ab^T}$, where $V_m := \overline{a^0 a^{0T}}$ is nonsingular for $m > m_0$, and $V_m \to V$, as $m \to \infty$. We show that

$$(16)\qquad (\overline{a^0 a^{0T}})^{-1}(\overline{a^0 a^{0T}} + \overline{\tilde{a} a^{0T}} + \overline{a^0 \tilde{a}^T} + \overline{\tilde{a}\tilde{a}^T - \mathbf{E}\tilde{a}\tilde{a}^T}) \xrightarrow{P} I_n,$$

$$(17)\qquad \text{and } (\overline{a^0 a^{0T}})^{-1}(\overline{\tilde{a} a^{0T}} X + \overline{a^0 \tilde{b}^T} + \overline{\tilde{a}\tilde{b}^T}) \xrightarrow{P} 0.$$

We deal with each summand in (16) separately.

We have $\|(\overline{a^0 a^{0T}})^{-1}(\overline{\tilde{a} a^{0T}})\| \leq \|V_m^{-1}\| \cdot \|\overline{\tilde{a} a^{0T}}\|$.

Since V_m is nonsingular matrix, $\|V_m^{-1}\| \leq const \cdot \lambda_{min}^{-1}(V_m)$. By Cauchy-Schwartz inequality we obtain

$$\mathbf{E}\|\overline{\tilde{a} a^{0T}}\|^2 = \mathbf{E}\|\frac{1}{m}\sum_{i=1}^m \tilde{a}_i a_i^{0T}\|^2 = \frac{1}{m^2}\sum_{j,k=1}^n \mathbf{E}(\sum_{i=1}^m \tilde{a}_{ij} a_{ik}^0)^2 \leq$$

$$\leq \frac{1}{m^2}\sum_{j,k=1}^n (\sum_{i=1}^m \mathbf{E}\tilde{a}_{ij}^2 \cdot \sum_{i=1}^m (a_{ik}^0)^2) \leq \frac{1}{m} \cdot \|S_{\tilde{a}}\| \cdot const,$$

therefore $\|\overline{\tilde{a}a^{0T}}\| = \frac{O_p(1)}{\sqrt{m}}$. By (ii) we have

$$\|(\overline{a^0a^{0T}})^{-1} \cdot \overline{a^0\tilde{a}^T}\| = \frac{O_p(1)}{\sqrt{m} \cdot \lambda_{min}(V_m)}. \tag{18}$$

$$\text{Similarly we have } \|(\overline{a^0a^{0T}})^{-1} \cdot \overline{\tilde{a}a^{0T}}\| = \frac{O_p(1)}{\sqrt{m} \cdot \lambda_{min}(V_m)}. \tag{19}$$

Next, from (i) we get $\mathbf{E}\|\overline{\tilde{a}\tilde{a}^T - \mathbf{E}\tilde{a}\tilde{a}^T}\|^2 = \frac{1}{m^2}\sum\limits_{j,k=1}^{n}\mathbf{E}(\sum\limits_{i=1}^{m}\tilde{a_{ij}}\tilde{a_{ik}} - \mathbf{E}\tilde{a_{ij}}\tilde{a_{ik}})^2 =$ $\frac{1}{m^2}\sum\limits_{j,k=1}^{n}\sum\limits_{i=1}^{m}\mathbf{E}(\tilde{a_{ij}}\tilde{a_{ik}} - \mathbf{E}\tilde{a_{ij}}\tilde{a_{ik}})^2 = \frac{O(1)}{m}$. Therefore

$$\|(\overline{a^0a^{0T}})^{-1} \cdot \overline{(\tilde{a}\tilde{a}^T - \mathbf{E}\tilde{a}\tilde{a}^T)}\| = \frac{O_p(1)}{\sqrt{m} \cdot \lambda_{min}(V_m)}. \tag{20}$$

By the assumption b) and (ii) we get

$$\mathbf{E}\|\overline{a^0\tilde{b}^T}\|^2 = \frac{1}{m^2}\sum_{j,k=1}^{n}\mathbf{E}(\sum_{i=1}^{m}a_{ij}{}^0\tilde{b}_{ik})^2 = \frac{O(1)}{m}.$$

$$\text{Thus } \|(\overline{a^0a^{0T}})^{-1} \cdot \overline{a^0\tilde{b}^T}\| = \frac{O_p(1)}{\sqrt{m} \cdot \lambda_{min}(V_m)}. \tag{21}$$

Similarly we obtain for the last residual:

$$\|(\overline{a^0a^{0T}})^{-1} \cdot \overline{\tilde{a}\tilde{b}^T}\| = \frac{O_p(1)}{\sqrt{m} \cdot \lambda_{min}(V_m)} \tag{22}$$

Therefore, equalities (18) - (20) yield the convergence (16), and the relations (19), (21) and (22) yield the convergence (17). Then (15) implies the desired convergence $\hat{X} \xrightarrow{P} X$ as $m \to \infty$. Now we prove the convergence (5).

We have

$$\overline{bb^T} = \overline{(X^Ta^0 + \tilde{b})(X^Ta^0 + \tilde{b})^T} = X^T\overline{a^0a^{0T}}X + X^T\overline{a^0\tilde{b}^T} + \overline{\tilde{b}a^{0T}}X + \overline{\tilde{b}\tilde{b}^T},$$

$$\overline{ba^T} = \overline{(X^Ta^0 + \tilde{b})a^T} = X^T\overline{a^0a^{0T}} + X^T\overline{a^0\tilde{a}^T} + \overline{\tilde{b}\tilde{a}^T} + \overline{\tilde{b}a^{0T}},$$

and $\hat{X} = X + o_p(1)$, then $\widehat{S_{\tilde{b}}} = X^T\overline{a^0\tilde{b}^T} + \overline{\tilde{b}\tilde{b}^T} - X^T\overline{a^0\tilde{a}^T}X - \overline{\tilde{b}\tilde{a}^T}X + o_p(1)$. From the proof of the first part of the theorem

$X^T\overline{a^0\tilde{b}^T} - X^T\overline{a^0\tilde{a}^T}X - \overline{\tilde{b}\tilde{a}^T}X = O_p(1)/\sqrt{m}$. Moreover $\overline{\tilde{b}\tilde{b}^T} \xrightarrow{P} \mathbf{E}\tilde{b}\tilde{b}^T = S_{\tilde{b}}$. As a result we obtain $\hat{S}_{\tilde{b}} \xrightarrow{P} S_{\tilde{b}}$, as $m \to \infty$. $\square$

Proof of Lemma 2. We substitute the estimator (3) into statistic (9):

$$(23)\quad T_m^0 = \overline{(b - \hat{X}^T s(a))e^{\lambda^T a}} = \overline{(b - (\bar{H}^{-1}\overline{ab^T})^T s(a))e^{\lambda^T a}} =$$
$$= \overline{(X^T a^0 + \tilde{b} - X^T\overline{a^0 a^T} \cdot \bar{H}^{-1}s(a) - \overline{\tilde{b}a^T} \cdot \bar{H}^{-1}s(a)) \cdot e^{\lambda^T a}} =$$
$$= \overline{\tilde{b}(e^{\lambda^T a} - a^T \cdot \bar{H}^{-1}\overline{s(a)e^{\lambda^T a}})} +$$
$$+ X^T(\overline{a^0 e^{\lambda^T a}} - \overline{a^0 a^T} \cdot \bar{H}^{-1}\overline{s(a)e^{\lambda^T a}}) =: F + X^T G.$$

First, we investigate the vector $\sqrt{m}F$.

Since $\bar{H} = \overline{aa^T - \mathbf{E}\tilde{a}\tilde{a}^T} \xrightarrow{P} V$, as $m \to \infty$, we denote $\overline{\Lambda} = \bar{H} - V$, $\overline{\Lambda} \approx 0$. The approximate equality " $\approx$ " means equality up to summands, converging to 0 in probability. Thus $\bar{H}^{-1} = (I_n + V^{-1}\overline{\Lambda}V^{-1})^{-1}V^{-1} = V^{-1} - V^{-1}\overline{\Lambda}V^{-1} + r_m$, where $\|r_m\| = \|\overline{\Lambda}\|^2 O_p(1)$. We show that $\sqrt{m} \cdot \|\overline{\Lambda}\|^2 \approx 0$.

From (i), and (ii), and (vi) we have $\mathbf{E}\|\bar{H} - V\|^2 =$

$$(24)\qquad = \mathbf{E}\|\overline{\tilde{a}a^{0T}} + \overline{a^0\tilde{a}^T} + \overline{\tilde{a}\tilde{a}^T - \mathbf{E}\tilde{a}\tilde{a}^T} + \overline{a^0 a^{0T}} - V\|^2 \le \frac{O(1)}{m} + \frac{o(1)}{\sqrt{m}}.$$

Therefore $\sqrt{m} \cdot \|\overline{\Lambda}\|^2 \approx 0$ and $\|r_m\| = o_p(1)/\sqrt{m}$. Moreover

$$\sqrt{m} \cdot \overline{\tilde{b}a^T} = \sqrt{m}(\overline{\tilde{b}\tilde{a}^T} + \overline{\tilde{b}a^{0T}}) = \sqrt{m} \cdot \frac{O_p(1)}{\sqrt{m}} = O_p(1),$$

$$\overline{s(a)e^{\lambda^T a}} = \overline{(a^0 + \tilde{a} - \frac{\mu_1}{\mu_0})e^{\lambda^T\tilde{a}}e^{\lambda^T a^0}} \xrightarrow{P} M(a^0 e^{\lambda^T a^0} \cdot \mu_0),$$

therefore $\overline{s(a)e^{\lambda^T a}} = O_p(1)$. Then we get

$\sqrt{m} \cdot \overline{\tilde{b}a^T}\bar{H}^{-1}\overline{s(a)e^{\lambda^T a}} \approx \sqrt{m} \cdot \overline{\tilde{b}a^T}V^{-1}M(a^0 e^{\lambda^T a^0})$. This implies

$$(25)\quad \sqrt{m}F \approx \sqrt{m} \cdot \overline{\tilde{b}(e^{\lambda^T a} - a^T V^{-1}M(a^0 e^{\lambda^T a^0}\mu_0))} = \sqrt{m} \cdot \overline{\tilde{b}(e^{\lambda^T a} - a^T f)},$$

where f is the vector defined in Lemma 2.

Next, consider $\sqrt{m}G$, where G comes from (23):

$$\sqrt{m}G \approx \sqrt{m} \cdot (\overline{a^0 e^{\lambda^T a}} - \overline{a^0 a^T}V^{-1}\overline{s(a)e^{\lambda^T a}} + \overline{a^0 a^T}V^{-1}\overline{\Lambda}V^{-1}\overline{s(a)e^{\lambda^T a}}).$$

Since $\sqrt[4]{m} \cdot \|\overline{\Lambda}\| \approx 0$ we have $m^{1/4}(\overline{a^0 a^T} - V) \approx 0$ and $\overline{s(a)e^{\lambda^T a}} = O_p(1)$. Then $\sqrt{m} \cdot \overline{a^0 a^T}V^{-1}\overline{\Lambda}V^{-1}\overline{s(a)e^{\lambda^T a}} \approx \sqrt{m} \cdot \overline{\Lambda}V^{-1}\overline{s(a)e^{\lambda^T a}} = \sqrt{m} \cdot (\overline{\bar{H}V^{-1}\overline{s(a)e^{\lambda^T a}} - \overline{s(a)e^{\lambda^T a}}})$, $\sqrt{m}G \approx \sqrt{m}(\overline{(a^0 - s(a))e^{\lambda^T a} + (H - \overline{a^0 a^T})V^{-1}\overline{s(a)e^{\lambda^T a}}})$. Since $\sqrt{m} \cdot \overline{H - a^0 a^T} = O_p(1)$ and $\overline{s(a)e^{\lambda^T a}} \xrightarrow{P} M(a^0 e^{\lambda^T a^0})$, we also have

$$(26)\qquad \sqrt{m}G \approx \sqrt{m} \cdot \overline{(a - s(a))e^{\lambda^T} + (H - a^0 a^T)f}.$$

Using (25) and (26), we obtain (10). □

Proof of Lemma 3. From the Lemma 2 we have

$$(27)\qquad \sqrt{m} \cdot T_m^0 \approx \frac{1}{\sqrt{m}}\sum_{i=1}^m z_i,$$

where $z_i =: \tilde{b_i}(e^{\lambda^T a_i} - {a_i}^T f) + X^T((a_i^0 - s(a_i))e^{\lambda^T a_i} + (H_i - {a_i}^0 {a_i}^T) f)$ are independent random vectors and $\mathbf{E} z_i = 0$. Represent the vectors z_i as

$$z_i = \tilde{b_i}(e^{\lambda^T a_i} - {a_i}^T f) + X^T [I_n, f^T \otimes I_n] \begin{bmatrix} (a_i^0 - s(a_i))e^{\lambda^T a_i} \\ vec(H_i) - vec({a_i}^0 {a_i}^T) \end{bmatrix}.$$

Then $\mathbf{E} z_i z_i^T = S_{\tilde{b}} \cdot \mathbf{E}(e^{\lambda^T a_i} - {a_i}^0 f)^2 +$

$$+ X^T [I_n, f^T \otimes I_n] \cdot \mathbf{cov} \begin{bmatrix} (a_i^0 - s(a_i))e^{\lambda^T a_i} \\ vec(H_i) - vec({a_i}^0 {a_i}^T) \end{bmatrix} [I_n, f^T \otimes I_n]^T X,$$

therefore $\lim\limits_{m \to \infty} \frac{1}{m} \sum\limits_{i=1}^m \mathbf{E} z_i z_i^T = \Sigma_T$. The limit exists due to conditions (i), (ii), (vi), and (viii). Conditions (vii), (ix) guarantee the following boundedness:

$$\exists \delta > 0 : \frac{1}{m} \sum_{i=1}^m \mathbf{E} \|z_i\|^{2+\delta} \le const.$$

Thus all the conditions of the CLT in Lyapunov form are satisfied, then

$$\frac{1}{\sqrt{m}} \sum_{i=1}^m z_i \xrightarrow{d} N(0, \Sigma_T).$$

From this and from (27) using Slutsky Lemma we get Lemma 3. □

Proof of Lemma 4. First we prove by induction that there exists a polynomial $p_1(a)$, $a \in \mathbb{R}^n$, of degree k such that

$$\mathbf{E}(p_1(a) e^{\lambda^T a}) = p(a^0) e^{\lambda^T a^0}. \tag{28}$$

1. Let $p(a^0)$ be a polynomial of degree 0. Since $\mathbf{E} e^{\lambda^T a} = e^{\lambda^T a^0} \mathbf{E} e^{\lambda^T \tilde{a}} = e^{\lambda^T a^0} \mu_0$, there exists a polynomial of degree 0, $p_1(a) = p(a^0)\mu_0^{-1}$.

2. Suppose that for arbitrary polynomial of degree less than k, $p(a^0)$, there exists $p_1(a)$ such that deg $p_1(a) < k$ and the equality (28) is satisfied.

3. Prove the existence of similar polynomial of degree k.

$\mathbf{E}(p(a) e^{\lambda^T a}) = p(a^0) e^{\lambda^T a^0} \mu_0 + e^{\lambda^T a^0} \mathbf{E} p^*(a^0, \tilde{a}) e^{\lambda^T \tilde{a}}$, where p^* is some polynomial of two variables. The expectation $\mathbf{E} p^*(a^0, \tilde{a}) e^{\lambda^T \tilde{a}}$ can be represented as $p_2(a^0) e^{\lambda^T a^0}$, where deg $p_2 <$ deg $p = k$. Therefore by part 2 of the proof, for $p_2(a^0)$ there exists a polynomial $p_1^*(a)$ of degree less than k, such that (28) is satisfied. Moreover, $\mathbf{E}(p(a) e^{\lambda^T a}/\mu_0 - p_1^*(a) e^{\lambda^T a}) = p(a^0) e^{\lambda^T a^0}$. Therefore, $\exists\ p_1(a) := p(a)/\mu_0 - p_1^*(a),\ \deg p_1 = k$.

Now, prove the convergence (12) for the constructed polynomial $p_1(a)$. In fact, we have to prove the next equality

$$\frac{1}{m} \sum_{i=1}^m p_1(a_i) e^{\lambda^T a_i} - \frac{1}{m} \sum_{i=1}^m p(a_i^0) e^{\lambda^T a_i^0} = o_p(1). \tag{29}$$

Consider the difference $\frac{1}{m}\sum\limits_{i=1}^{m}(p_1(a_i)e^{\lambda^T a_i} - p(a_i^0)e^{\lambda^T a_i^0}) =: \frac{1}{m}\sum\limits_{i=1}^{m} z_i$, $\mathbf{E}z_i = 0$, $\mathbf{D}(\frac{1}{m}\sum\limits_{i=1}^{m} z_i) = \mathbf{E}(\frac{1}{m}\sum\limits_{i=1}^{m} z_i)^2 = \frac{1}{m^2}\sum\limits_{i=1}^{m}\mathbf{E}z_i^2$,

$$\mathbf{E}z_i^2 = \mathbf{E}(p_1(a_i)e^{\lambda^T a_i})^2 - (p(a_i^0)e^{\lambda^T a_i^0})^2 \leq \mathbf{E}(p_1^2(a_i)e^{2\lambda^T a_i}),\ i = \overline{1,m}.$$

We have $\mathbf{E}z_i^2 \leq const \cdot \mathbf{E}[(1 + \|a_i\|^{2k})e^{2\lambda^T a_i}] \leq const \cdot \mathbf{E}[(1 + \|a_i^0\|^{2k} + \|\tilde{a}_i\|^{2k})e^{2\lambda^T a_i^0}e^{2\lambda^T \tilde{a}_i}] \leq (1 + \|a_i^0\|^{2k})e^{2\lambda^T a_i^0} \cdot const,\ i = \overline{1,m}$.

Then by condition (xvi) we get as $m \to \infty$,

$$\frac{1}{m^2}\sum_{i=1}^{m}\mathbf{E}z_i^2 \leq \frac{const}{m^2} \cdot \sum_{i=1}^{m}(1 + \|a_i^0\|^{2k})e^{2\lambda^T a_i^0} = \frac{1}{m}\overline{(1 + \|a^0\|^{2k})e^{2\lambda^T a^0}} \to 0,$$

Thus we obtain (29), and as a result we get the convergence (12). □

Proof of Theorem 1. From the conditions of Theorem 1 we get that Σ_T is positive matrix. Then $m \cdot \|\Sigma_T^{-1/2}T_m^0\|^2 \xrightarrow{d} \chi_p^2$. Since $\hat{\Sigma}_T$ is the consistent estimator of Σ_T, we have $T_m^2 = m \cdot \|\hat{\Sigma}_T^{-1/2}T_m^0\|^2 \xrightarrow{d} \chi_p^2$ under null hypothesis. □

Proof of Theorem 2. Assume the hypothesis $\mathbf{H_{1,m}}$. Then

$$\hat{X} = \bar{H}^{-1}\overline{ab^T} + \bar{H}^{-1}\frac{1}{\sqrt{m}} \cdot \overline{ag(a^0)^T}. \tag{30}$$

Since $\bar{H}^{-1}\overline{ab^T} \xrightarrow{P} X$ under $\mathbf{H_0}$, we have $\bar{H}^{-1} = O_p(1)$ and $\frac{1}{m}\overline{a^0 g(a^0)^T} \to 0$, $\mathbf{E}\|\frac{1}{\sqrt{m}} \cdot \overline{\tilde{a}g(a^0)^T}\|^2 \to 0$, as $m \to \infty$. Therefore from (30) we obtain $\hat{X} \xrightarrow{P} X$ under $H_{1,m}$. However for the statistic T_m^0 we have

$$\sqrt{m} \cdot T_m^0|_{H_{1,m}} = \sqrt{m} \cdot T_m^0|_{H_0} + (\overline{g(a^0)e^{\lambda^T a}} - \overline{g(a^0)a^T}\bar{H}^{-1}\overline{se^{\lambda^T a}}), \tag{31}$$

where $T_m^0|_{H_{1,m}}$, and $T_m^0|_{H_0}$ are the values of T_m^0 under the corresponding hypotheses $H_{1,m}$ and H_0.

Now, consider the last summand in (31). By conditions (xiii) and (xiv) we have $\overline{g(a^0)e^{\lambda^T a}} \approx \overline{g(a^0)\mathbf{E}e^{\lambda^T a}} = \mu_0\overline{g(a^0)e^{\lambda^T a^0}} \to \mu_0 M(g(a^0)e^{\lambda^T a^0})$ and $\overline{se^{\lambda^T a}} \xrightarrow{P} \mu_0 M(a^0 e^{\lambda^T a^0})$, $\bar{H}^{-1} \xrightarrow{P} V^{-1}$, $\overline{g(a^0)a^T} \approx \overline{g(a^0)\mathbf{E}a^T} = \overline{g(a^0)a^{0T}} \to M(g(a^0)a^{0T})$, as $m \to \infty$. Relation (31) yields

$$\sqrt{m} \cdot T_m^0|_{H_{1,m}} \xrightarrow{d} N(C, \Sigma_T), \tag{32}$$

where C is the vector defined in (14).

The conditions of Theorem 1 are satisfied, therefore $\Sigma_T > 0$. And from (32) we have the following convergence

$$m \cdot \|\Sigma_T^{-1/2} \cdot T_m^0|_{H_{1,m}}\|^2 \xrightarrow{d} \chi_p^2(\|\Sigma_T^{-1/2}C\|). \tag{33}$$

Further, due to (xii) and (xiii) we have $\hat{S}_{\tilde{b}} \xrightarrow{P} S_{\tilde{b}}$, under $\mathbf{H_{1,m}}$, and $\hat{\Sigma}_T \xrightarrow{P} \Sigma_T$ under $\mathbf{H_{1,m}}$. (we used only the observations $a_i,\ i = \overline{1,m}$, in the construction of $\hat{\Sigma}_T$, and they do not change under local alternative $\mathbf{H_{1,m}}$). Thus by relation (33). We have $T_m^2|_{H_{1,m}} \xrightarrow{d} \chi_p^2(\|\Sigma_T^{-1/2} C\|)$. $\square$

References

1. Kukush, A., and Van Huffel, S., *Consistency of element-wise weighted total least squares estimator in multivariate errors-in-variables model* $AX = B$, Metrika, **59**, (2004), no.1, 75-97.
2. Kukush, A., Markovsky, I., and Van Huffel, S., *Consistency of the structured total least squares estimator in a multivariate errors-in-variables model,* Journal of Statistical Planning and Inference, **133**, (2005), no.2, 315-358.
3. Kukush A., Markovsky I., and Van Huffel, S., *Estimation in a linear multivariate measurement error model with clustering in the regressor,* Internal Report 05-170. ESAT-SISTA. K.U.Leuven (Leuven, Belgium), (2005),
4. Zhu, L., Cui, H., and K.W.Ng., *Testing lack-of-fit for linear errors-in-variables model.* Acta Appl. Math. (to appear).
5. Kukush, A.G., Cheng, C.-L., *A goodness-of-fit test for a polynomial errors-in-variables model,* Ukrainian Mathematical Journal, **56**, (2004), no.4, 527-543.
6. Cheng, C.-L., and Schneeweiss, H., *Polynomial regression with errors in the variables,* J. R. Statist. Soc. B, **60**, (1998), 189-199.

Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: alexander_kukush@univ.kiev.ua

Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: poleha@bigmir.net