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AN EXAMPLE OF A STOCHASTIC DIFFERENTIAL EQUATION WITH THE PROPERTY OF WEAK NON-UNIQUENESS OF A SOLUTION

A family of one-dimensional diffusion processes is constructed such that each one of this family is a weak solution to some stochastic differential equation. It turns out that the property of weak uniqueness of a solution to this equation is failed.

Introduction

For some given constants $b_1 > 0$ and $b_2 > 0$, let $b(x)$ for $x \in \mathbb{R}$ ($\mathbb{R}$ is a real line) denote the function on $\mathbb{R}$ taking on the value b_1 for $x < 0$, the value b_2 for $x > 0$, and the value zero for $x = 0$. Given a real constant A, consider the following stochastic differential equation

$$dx(t) = A1\!\!\mathrm{I}_{\{0\}}(x(t))dt + d\xi(t), \tag{1}$$

where $1\!\!\mathrm{I}_\Gamma(x), \Gamma \subset \mathbb{R}, x \in \mathbb{R}$, stands for an indicator function and $\xi(t), t \geq 0$, is a square integrable martingale, whose characteristic is given by the integral

$$\langle \xi \rangle_t = \int_0^t b(x(s))ds, \ t \geq 0. \tag{2}$$

We will show that, for any given A (the constants b_1 and b_2 will be fixed throughout this paper), there exist infinitely many diffusion processes satisfying Eq. (1); each one of them is determined by fixing two parameters: $q \in [-1, 1]$ and $r > 0$. The first one characterizes the property of the point $x = 0$ to be "non-symmetric", and the second one characterizes the property of this point to be "sticky" for the corresponding process.

More precisely, if $A = 0$, then we should put $q = q_0 \equiv (b_1 - b_2)/(b_1 + b_2)$ and an arbitrary positive number could be taken for the value of r. If $A > 0$, then the value of q could be chosen arbitrarily from the interval $(q_0, 1]$ and, after that, we should put

$$r = \frac{(\sqrt{b_1} + \sqrt{b_2})[(1+q)b_2 - (1-q)b_1]}{2A[(1+q)\sqrt{b_2} + (1-q)\sqrt{b_1}]}. \tag{3}$$

If $A < 0$, then the value of r should be determined by the same formula, but this time the value of q could be taken arbitrarily from the interval $[-1, q_0)$ (see Section 2 below).

In particular, if $A > 0$ and $b_1 = b_2 = 1$, the solution of (1) corresponding to $q = 1$ and $r = A^{-1}$ has the following property: its part in the region $[0, +\infty)$ coincides with the slowly reflecting Brownian motion (see [1]). This solution can be singled out by the

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requirement of the property $x(t) \geq 0$ for all $t \geq 0$. As proven in [1], such a solution is weakly unique, though there is no pathwise uniqueness in this situation.

Some examples of stochastic differential equations, for which the property of weak uniqueness was failed, were constructed in [5], [8], [13]. Our example (described below in Section 2) is a result of a random change of time in a (generalized) diffusion process that itself can be described as a solution to the stochastic differential equation

$$d\widetilde{x}(t) = \widetilde{q}\delta(\widetilde{x}(t))dt + d\widetilde{\xi}(t), \tag{4}$$

where $\delta(x), x \in \mathbb{R}$, is some kind of a (non-symmetric) Dirac function (see Section 1); the constant $\widetilde{q}$ is given by the right-hand side of (3) multiplied by A in the case of $A \neq 0$ and $\widetilde{q} = 0$ if $A = 0$ (in this case, $q = q_0$); $\widetilde{\xi}(t), t \geq 0$, is a square integrable martingale with its characteristic given by

$$\langle \widetilde{\xi} \rangle_t = \int_0^t b(\widetilde{x}(s))ds, t \geq 0 \tag{5}$$

(see Remark 1 in Section 1).

A.V. Skorokhod was first to make use of a random change of time in order to construct a slowly reflecting stochastic process from such one, for which the reflection was instantaneous (see [7], §24). The main distinction of our construction is admitting the boundary to be permeable: notice that the point $x = 0$ is reflecting to the right (respectively, to the left) for the solution of Eq. (4) in the case of $q = +1$ (respectively, $q = -1$); in the rest cases (that is, for $q \in (-1, 1)$), the point $x = 0$ is "transparent" for the corresponding process to move into either direction.

It is interesting to have a look at our example from the point of view of the general theory of one-dimensional stochastic differential equations developed in [6]. It turns out that Eq. (1) for $A = \widetilde{q}/r$ and Eq. (4) can be written in the form

$$y(t) = y(0) + \widetilde{q}L^{y(\cdot)}(t, 0) + \int_0^t b(y(s))^{1/2}dw(s), \ t \geq 0, \tag{6}$$

where $w(\cdot)$ is a standard Wiener process in $\mathbb{R}$ and $L^{y(\cdot)}(t, y)$ is the (non-symmetric) local time for the unknown process $y(\cdot)$ at the point $y \in \mathbb{R}$ on the interval $[0, t]$ (see [6]). As shown in [6], this equation has many weak solutions, and the so-called *fundamental solution* can be pointed out such that any other solution with the strong Markov property is obtained from the fundamental one by a random change of time. The solution $\widetilde{x}(\cdot)$ to Eq. (4) appears to be the fundamental solution to Eq. (6), and our construction can thus serve as an illustrative example to the theory of [6].

Notice, finally, that the processes considered in this paper were investigated from various points of view in [3], [4], [9], [10], [11].

1. The process that is a result of pasting together two Brownian motions

We fix numbers $b_1 > 0, b_2 > 0$, and $q \in [-1, 1]$ and put

$$d = (1-q)\sqrt{b_1} + (1+q)\sqrt{b_2}, \ q_1 = d^{-1}(1-q)\sqrt{b_1}, q_2 = d^{-1}(1+q)\sqrt{b_2},$$
$$c = 2(\sqrt{b_1} + \sqrt{b_2})^{-1}, \ \widetilde{q} = c^{-1}(q_2\sqrt{b_2} - q_1\sqrt{b_1}).$$

Denote, by D_1 and D_2, respectively, the left and right halves of a real line: $D_i = \{x \in \mathbb{R} : (-1)^i x > 0\}, i = 1, 2$. We define a function $\widetilde{g}(t, x, y)$ for $t > 0, x \in \mathbb{R}$, and $y \in \mathbb{R}$ by setting

$$\widetilde{g}(t,x,y) = (2\pi b_j t)^{-1/2}\Big[\exp\Big\{-\frac{1}{2t}\Big(\frac{y}{\sqrt{b_j}} - \frac{x}{\sqrt{b_i}}\Big)^2\Big\} + \\ + (q_2 - q_1)\operatorname{sign} y \exp\Big\{-\frac{1}{2t}\Big(\frac{|y|}{\sqrt{b_j}} + \frac{|x|}{\sqrt{b_i}}\Big)^2\Big\}\Big]$$

if $x \in D_i \cup \{0\}, i = 1, 2$, and $y \in D_j, j = 1, 2$. For $y = 0$, we set

$$\widetilde{g}(t,x,0) = \frac{c}{2}[\sqrt{b_1}\widetilde{g}(t,x,0-) + \sqrt{b_2}\widetilde{g}(t,x,0+)], t > 0, x \in \mathbb{R}.$$

It is not difficult to verify that the function $\widetilde{g}$ possesses the following properties:

1) $\widetilde{g}(t,x,y) \geq 0$ for all $t > 0, x \in \mathbb{R}$, and $y \in \mathbb{R}$;

2) $\int_{\mathbb{R}} \widetilde{g}(t,x,y)dy = 1$ for all $t > 0$ and $x \in \mathbb{R}$;

3) $\widetilde{g}(s+t,x,y) = \int_{\mathbb{R}} \widetilde{g}(s,x,z)\widetilde{g}(t,z,y)dz$ for all $t > 0, s > 0, x \in \mathbb{R}$, and $y \in \mathbb{R}$;

4) $\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (y-x)^4 g(t,x,y)dy = O(t^2)$ as $t \downarrow 0$.

These properties imply the existence of a continuous Markov process $(\widetilde{x}(t), \widetilde{\mathcal{M}}_t, \mathbb{P}_x)$ in $\mathbb{R}$ (we use notation from [2]) such that the relation

$$E_x\varphi(\widetilde{x}(t)) = \int_{\mathbb{R}} \varphi(y)\widetilde{g}(t,x,y)dy$$

holds true for $t > 0, x \in \mathbb{R}$, and an arbitrary bounded Borel function φ on $\mathbb{R}$.

By some not very difficult computations, one can get the following formulae:

a) $\mathbb{P}_0(\{\widetilde{x}(t) \in D_i\}) = q_i$ for $t > 0$ and $i = 1, 2$;

b) $E_x(\widetilde{x}(t) - \widetilde{x}(0)) = \widetilde{q}f_t(x)$ for $t > 0$ and $x \in \mathbb{R}$, where we put

$$f_t(x) = c\int_0^t \exp\Big\{-\frac{x^2}{2b_i\tau}\Big\}\frac{d\tau}{\sqrt{2\pi\tau}}$$

for $t > 0$ and $x \in D_i \cup \{0\}, i = 1, 2$;

c) for $t > 0$ and $x \in D_i, i = 1, 2$,

$$E_x(\widetilde{x}(t) - \widetilde{x}(0))^2 \\ = tb_i - 2\widetilde{q}xf_t(x) + (-1)^{i+1}2q_{3-i}(b_2 - b_1)\sqrt{\frac{2}{\pi}}\int_0^t \exp\Big\{-\frac{x^2}{2b_i\tau}\Big\}\sqrt{t-\tau}\frac{d\tau}{\sqrt{2\pi\tau}};$$

in the case of $x = 0$, we have that, for $t > 0$,

$$E_0\widetilde{x}(t)^2 = t(q_1b_1 + q_2b_2).$$

The relations in c) show that, for $x \in \mathbb{R}$,

$$\lim_{t\downarrow 0} t^{-1}E_x(\widetilde{x}(t) - \widetilde{x}(0))^2 = \widetilde{b}(x), \tag{7}$$

where

$$\widetilde{b}(x) = \sum_{i=1}^2 b_i 1\!\!1_{D_i}(x) + (q_1b_1 + q_2b_2)1\!\!1_{\{0\}}(x).$$

So, the function $\widetilde{b}(\cdot)$ is the diffusion coefficient of the process $\widetilde{x}(t), t \geq 0$. In order to calculate the drift coefficient of this process, we note that

$$\lim_{t\downarrow 0}\int_{\mathbb{R}} t^{-1}f_t(x)\varphi(x)dx = \frac{c}{2}[\sqrt{b_1}\varphi(0-) + \sqrt{b_2}\varphi(0+)] \tag{8}$$

for an arbitrary bounded Borel function φ on $\mathbb{R}$ such that the limits

$$\varphi(0\pm) = \lim_{x \to 0\pm} \varphi(x)$$

exist. Relation (8) means that the function $t^{-1} f_t$ converges, as $t \downarrow 0$, to a non-symmetric Dirac function δ, whose action on a test function φ is determined by the right-hand side of (8). Taking into account relation b), we can say that the drift coefficient

$$\widetilde{a}(x) = \lim_{t \downarrow 0} t^{-1} E_x(\widetilde{x}(t) - \widetilde{x}(0)), x \in \mathbb{R},$$

of the process $\widetilde{x}(t), t \geq 0$, coincides with the generalized function $\widetilde{q}\delta(x)$. In other words, if $\widetilde{q} \neq 0$, the process $\widetilde{x}(\cdot)$ is a generalized diffusion process.

We now give a martingale characterization of the process $(\widetilde{x}(t), \widetilde{\mathcal{M}}_t, \mathbb{P}_x)$. We observe at first that the function $f_t(x)$ defined above satisfies the relation

$$\int_{\mathbb{R}} f_t(y) \widetilde{g}(s, x, y) dy = f_{t+s}(x) - f_s(x)$$

for all $t \geq 0, s > 0$, and $x \in \mathbb{R}$. In addition, the relation

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}} f_t(x) = 0$$

is fulfilled. This implies (see Theorem 6.6 in [2]) the existence of an additive homogeneous continuous non-negative functional $(\eta_t)_{t \geq 0}$ of the process $(\widetilde{x}(t), \widetilde{\mathcal{M}}_t, \mathbb{P}_x)$ such that $E_x \eta_t = f_t(x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Moreover, as shown in [2], this functional is the limit for the family of integral functionals:

$$\eta_t = \underset{h \downarrow 0}{\text{l.i.m.}} \int_0^t h^{-1} f_h(\widetilde{x}(s)) ds, t \geq 0.$$

Since $h^{-1} f_h \to \delta$, as $h \downarrow 0$, the functional η_t can be written in the form

$$\eta_t = \int_0^t \delta(\widetilde{x}(s)) ds, t \geq 0.$$

We now put, for $t \geq 0$,

$$\widetilde{\xi}(t) = \widetilde{x}(t) - \widetilde{x}(0) - \widetilde{q}\eta_t.$$

Relation b) above shows that $E_x \widetilde{\xi}(t) = 0$ for any $t \geq 0$ and $x \in \mathbb{R}$. This relation means that the process $\widetilde{\xi}(\cdot)$ is a martingale with respect to $(\widetilde{\mathcal{M}}_t, \mathbb{P}_x)$ for every $x \in \mathbb{R}$. It is evident that this martingale is square integrable, and relations c) above show that

$$E_x \widetilde{\xi}(t)^2 = E_x \int_0^t \widetilde{b}(\widetilde{x}(s)) ds, t \geq 0, x \in \mathbb{R}.$$

Therefore, the characteristic of the martingale $\widetilde{\xi}(\cdot)$ can be written in the form

$$\langle \widetilde{\xi} \rangle_t = \int_0^t \widetilde{b}(\widetilde{x}(s)) ds, t \geq 0.$$

All these statements allow us to assert that there exists a one-dimensional standard Wiener process $(w(t))_{t \geq 0}$ (it is a functional of the process $\widetilde{x}(\cdot)$) such that

$$\widetilde{x}(t) = \widetilde{x}(0) + \widetilde{q}\eta_t + \int_0^t \widetilde{b}(\widetilde{x}(s))^{1/2} dw(s), t \geq 0.$$

In other words, the process $\widetilde{x}(\cdot)$ is the solution to the stochastic differential equation

$$d\widetilde{x}(t) = \widetilde{q}\delta(\widetilde{x}(t)) dt + \widetilde{b}(\widetilde{x}(t))^{1/2} dw(t).$$

Remark 1. This stochastic differential equation can be written in the form (4) (that is, with the function $b(\cdot)$ instead of $\tilde{b}(\cdot)$), since the time that the process $\widetilde{x}(\cdot)$ spends at the point $x = 0$ is a.s. equal to zero.

Remark 2. The process $(\widetilde{x}(t), \widetilde{\mathcal{M}}_t, \mathbb{P}_x)$ described above is slightly different from those considered in [3], [4], [9], [11].

2. The random change of time

We now fix a positive number r and put

$$\zeta_t = \inf\{s \geq 0 : s + r\eta_s \geq t\}, x(t) = \widetilde{x}(\zeta_t), \mathcal{M}_t = \widetilde{\mathcal{M}}_{\zeta_t}$$

for $t \geq 0$.

Proposition 1. *The process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ is a continuous strong Markov process in $\mathbb{R}$ such that, for $t > 0$, $x \in \mathbb{R}$, and any continuous bounded function φ on $\mathbb{R}$, the relation*

$$E_x\varphi(x(t)) = \varphi(0)h(t,x) + \int_{\mathbb{R}} g(t,x,y)\varphi(y)dy \tag{9}$$

holds true, where

$$h(t,x) = \frac{2}{\sqrt{2\pi t}}\int_0^\infty \exp\left\{-\frac{2\theta}{cr} - \frac{1}{2t}\left(\theta + \frac{|x|}{\sqrt{b_i}}\right)^2\right\}d\theta, x \in D_i \cup \{0\}, i = 1,2,$$

$$g(t,x,y) = \frac{1}{\sqrt{2\pi t b_j}}\left[\exp\left\{-\frac{1}{2t}\left(\frac{y}{\sqrt{b_j}} - \frac{x}{\sqrt{b_i}}\right)^2\right\} - \exp\left\{-\frac{1}{2t}\left(\frac{|y|}{\sqrt{b_j}} + \frac{|x|}{\sqrt{b_i}}\right)^2\right\}\right] +$$
$$+ \frac{4(1+q\operatorname{sign} y)}{cdr\sqrt{2\pi t}}\int_0^\infty \exp\left\{-\frac{2\theta}{cr} - \frac{1}{2t}\left(\theta + \frac{|x|}{\sqrt{b_i}} + \frac{|y|}{\sqrt{b_j}}\right)^2\right\}d\theta$$

for $x \in D_i \cup \{0\}, i = 1,2$ and $y \in D_j, j = 1,2$; for $y = 0$, we have

$$g(t,x,0) = \frac{c}{2}[\sqrt{b_1}g(t,x,0-) + \sqrt{b_2}g(t,x,0+)].$$

Proof. The fact that the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ is a strong Markov one is a consequence of Theorem 10.11 from [2]. To prove (9), we define the function $g_p(t,x,y)$ for $p > 0, t > 0, x \in \mathbb{R}, y \in \mathbb{R}$ from the relation

$$E_x\varphi(\widetilde{x}(t))\exp\{-pr\eta_t\} = \int_{\mathbb{R}} \varphi(y)g_p(t,x,y)dy$$

valid for all $p > 0, t > 0, x \in \mathbb{R}$, and any continuous bounded function φ on $\mathbb{R}$. The function g_p can be found from the equation (see [12])

$$g_p(t,x,y) = \widetilde{g}(t,x,y) - pr\int_0^t \widetilde{g}(\tau,x,0)g_p(t-\tau,0,y)d\tau, \tag{10}$$

where $p > 0, t > 0, x \in \mathbb{R}, y \in \mathbb{R}$ (r is a fixed parameter).

We put

$$R_p(x,y) = \int_0^\infty e^{-pt}g_p(t,x,y)dt,\ G_p(x,y) = \int_0^\infty e^{-pt}\widetilde{g}(t,x,y)dt$$

for $p > 0, x \in \mathbb{R}$, and $y \in \mathbb{R}$. Then Eq. (10) implies the following equation for the function R_p

$$R_p(x,y) = G_p(x,y) - prG_p(x,0)R_p(0,y).$$

It is not difficult to find out the solution of this equation. It can be written down in the following form (for $p > 0, x \in D_i \cup \{0\}, i = 1, 2$, and $y \in D_j, j = 1, 2$):

$$R_p(x,y) = \frac{1}{\sqrt{2pb_j}}\Big[\exp\Big\{-\Big|\frac{y}{\sqrt{b_j}} - \frac{x}{\sqrt{b_i}}\Big|\sqrt{2p}\Big\} - \tag{11}$$
$$- \exp\Big\{-\Big(\frac{|y|}{\sqrt{b_j}} + \frac{|x|}{\sqrt{b_i}}\Big)\sqrt{2p}\Big\}\Big] +$$
$$+ \frac{4(1+q\,\mathrm{sign}\,y)}{cdr\sqrt{2p}}\int_0^\infty \exp\Big\{-\frac{2\theta}{cr} - \Big(\theta + \frac{|x|}{\sqrt{b_i}} + \frac{|y|}{\sqrt{b_j}}\Big)\sqrt{2p}\Big\}d\theta.$$

If $y = 0$, we have

$$R_p(x,0) = \frac{c}{2}[R_p(x,0+)\sqrt{b_2} + R_p(x,0-)\sqrt{b_1}] = \tag{12}$$
$$= \frac{2}{r\sqrt{2p}}\int_0^\infty \exp\Big\{-\frac{2\theta}{cr} - \Big(\theta + \frac{|x|}{\sqrt{b_i}}\Big)\sqrt{2p}\Big\}d\theta.$$

Making use of the following well-known formula

$$\int_0^\infty (2\pi t)^{-1/2}\exp\big\{-pt - \frac{\alpha^2}{2t}\big\}dt = (2p)^{-1/2}\exp\{-|\alpha|\sqrt{2p}\}$$

valid for all real α and $p > 0$, we can rewrite (11) and (12) in the form

$$R_p(x,y) = \int_0^\infty e^{-pt}g(t,x,y)dt,\ rR_p(x,0) = \int_0^\infty e^{-pt}h(t,x)dt. \tag{13}$$

On the other hand, for $p > 0, x \in \mathbb{R}$, and a continuous bounded function φ on $\mathbb{R}$, we have

$$\int_0^\infty e^{-pt}E_x\varphi(x(t))dt = E_x\int_0^\infty e^{-pt}\varphi(\widetilde{x}(\zeta_t))dt =$$
$$= E_x\int_0^\infty e^{-p(t+r\eta_t)}\varphi(\widetilde{x}(t))dt + rE_x\int_0^\infty e^{-p(t+r\eta_t)}\varphi(\widetilde{x}(t))d\eta_t =$$
$$= \int_0^\infty e^{-pt}E_x\varphi(\widetilde{x}(t))e^{-pr\eta_t}dt + r\int_0^\infty e^{-pt}E_x\varphi(\widetilde{x}(t))\delta(\widetilde{x}(t))e^{-pr\eta_t}dt.$$

Taking into account the definition of R_p, we get

$$\int_0^\infty e^{-pt}E_x\varphi(x(t))dt = \int_{\mathbb{R}}\varphi(y)R_p(x,y)dy + r\varphi(0)R_p(x,0).$$

This equality and (13) imply (9). The proposition is proved.

Denote, by $P(t,x,\Gamma)$ for $t > 0, x \in \mathbb{R}$, and $\Gamma \in \mathcal{B}(\mathbb{R})$, the transition probability of the process $(x(t), \mathcal{M}_t, \mathbb{P}_x)$. Then we have the equality

$$P(t,x,\Gamma) = h(t,x)\mathbb{I}_\Gamma(0) + \int_\Gamma g(t,x,y)dy.$$

Remark 3. It is not difficult to see that the relations

$$\lim_{r\downarrow 0} h(t,x) = 0,\ \lim_{r\downarrow 0} g(t,x,y) = \widetilde{g}(t,x,y)$$

are held for all $t > 0, x \in \mathbb{R}$, and $y \in \mathbb{R}$. On the other hand, we have the relations

$$\lim_{r\uparrow+\infty} h(t,x) = \frac{2}{\sqrt{2\pi t}}\int_{|x|/\sqrt{b_i}}^\infty \exp\Big\{-\frac{\theta^2}{2t}\Big\}d\theta, x \in D_i \cup \{0\}, i = 1, 2;$$

$$\lim_{r\uparrow+\infty} g(t,x,y) =$$

$$= \frac{1}{\sqrt{2\pi t b_j}}\left[\exp\left\{-\frac{1}{2t}\left(\frac{y}{\sqrt{b_j}}-\frac{x}{\sqrt{b_i}}\right)^2\right\}-\exp\left\{-\frac{1}{2t}\left(\frac{|y|}{\sqrt{b_j}}+\frac{|x|}{\sqrt{b_i}}\right)^2\right\}\right]$$

for $t>0, x\in D_i\cup\{0\}, i=1,2$, and $y\in D_j, j=1,2$. These relations mean that, for $r=+\infty$, the part of the process $x(\cdot)$ in $D_i\cup\{0\}, i=1,2$, coincides with the Brownian motion there (that is, with the process $\sqrt{b_i}w(\cdot)$, where $w(\cdot)$ is a standard Wiener process) with the absorbing point $x=0$.

Let us now fix a finite $r>0$ and take note of the following properties of the corresponding process $x(\cdot)$:

A) $\mathbb{P}_0(\{x(t)\in D_i\}) = q_i(1-h(t,0)), t>0, i=1,2.$

B) $E_x\int_0^t \mathrm{I\!I}_{\{0\}}(x(s))ds = \int_0^t h(s,x)ds, t\geq 0, x\in\mathbb{R}.$

C) The process $(x(t),\mathcal{M}_t,\mathbb{P}_x)$ is a classical diffusion process with the drift coefficient $a(x)=\frac{\widetilde{q}}{r}\mathrm{I\!I}_{\{0\}}(x)$ and the diffusion coefficient $b(x)=\sum_{i=1}^2 b_i\mathrm{I\!I}_{D_i}(x)$. This assertion follows from the relations:

(i) $\lim_{t\downarrow 0} t^{-1}P(t,x,(x-\varepsilon,x+\varepsilon)^c)=0$
for all $\varepsilon>0$ and $x\in\mathbb{R}$, where $(x-\varepsilon,x+\varepsilon)^c=\mathbb{R}\setminus(x-\varepsilon,x+\varepsilon)$;

(ii) $\lim_{t\downarrow 0} t^{-1}\int_{\mathbb{R}}(y-x)P(t,x,dy)=a(x)$
for all $x\in\mathbb{R}$;

(iii) $\lim_{t\downarrow 0} t^{-1}\int_{\mathbb{R}}(y-x)^2P(t,x,dy)=b(x)$
for all $x\in\mathbb{R}$.

Remark 4. Property C) shows that the process $(x(t),\mathcal{M}_t,\mathbb{P}_x)$ is an example of the diffusion process, whose local characteristics (that is, the coefficients $a(\cdot)$ and $b(\cdot)$) do not determine its transition probability uniquely (see [11]).

We now give a martingale characterization of the process $(x(t),\mathcal{M}_t,\mathbb{P}_x)$. For $t\geq 0$, we put

$$\xi(t)=x(t)-x(0)-\frac{\widetilde{q}}{r}\int_0^t \mathrm{I\!I}_{\{0\}}(x(s))ds.$$

Proposition 2. *The process $(\xi(t))_{t\geq 0}$ is a square integrable martingale with respect to $(\mathcal{M}_t,\mathbb{P}_x)$ for every $x\in\mathbb{R}$, and its square characteristic is determined by formula (2).*

Proof. It is sufficient to verify that the equalities

$$E_x\xi(t)=0, E_x\xi(t)^2=E_x\int_0^t b(x(s))ds \tag{14}$$

hold true for all $x\in\mathbb{R}$ and $t\geq 0$.

The first relation in (14) is equivalent to the following one:

$$\int_{\mathbb{R}} yg(t,x,y)dy = x+\frac{\widetilde{q}}{r}\int_0^t h(s,x)ds.$$

Its validity is established by taking the Laplace transformation. To prove the second relation in (14), we notice that

$$E_x\xi(t)^2=E_x(x(t)-x(0))^2+\frac{2\widetilde{q}x}{r}\int_0^t h(s,x)ds, \tag{15}$$

since

$$E_x(x(t)-x(0))\int_0^t \mathrm{I\!I}_{\{0\}}(x(s))ds = -x\int_0^t h(s,x)ds+$$
$$+\frac{\widetilde{q}}{2r}E_x\Big(\int_0^t \mathrm{I\!I}_{\{0\}}(x(s))ds\Big)^2.$$

Taking into account the relation

$$E_x(x(t)-x(0))^2 = E_x x(t)^2 - x^2 - \frac{2\widetilde{q}x}{r}\int_0^t h(s,x)ds,$$

we get

$$E_x\xi(t)^2 = E_x x(t)^2 - x^2$$

from (15). Noticing that

$$E_x x(t)^2 = \int_{\mathbb{R}} y^2 P(t,x,dy) = \int_{\mathbb{R}} y^2 g(t,x,y)dy,$$

we arrive at the conclusion that the second relation in (14) is equivalent to the equality

$$\int_{\mathbb{R}} y^2 g(t,x,y)dy - x^2 = \int_0^t ds \int_{\mathbb{R}} b(y)g(s,x,y)dy.$$

The validity of this equality can be verified by taking the Laplace transformation. The proposition is proved.

As a consequence of Proposition 2, we have that the paths of the process $(x(t),\mathcal{M}_t,\mathbb{P}_x)$ are the solution to the stochastic differential equation (1) with $A=\widetilde{q}/r$. As mentioned in Introduction, there are infinitely many possibilities to choose $q\in[-1,1]$ and $r>0$ in such a way that the ratio $\widetilde{q}/r$ is equal to a fixed constant A. It is obvious that $A=0$ for $q=q_0$ ($\widetilde{q}=0$ for $q=q_0$), $A>0$ for $q\in(q_0,1]$, and $A<0$ for $q\in[-1,q_0)$.

In particular, if $A>0$, we can put $q=+1$ ($\widetilde{q}=2^{-1}\sqrt{b_2}(\sqrt{b_1}+\sqrt{b_2})$ in this case) and $r=(2A)^{-1}\sqrt{b_2}(\sqrt{b_1}+\sqrt{b_2})$. One can observe that $g(t,x,y)=0$ for $y<0$ in this case, so, if $x(0)\geq 0$, then $x(t)\geq 0$ for all $t\geq 0$. If, additionaly, $b_1=b_2=1$, then the corresponding process is called a slowly reflecting Brownian motion; it was investigated in [1].

Remark 5. For $t>0, x\in\mathbb{R}$, and a continuous bounded function φ on $\mathbb{R}$, we put

$$u(t,x,\varphi)=E_x\varphi(x(t)).$$

This function satisfies the equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}b_i\frac{\partial^2 u}{\partial x^2}$$

in the region $t>0, x\in D_i$ for $i=1,2$, and the initial condition

$$\lim_{t\downarrow 0} u(t,x,\varphi)=\varphi(x), x\in\mathbb{R}.$$

In addition, the relations $u(t,0+,\varphi)=u(t,0-,\varphi)=u(t,0,\varphi)$ and

$$\frac{\partial u(t,0,\varphi)}{\partial t}=\frac{1}{cr}\left[q_2\sqrt{b_2}\frac{\partial u(t,0+,\varphi)}{\partial x}-q_1\sqrt{b_1}\frac{\partial u(t,0-,\varphi)}{\partial x}\right]$$

are valid for $t>0$.

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