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DURATION OF STAY INSIDE AN INTERVAL BY THE POISSON PROCESS WITH A NEGATIVE EXPONENTIAL COMPONENT

Several two-boundary problems for the Poisson process with an exponential component are solved in the present article. The integral transforms of the joint distribution of the epoch of the first exit from the interval and the value of the overshoot through boundaries at the epoch of the exit are obtained. The Laplace transform of the total duration time of the process's stay inside the interval is found.

1. Main definitions and auxiliary results

Let $\xi(t) \in \mathbb{R}$, $t \geq 0$ be a homogeneous process with independent increments [1] with the cumulant

$$k(p) = \frac{1}{2}\,p^2\sigma^2 - \alpha p + \int_{-\infty}^{\infty} \left(e^{-px} - 1 + \frac{px}{1+x^2} \right) \Pi(dx), \qquad \Re p = 0. \tag{1}$$

Here and in the sequel, the paths of the process are supposed to be right-continuous and $\xi(0) = 0$. Note that the introduced process is a space homogeneous and a strong Markov process [2]. This property will be used continually while setting up equations. The method we propose to solve two-boundary problems is based on the application of integral transforms of the one-boundary functionals of the process which were determined by B.A. Rogozin, E.A. Pečerskii, A.A. Borovkov, V.M. Zolotarev and others. Let us define the one-boundary functionals we require. For $x \geq 0$, we introduce the random variables

$$\tau^{x} = \inf\{t : \xi(t) > x\}, \quad T^{x} = \xi(\tau^x) - x, \quad \tau_x = \inf\{t : \xi(t) < -x\}, \quad T_x = -\xi(\tau_x) - x$$

the first passage time of the level x and the value of the overshoot through this level at the epoch of passage, the first passage time of the level $-x$ and the value of the overshoot this level at the epoch of the passage. The integral transforms of the joint distribution of $\{\tau^x, T^x\}$, $\{\tau_x, T_x\}$ for $s > 0$, $\Re p \geq 0$ satisfy the following equalities

$$\mathbf{E}\,[\,e^{-s\tau^x} \exp\{-pT^x\}\,] = \left(\mathbf{E}\,e^{-p\xi^+(\nu_s)}\right)^{-1} \mathbf{E}\,[\,e^{-p(\xi^+(\nu_s)-x)};\ \xi^+(\nu_s) > x\,],$$

$$\mathbf{E}\,[\,e^{-s\tau_x} \exp\{-pT_x\}\,] = \left(\mathbf{E}\,e^{p\xi^-(\nu_s)}\right)^{-1} \mathbf{E}\,[\,e^{p(\xi^-(\nu_s)+y)};\ -\xi^-(\nu_s) > x\,], \tag{2}$$

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where $\xi^+(t) = \sup_{u\le t} \xi(u)$, $\xi^-(t) = \inf_{u\le t} \xi(u)$, ν_s is the exponential variable with parameter $s > 0$ independent of the process, $\mathbf{P}[\,\nu_s > t\,] = \exp\{-st\}$, and

$$\mathbf{E}\exp\{\,-p\,\xi^{\pm}(\nu_s)\,\} = \exp\left\{\int_0^\infty \frac{1}{t}\,e^{-st}\,\mathbf{E}\,[\,e^{\,-p\xi(t)} - 1;\ \pm\,\xi(t) > 0\,]\,dt\right\}, \quad \pm\Re\,p \ge 0.$$

Formulae (2) were obtained by E.A. Peĉerskii and B.A. Rogozin in [3]. A simple proof of these equalities is given in [4]. In this framework we solve several two-boundary problems for the Poisson process with a negative exponential component. Let us give the rigorous definition of the process.

Let $\eta \in (0,\infty)$ be a positive random variable, and γ be an exponential variable with parameter $\lambda > 0 : \mathbf{P}\,[\,\gamma > x\,] = e^{-\lambda x}$, $x \ge 0$. Introduce a random variable $\xi \in \mathbb{R}$ by its distribution function

$$F(x) = a\,e^{x\lambda}\,\mathbf{I}_{\{x\le 0\}} + (a + (1-a)\,\mathbf{P}\,[\,\eta < x\,])\,\mathbf{I}_{\{x>0\}}, \qquad a \in (0,1), \quad \lambda > 0.$$

Define a right-continuous step Poisson process $\xi(t) \in \mathbb{R}$, $t \ge 0$ with cumulant

$$k(p) = c\int_{-\infty}^{\infty} (e^{-xp} - 1)\,dF(x) = a_1\frac{p}{\lambda - p} + a_2(\mathbf{E}\,e^{-p\eta} - 1), \qquad c > 0, \quad \Re\,p = 0, \tag{3}$$

where $a_1 = ac$, $a_2 = (1-a)c$. Note that the jumps of the process $\xi(t)$, $t \ge 0$, occur at the time epochs that are exponentially distributed with parameter c. With the probability $1-a$, there occur positive jumps with value distributed as η, and, with the probability a, there occur negative jumps (whose value γ is exponentially distributed with parameter λ). Here and in the sequel, we will call such a process as the Poisson process with a negative exponential component.

The first term of (3) is the simplest case of a rational function, while the second is nothing but the cumulant of a monotone Poisson process with positive jumps of value η. It is well known fact (see, for instance, [5],[6]) that, in this case, the equation $k(p) - s = 0$, $s > 0$, has a unique root $c(s) \in (0,\lambda)$ in the semi-plane $\Re\,p > 0$. Moreover, for the integral transforms of the random variables $\xi^+(\nu_s)$, $\xi^-(\nu_s)$, the following formulae hold:

$$\mathbf{E}\,e^{-p\xi^-(\nu_s)} = \frac{c(s)}{\lambda}\,\frac{\lambda - p}{c(s) - p}, \qquad \Re\,p \le 0, \tag{4}$$

$$\mathbf{E}\,e^{-p\xi^+(\nu_s)} = \frac{s\lambda}{c(s)}\,(p - c(s))\,R(p,s), \qquad \Re\,p \ge 0,$$

where

$$R(p,s) = \left(a_1 p + (p-\lambda)[s - a_2(\,E\,e^{-p\eta} - 1)]\right)^{-1}, \qquad \Re\,p \ge 0, \quad p \ne c(s) \tag{5}$$

It follows from (2) and (4) after some calculations that the integral transforms of the joint distributions $\{\tau_x,\,T_x\}$, $\{\tau^x,\,T^x\}$ of the Poisson process with a negative exponential component satisfy the equalities

$$\mathbf{E}[\,e^{-s\tau_x};\,T_x \in du\,] = (\lambda - c(s))\,e^{-xc(s)}\,e^{-\lambda u}\,du = \mathbf{E}\,e^{-s\tau_x}\,\mathbf{P}\,[\,\gamma \in du\,], \tag{6}$$

$$\int_0^\infty e^{-px}\,\mathbf{E}\,e^{\,-s\tau^x - z\xi(\tau^x)}\,dx = \frac{1}{p}\left(1 - \frac{p+z-c(s)}{z - c(s)}\,\frac{R(p+z,s)}{R(z,s)}\right), \quad \Re\,p > 0,\ \Re\,z \ge 0.$$

The first equality of (6) yields that τ_x and T_x are independent. Moreover, for all $x \ge 0$, the value of the overshoot through the lower level T_x is exponentially distributed with parameter λ. This fact serves as a characterizing feature of the Poisson process with a negative exponential component.

Observe that the function $R(p,s)$ defined by (5) is analytic in the semi-plane $\Re p > c(s)$, and $\lim_{p\to\infty} R(p,s) = 0$. Therefore, it allows the representation in the form of an absolutely convergent Laplace integral [7]

$$R(p,s) = \int_0^\infty e^{-px} R_x(s)\,dx, \qquad \Re p > c(s). \tag{7}$$

We call the function $R_x(s)$, $x \ge 0$, as the resolvent of the Poisson process with a negative exponential component. We assume that $R_x(s) = 0$, for $x < 0$. Notice that $R_0(s) = \lim_{p\to\infty} p\,R(p,s) = (c+s)^{-1}$, and equality (4) implies

$$\mathbf{P}\,[\,\xi^-(\nu_s) = 0\,] = \frac{c(s)}{\lambda}, \qquad \mathbf{P}\,[\,\xi^+(\nu_s) = 0\,] = \frac{\lambda}{c(s)}\,\frac{s}{s+c}.$$

The second formula of (4) yields

$$R(p,s) = \frac{c(s)}{s\lambda}\,\frac{1}{p-c(s)}\,\mathbf{E}\,e^{-p\xi^+(\nu_s)}, \qquad \Re p > c(s). \tag{8}$$

The functions

$$\frac{1}{p-c(s)} = \int_0^\infty e^{-u(p-c(s))}du, \qquad \Re p > c(s),$$
$$\mathbf{E}\,e^{-p\xi^+(\nu_s)} = \int_0^\infty e^{-up} d\,\mathbf{P}\,[\,\xi^+(\nu_s) < u\,], \qquad \Re p \ge 0,$$

which enter the right-side of (8) are the Laplace transforms for $\Re p > c(s)$. Therefore, the original functions of the left- and right-hand sides of (8) coincide, and

$$R_x(s) = \frac{c(s)}{s\lambda} \int_{-0}^x e^{c(s)(x-u)} d\,\mathbf{P}\,[\,\xi^+(\nu_s) < u\,], \qquad x \ge 0. \tag{9}$$

The latter formula is another representation for the resolvent of the Poisson process with a negative exponential component. Note that the representation for the resolvent similar to (9) for a semi-continuous process with independent increments was obtained by V.M. Shurenkov and V.N. Suprun in [8,9]. Representation (9) implies that the function $R_x(s)$, $x \ge 0$ is positive, monotone, continuous, and increasing, and $R_x(s) < A(s)\exp\{xc(s)\}$, $A(s) < \infty$. Consequently,

$$\int_0^\infty R_x(s) e^{-\alpha x} dx < \infty, \qquad \alpha > c(s).$$

Moreover, in the neighbourhood of any $x \ge 0$, the function $R_x(s)$ has bounded variation. Hence, the inversion formula [7] is valid:

$$R_x(s) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{xp} R(p,s)\,dp, \qquad \alpha > c(s). \tag{10}$$

The latter equality together with (7) determines the resolvent of the Poisson process with a negative exponential component. Now we state the main results.

2 Exit from an interval

In this section, the integral transform of the joint distribution of the instant of the first exit from an interval by the Poisson process with a negative exponential component and the value of the overshoot through a boundary are determined. These integral transforms will be obtained as a corollary from a general theorem for a homogeneous process with independent increments.

Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, $\xi(0) = 0$ be a homogeneous process with independent increments with cumulant (1), $B > 0$ is fixed. Introduce the random variable

$$\chi(y) = \inf\{\, t: \; y + \xi(t) \notin [0, B] \,\}, \qquad y \in [0, B]$$

the instant of the first exit by the process $y + \xi(t)$ from the interval $[0, B]$. Introduce events $A^B = \{\, \xi(\chi(y)) > B \,\}$— the exit from the interval by the process takes place through the upper level; $A_0 = \{\, \xi(\chi(y)) < 0 \,\}$— the exit from the interval by the process takes place through the lower level. Denote, by

$$X(y) = (\xi(\chi(y)) - B)\, \mathbf{I}_{A^B} + (-\xi(\chi(y)))\, \mathbf{I}_{A_0}, \qquad \mathbf{P}\,[\, A^B + A_0 \,] = 1,$$

the value of the overshoot through a boundary at the epoch of the exit from the interval by the given process. Here, $\mathbf{I}_A = \mathbf{I}_A(\omega)$ is an indicator of the event A. The following theorem is true.

Theorem 1 [4]. *Let $\xi(t) \in \mathbb{R}$, $t \geq 0$ be a homogeneous process with independent increments with cumulant (1), $B > 0$ is fixed, $y \in [0, B]$, $x = B - y$, $\xi(0) = 0$, and*

$$\chi(y) = \inf\{\, t: \; y + \xi(t) \notin [0, B] \,\}, \qquad X(y) = (\xi(\chi(y)) - B)\, \mathbf{I}_{A^B} + (-\xi(\chi(y)))\, \mathbf{I}_{A_0}$$

the instant of the first exit by the process $y + \xi(t)$ from the interval $[0, B]$ and the value of the overshoot through a boundary at the epoch of the first exit from the interval by the given process. The Laplace transforms of the joint distribution of the variables $\{\, \chi(y), X(y) \,\}$ for $s > 0$ satisfy the formulae

$$\mathbf{E}\,[\, e^{-s\chi(y)}; \; X(y) \in du, \; A^B \,] = f_+^s(x, du) + \int_0^\infty f_+^s(x, dv)\, \mathbf{K}_+^s(v, du),$$

$$\mathbf{E}\,[\, e^{-s\chi(y)}; \; X(y) \in du, \; A_0 \,] = f_-^s(y, du) + \int_0^\infty f_-^s(y, dv)\, \mathbf{K}_-^s(v, du), \tag{11}$$

where

$$f_+^s(x, du) = \mathbf{E}\,[\, e^{-s\tau^x}; \; T^x \in du \,] - \int_0^\infty \mathbf{E}\,[\, e^{-s\tau_y}; \; T_y \in dv \,]\, \mathbf{E}\,[\, e^{-s\tau^{v+B}}; \; T^{v+B} \in du \,],$$
$$f_-^s(y, du) = \mathbf{E}\,[\, e^{-s\tau_y}; \; T_y \in du \,] - \int_0^\infty \mathbf{E}\,[\, e^{-s\tau^x}; \; T^x \in dv \,]\, \mathbf{E}\,[\, e^{-s\tau_{v+B}}; \; T_{v+B} \in du \,];$$

and $\mathbf{K}_\pm^s(v, du) = \sum_{n=1}^\infty K_\pm^{(n)}(v, du, s)$, $\quad v \geq 0$, is the series of successive iterations;

$$K_\pm^{(1)}(v, du, s) = K_\pm(v, du, s), \quad K_\pm^{(n+1)}(v, du, s) = \int_0^\infty K_\pm^{(n)}(v, dl, s)\, K_\pm(l, du, s) \tag{12}$$

are the successive iterations ($n \in \mathbf{N}$) of the kernels $K_\pm(v, du, s)$ which are given by the defining equalities

$$K_+(v, du, s) = \int_0^\infty \mathbf{E}\,[\, e^{-s\tau_{v+B}}; \; T_{v+B} \in dl \,]\, \mathbf{E}\,[\, e^{-s\tau^{l+B}}; \; T^{l+B} \in du \,],$$

$$K_-(v, du, s) = \int_0^\infty \mathbf{E}\,[\, e^{-s\tau^{v+B}}; \; T^{v+B} \in dl \,]\, \mathbf{E}\,[\, e^{-s\tau_{l+B}}; \; T_{l+B} \in du \,]. \tag{13}$$

We apply now the formulae of the theorem in the case where the underlying process is the Poisson process with an exponentially distributed negative component.

Corollary 1. *Let $\xi(t) \in \mathbb{R}$, $t \geq 0$ be a Poisson process with a negative exponential component with cumulant (3), $B > 0$, $y \in [0, B]$, $x = B - y$, $\xi(0) = 0$, and*

$$\chi(y) = \inf\{\, t: \; y + \xi(t) \notin [0, B]\,\}, \qquad X(y) = (\xi(\chi(y)) - B)\,\mathbf{I}_{A^B} + (-\xi(\chi(y)))\,\mathbf{I}_{A_0}$$

the instant of the first exit from the interval and the value of the overshoot through one of the boundaries. Then, for $s > 0$,

1) the integral transforms of the joint distribution $\{\, \chi(y),\, X(y)\,\}$ satisfy the equalities

(14)
$$\mathbf{E}[e^{-s\chi(y)}; X(y) \in du, A_0] = (\lambda - c(s))\, e^{-yc(s)-\lambda u} \left(1 - \mathbf{E}[\, e^{-s\tau^x - c(s)\xi(\tau^x)}\,]\right) K(s)^{-1} du,$$

$$\mathbf{E}[e^{-s\chi(y)}; X(y) \in du, A^B] = \mathbf{E}[e^{-s\tau^x}; T^x \in du] - \mathbf{E}\,[e^{-s\chi(y)}; A_0]\, \mathbf{E}\,[e^{-s\tau^{\gamma+B}}; T^{\gamma+B} \in du],$$

where

$$K(s) = 1 - \mathbf{E}\exp\{-s\tau_B\}\; \mathbf{E}\exp\{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}\},$$
$$\mathbf{E}\exp\{-s\tau^{\gamma+B} - c(s)T^{\gamma+B}\} = \int_0^\infty \lambda\, e^{-\lambda u} \mathbf{E}\exp\{-s\tau^{u+B} - c(s)T^{u+B}\}\, du,$$

and, in particular,

$$\mathbf{E}[\, e^{-s\chi(y)}; A_0] = \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)} \left(1 - \mathbf{E}[\, e^{-s\tau^x - c(s)\xi(\tau^x)}\,]\right) K(s)^{-1},$$

$$\mathbf{E}[e^{-s\chi(y)}; A^B] = \mathbf{E}\exp\{-s\tau^x\} - \mathbf{E}[e^{-s\chi(y)}; A_0]\; \mathbf{E}\exp\{-s\tau^{\gamma+B}\}; \tag{15}$$

2) for the Laplace transforms of the random variable $\chi(y)$, the following representations hold:

$$\mathbf{E}[e^{-s\chi(y)}; X(y) \in du, A_0] = e^{-\lambda(u+B)}\, \frac{R_x(s)}{\hat{R}_B(\lambda, s)}\, du, \quad \mathbf{E}[e^{-s\chi(y)}; A_0] = \frac{1}{\lambda}\, e^{-\lambda B}\, \frac{R_x(s)}{\hat{R}_B(\lambda, s)},$$

$$\mathbf{E}[\, e^{-s\chi(y)}; A^B] = 1 - \frac{R_x(s)}{\hat{R}_B(\lambda, s)} \left[\frac{1}{\lambda}\, e^{-\lambda B} + s\lambda\, \hat{S}_B(\lambda, s)\right] + s\lambda\, S_x(s),$$

$$\int_0^\infty e^{-st}\mathbf{P}\,[\chi(y) > t]\, dt = \lambda\, \frac{R_x(s)}{\hat{R}_B(\lambda, s)}\, \hat{S}_B(\lambda, s) - \lambda\, S_x(s), \tag{16}$$

where $R_x(s)$, $x \geq 0$, is the resolvent of the Poisson process with a negative exponential component defined by (10),

$$S_x(s) = \int_0^x R_u(s)\, du, \quad \hat{R}_B(\lambda, s) = \int_B^\infty e^{-\lambda u} R_u(s)\, du, \quad \hat{S}_B(\lambda, s) = \int_B^\infty e^{-\lambda u} S_u(s)\, du.$$

Proof. We use the results of Theorem 1 for the Poisson process with a negative exponential component which takes a simplified form in this case. Using equality (6) and defining formulae (13) of the kernels $K_\pm(v, du, s)$ yields

$$K_+(v, du, s) = \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(v+B)}\, \mathbf{E}\,[\, e^{-s\tau^{\gamma+B}};\; T^{\gamma+B} \in du\,],$$
$$K_-(v, du, s) = (\lambda - c(s))\, e^{-c(s)B-\lambda u}\, \mathbf{E}\,[\, e^{-s\tau^{v+B} - c(s)T^{v+B}}\,]\, du,$$

where γ is the exponentially distributed random variable with parameter λ. The latter equalities imply the following form of the successive iterations $K_{\pm}^{(n)}(v,du,s)$, $n\in\mathbf{N}$, of the kernels $K_{\pm}(v,du,s)$:

$$\begin{aligned} K_{-}^{(n)}(v,du,s) =&\mathbf{E}\,[\,e^{-s\tau^{v+B}-c(s)T^{v+B}}\,]\mathbf{E}\,e^{-s\tau_B}\ \ (1-K(s))^{n-1}\ \ \lambda\,e^{-\lambda u}\,du,\\ K_{+}^{(n)}(v,du,s) =&e^{-vc(s)}\ \mathbf{E}\,e^{-s\tau_B}\ \ (1-K(s))^{n-1}\ \ \mathbf{E}[\,e^{-s\tau^{\gamma+B}};\ T^{\gamma+B}\in\ du\,]. \end{aligned}$$

The series $\mathbf{K}_{\pm}^{s}(v,du)$ of successive iterations $K_{\pm}^{(n)}(v,du,s)$ are, in this case, geometric series, and their sums are given by

$$\begin{aligned} \mathbf{K}_{-}^{s}(v,du) &= \sum_{n=1}^{\infty} K_{-}^{(n)}(v,du,s) = \mathbf{E}[\,e^{-s\tau^{v+B}-c(s)T^{v+B}}\,]\ \ \mathbf{E}\,e^{-s\tau_B}\ K(s)^{-1}\lambda\ e^{-\lambda u}\,du,\\ \mathbf{K}_{+}^{s}(v,du) &= \sum_{n=1}^{\infty} K_{+}^{(n)}(v,du,s) = e^{-vc(s)}\ \mathbf{E}\,e^{-s\tau_B}\ K(s)^{-1}\ \mathbf{E}[\,e^{-s\tau^{\gamma+B}};\ T^{\gamma+B}\in\ du\,]. \end{aligned}$$

Substituting the obtained expressions for $\mathbf{K}_{\pm}^{s}(v,du)$ into equalities (11) of Theorem 1, we get formulae (14) of Corollary 1. Integrating (14) with respect to $u\in\mathbb{R}_+$ yields equalities (15) of Corollary 1. Further, utilizing representation (10) for the resolvent and formulae (6), we obtain the resolvent representations for the functions $\mathbf{E}\exp\{-s\tau^{x}\}$, $\mathbf{E}\exp\{-s\tau^{x}-c(s)\xi(\tau^{x})\}$:

$$\begin{aligned} &\mathbf{E}\exp\{-s\tau^{x}\} = 1-\frac{s\lambda}{c(s)}\ R_x(s)+s\lambda\,S_x(s),\\ &\mathbf{E}\exp\{-s\tau^{x}-c(s)\xi(\tau^{x})\} = 1-e^{-xc(s)}\ R_x(s)\ r(c(s),s), \end{aligned}$$

where

$$S_x(s)=\int_0^x R_u(s)\,du, \qquad r(c(s),s)=\frac{d}{dp}R(p,s)^{-1}\Big|_{p=c(s)}.$$

Substituting the resolvent representations into (15) yields formulae (16) of Corollary 1. Note that, for a random walk with a geometrically distributed negative component, the resolvent representations similar to (16) were obtained in [10].

Remark 1. The formulae of Corollary 1 can be also obtained after solving the following system of integral equations:

$$\begin{aligned} \mathbf{E}\,[\,e^{-s\tau^{x}};\ T^{x}\in du\,] =&\mathbf{E}\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A^{B}\,]\\ &+\int_0^{\infty}\mathbf{E}\,[\,e^{-s\chi(y)};\ X(y)\in dv,\ A_0\,]\ \mathbf{E}\,[\,e^{-s\tau^{v+B}};\ T^{v+B}\in du\,],\\ \mathbf{E}\,[\,e^{-s\tau_y};\ T_y\in du\,] =&\mathbf{E}\,[\,e^{-s\chi(y)};\ X(y)\in du,\ A_0\,]\\ &+\int_0^{\infty}\mathbf{E}\,[\,e^{-s\chi(y)};\ X(y)\in dv,\ A^{B}\,]\ \mathbf{E}\,[\,e^{-s\tau_{v+B}};\ T_{v+B}\in du\,]. \end{aligned} \tag{17}$$

It takes a simple form for the Poisson process with a negative exponential component. This system was solved in [4] for a process with the general cumulant (1).

Remark 2. We have obtained the integral transform of the joint distribution $\{\chi(y),\ X(y)\}$ for the process with independent increments with the general cumulant (1) and, in particular, for a Poisson process with exponential component. The value of the overshoot $X(y)$ has a probabilistic interpretation, since it is expressed in terms of one-boundary functionals T^{x}, T_x. This fact allows us to effectively apply the distribution of $\{\chi(y),\ X(y)\}$ to solve other exit problems.

3 Duration of a stay inside the interval by the Poisson process with a negative exponential component

In this section, we obtain the Laplace transform of the total duration of a stay of the process inside the interval. Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, $\xi(0) = 0$, be the Poisson process with a negative exponential component. Denote, by

$$\sigma_y(t) = \int_0^t \mathbf{I}\{\, y + \xi(u) \in [0, B]\,\}\, du, \qquad y \in \mathbb{R},$$

the total duration of a stay of the process $y + \xi(\cdot)$ inside $[0, B]$ up to the moment t. We will determine

$$C_a^s(y) = \mathbf{E} \exp\{-a\sigma_y(\nu_s)\,\}, \qquad y \in R, \quad a \geq 0,$$

the Laplace transform of the total duration of a stay of the process $y + \xi(\cdot)$ inside the interval $[0, B]$ on the exponential time interval $[0, \nu_s]$. We require several auxiliary functions to solve the stated problem. Let $y \geq 0$, $\xi(0) = 0$, and let

$$\tau_y = \inf\{\, t : \; y + \xi(t) < 0\,\} \quad \text{and} \quad \sigma_y = \sigma_y(\tau_y)$$

be, respectively, the instant of the first exit from the upper semi-plane by the process $y + \xi(\cdot)$ and the total duration of a stay of the process inside $[0, B]$ on the time interval $[0, \tau_y]$. On the event $\{\, \tau_y = \infty\,\}$, we set, by definition, $\sigma_y = \infty$. The following statement is true.

Lemma 1. *Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, $\xi(0) = 0$, be the Poisson process with a negative exponential component with cumulant (3). Then, for the integral transform*

$$D_a^s(y) = \mathbf{E} \exp\{-s\tau_y - a\sigma_y\,\}, \qquad y \geq 0, \quad a \geq 0,$$

of the joint distribution $\{\, \tau_y, \sigma_y\,\}$, the following equality holds for $s > 0$:

$$(18) \quad D_a^s(y) = \Big[V_a^s(B - y) - a\, R_{B-y}(s + a)\, e^{-c(s)(B-y)}\Big]\, V_a^s(B)^{-1} \mathbf{E}\, e^{-s\tau_y}, \qquad y \geq 0.$$

Here, $R_x(s) = 0$ for $x < 0$, and the continuous function $V_a^s(x)$, $x \in \mathbb{R}$, is given by the equality

$$(19) \quad V_a^s(x) = 1 + a(\lambda - c(s)) \int_0^x e^{-uc(s)} R_u(s + a)\, du, \quad x \geq 0, \quad V_a^s(x) = 1, \quad x < 0.$$

Proof. For the functions $D_a^s(y)$, $y \geq 0$, according to the total probability law and the strong Markov property of the underlying process, we can write the system

$$D_a^s(y) = \mathbf{E}\,[e^{-(s+a)\chi(y)};\; A_0\,] + \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi(y)};\; X(y) \in dv, A^B\,]\, \mathbf{E}\,[e^{-s\tau_v};\; T_v > B\,]$$

$$+ \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi(y)};\; X(y) \in dv, A^B\,] \int_0^B \mathbf{E}\,[e^{-s\tau_v};\; T_v \in du\,]\, D_a^s(B - u), \quad y \in [0, B],$$

$$D_a^s(y) = \mathbf{E}\,[e^{-s\tau_{y-B}};\; T_{y-B} > B\,] + \int_0^B \mathbf{E}\,[e^{-s\tau_{y-B}};\; T_{y-B} \in du\,]\, D_a^s(B - u), \qquad y > B.$$

The first equation of this system represents the fact that the total duration of a stay of the process $y + \xi(\cdot)$, $y \in [0, B]$, inside $[0, B]$ on the time interval $[0, \tau_y]$ is realized either on the path which does not intersect the upper level (the first term on the right-hand side of the equation), or on the path which does intersect the upper level and then overleaps the interval $[0, B]$ (the second term on the right-hand side of the equation), or on the path which intersects the upper level and then returns to the interval $[0, B]$ (the third term on the right-hand side of the equation). The second equation is written analogously.

Now, using the first equality of (6) and equalities (16) of Corollary 1, we get, after some simplifications,

$$D_a^s(y) = \frac{1}{\lambda} e^{-\lambda B} \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)} + (\lambda - c(s))\, E_y^{s+a}(c(s)) \left(\frac{1}{\lambda} e^{-\lambda B} + \tilde{D}_a^s(\lambda) \right), \quad y \in [0, B],$$

(20)

$$D_a^s(y) = \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(y-B)} e^{-\lambda B} + (\lambda - c(s))\, e^{-c(s)(y-B)} \tilde{D}_a^s(\lambda), \quad y > B,$$

where

$$\tilde{D}_a^s(\lambda) = \int_0^B e^{-\lambda u} D_a^s(B-u)\, du,$$
$$E_y^{s+a}(c(s)) = \mathbf{E}\,[\, e^{-(s+a)\chi(y) - c(s)X(y)}; A^B \,].$$

Note that it is now sufficient to determine the function $\tilde{D}_a^s(\lambda)$. Then, by means of (20), we will derive the function $D_a^s(y)$, $y \ge 0$. The function $\tilde{D}_a^s(\lambda)$ can be obtained from the first equation of system (20). To do this, we first make auxiliary calculations. Denote

$$T_x^s(z) = \mathbf{E} \exp\{-s\tau^x - zT^x\}, \qquad x \ge 0, \quad \Re z \ge 0.$$

Utilizing the second equality of (6) and the defining resolvent (10) yields

$$T_x^{s+a}(c(s)) = e^{xc(s)} + a \frac{\lambda - c(s)}{c(s) - c(s+a)} R_x(s+a)$$
$$+ a(\lambda - c(s)) \int_0^x e^{-c(s)(u-x)} R_u(s+a)\, du, \qquad x \ge 0.$$

Using the latter equality and relations (14) and (16) of Corollary 1, we find

$$E_y^{s+a}(c(s)) = e^{xc(s)} V_a^s(B-y)$$
$$- \frac{e^{-B(\lambda - c(s))}}{\lambda - c(s)} \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)} V_a^s(B) - aR_{B-y}(s+a), \qquad y \in [0, B]. \tag{21}$$

Multiplying this equality by $e^{-\lambda(B-y)}$ and integrating with respect to $y \in [0, B]$, we get

$$(\lambda - c(s)) \int_0^B e^{-\lambda(B-y)} E_y^{s+a}(c(s))\, dy = 1 - V_a^s(B) \left[1 + \frac{\check{R}_B(\lambda, s+a)}{\hat{R}_B(\lambda, s+a)} \right] e^{-B(\lambda - c(s))}, \tag{22}$$

where $\check{R}_B(\lambda, s+a) = \int_0^B e^{-\lambda u} R_u(s+a)\, du$. Further, multiplying the first equation of (20) by $e^{-\lambda(B-y)}$ and integrating it with respect to $y \in [0, B]$, we get

$$\tilde{D}_a^s(\lambda) = \frac{1}{\lambda} e^{-\lambda B} \frac{\check{R}_B(\lambda, s+a)}{\hat{R}_B(\lambda, s+a)} +$$
$$+ \left(\frac{1}{\lambda} e^{-\lambda B} + \tilde{D}_a^s(\lambda) \right) \left(1 - V_a^s(B) \left[1 + \frac{\check{R}_B(\lambda, s+a)}{\hat{R}_B(\lambda, s+a)} \right] e^{-B(\lambda - c(s))} \right)$$

which is a linear equation for the function $\tilde{D}_a^s(\lambda)$. It yields

$$\tilde{D}_a^s(\lambda) + \frac{1}{\lambda} e^{-\lambda B} = \frac{1}{\lambda} e^{-c(s)B} V_a^s(B)^{-1}.$$

Substituting this expression for the function $\tilde{D}_a^s(\lambda)$ into equalities (20) and taking into account the fact that $R_x(s) = 0$ and $V_a^s(x) = 1$ for $x < 0$, we get the equality of the lemma.

Now, for $y \geq 0$, we determine the function

$$Q_a^s(y) = \mathbf{E}\,[\,e^{-a\sigma_y(\nu_s)};\ \tau_y > \nu_s\,], \qquad y \geq 0, \quad a \geq 0,$$

being the Laplace transform of the total duration of a stay of the process $y + \xi(\cdot)$ inside $[0, B]$ on the time interval $[0, \nu_s]$ on the event $\{\tau_y > \nu_s\}$. The following statement is true.

Lemma 2. *Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, $\xi(0) = 0$ be a Poisson process with a negative exponential component with cumulant (3). Then the function $Q_a^s(y)$, $y \geq 0$, satisfies, for $s > 0$, the equality*

$$Q_a^s(y) = v_a^s(B - y) - aR_{B-y}(s + a) - v_a^s(B)\, D_a^s(y), \qquad y \geq 0, \tag{23}$$

where $R_x(s) = 0$ for $x < 0$, the function $D_a^s(y)$, $y \geq 0$, is given by (18), and the continuous function $v_a^s(x)$, $x \in R$, is given by the formula

$$v_a^s(x) = 1 + a\lambda \int_0^x R_u(s + a)\, du, \quad x \geq 0, \qquad v_a^s(x) = 1, \quad x < 0. \tag{24}$$

Proof. In accordance with the total probability law and due to the fact that $\chi(y)$, τ_y are the Markov times, the following system of equations for the function $Q_a^s(y)$, $y \geq 0$, is valid:

$$Q_a^s(y) = \frac{s}{s+a}\left(1 - \mathbf{E}\, e^{-(s+a)\chi(y)}\right) + \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi(y)}; X(y) \in dv, A^B]\,(1 - \mathbf{E}\, e^{-s\tau_v})$$
$$+ \int_0^\infty \mathbf{E}\,[e^{-(s+a)\chi(y)}; X(y) \in dv, A^B] \int_0^B \mathbf{E}\,[e^{-s\tau_v}; T_v \in du]\, Q_a^s(B - u), \quad y \in [0, B],$$
$$Q_a^s(y) = 1 - \mathbf{E}\, e^{-s\tau_{y-B}} + \int_0^B \mathbf{E}\,[e^{-s\tau_{y-B}}; T_{y-B} \in du]\, Q_a^s(B - u), \qquad y > B.$$

Let us interpret the obtained equations. For the first equation, we have that the duration of a stay inside $[0, B]$ on the event $\{\tau_y > \nu_s\}$ can be realized on the following disjunctive events: 1) sample paths of the process do not leave the interval $[0, B]$ (the first term), 2) the paths intersect the upper level B and do not return to the interval (the second term), 3) the paths leave the interval $[0, B]$ through the upper level B and then return to the interval $[0, B]$ (the third term of equation). The second equation is set up analogously. After taking into account the first equality of (6) and equalities (16), it follows from the latter system that

$$Q_a^s(y) = 1 - \frac{1}{\lambda} e^{-\lambda B} \frac{R_{B-y}(s + a)}{\hat{R}_B(\lambda, s + a)} - a\lambda \frac{R_{B-y}(s + a)}{\hat{R}_B(\lambda, s + a)} \hat{S}_B(\lambda, s + a)$$
$$+ a\lambda S_{B-y}(s + a) + (\lambda - c(s))\, E_y^{s+a}(c(s)) \left(\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda}\right), \qquad y \in [0, B],$$
$$Q_a^s(y) = 1 - \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(y-B)} + (\lambda - c(s))\, e^{-c(s)(y-B)} \tilde{Q}_a^s(\lambda), \qquad y > B, \tag{25}$$

where the function $E_y^{s+a}(c(s)) = \mathbf{E}\,[\,e^{-(s+a)\chi(y) - c(s)X(y)};\ A^B\,]$ is determined by (21), and

$$\tilde{Q}_a^s(\lambda) = \int_0^B e^{-\lambda u} Q_a^s(B - u)\, du,$$
$$\hat{S}_B(\lambda, s + a) = \int_B^\infty e^{-\lambda u} S_u(s + a)\, du.$$

The function $\tilde{Q}_a^s(\lambda)$ can be determined straightforwardly from the first equation of (25), since we have already given the auxiliary calculations in the proof of the previous lemma. Multiplying the first equation of (25) by $e^{-\lambda(B-y)}$ and integrating it with respect to $y \in [0, B]$, we get

(26)
$$\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda} = -\frac{1}{\lambda}e^{-\lambda B}\left[1 + \frac{R_{B-y}(s+a)}{\hat{R}_B(\lambda, s+a)}\right] v_a^s(B) + (\lambda - c(s))\,\tilde{E}^{s+a}(c(s))\left(\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda}\right)$$

which is a linear equation for the function $\tilde{Q}_a^s(\lambda)$. The function

$$\tilde{E}^{s+a}(c(s)) = \int_0^B \exp\{-\lambda(B-y)\} E_y^{s+a}(c(s))\,dy$$

is determined by (22), and the function $v_a^s(x)$, $x \in R$, is defined by equality (24) of Lemma 2. To obtain this equation, we have also used the obvious identities

$$\lambda \hat{S}_B(\lambda, s+a) = \hat{R}_B(\lambda, s+a) + S_B(s+a)\,e^{-\lambda B},$$
$$\lambda \int_0^B e^{-\lambda u} S_u(s+a)\,du = \int_0^B e^{-\lambda u} R_u(s+a)\,du - S_B(s+a)\,e^{-\lambda B}.$$

Utilizing equality (22), we find

$$\tilde{Q}_a^s(\lambda) - \frac{1}{\lambda} = -\frac{v_a^s(B)}{V_a^s(B)}\,\frac{1}{\lambda}\,e^{-c(s)B}$$

from (26). Substituting the latter expression for the function $\tilde{Q}_a^s(\lambda)$ into (25), and taking into account the fact that $v_a^s(x) = V_a^s(x) = 1$ for $x < 0$, we obtain the equality of Lemma 2.

Thus, the auxiliary functions $D_a^s(y)$, $Q_a^s(y)$, $y \geq 0$, are now determined by equalities (18) and (23). Hence, we can now obtain the integral transforms of the distribution of the duration of a stay inside the interval by the process.

Theorem 2. *Let $\xi(t) \in \mathbb{R}$, $t \geq 0$, $\xi(0) = 0$ be the Poisson process with a negative exponential component with cumulant (3), $B > 0$, $a \geq 0$, and let*

$$\sigma_y(t) = \int_0^t \mathbf{I}\{\, y + \xi(u) \in [0, B]\,\}\,du,$$
$$C_a^s(y) = s\int_0^\infty e^{-st}\mathbf{E}\exp\{-a\sigma_y(t)\,\}\,dt, \qquad y \in \mathbb{R},$$

be the total duration of a stay of the process $y + \xi(\cdot)$ inside the interval $[0, B]$ on the time interval $[0, t]$ and the integral transform of the distribution of $\sigma_y(t)$. Then, for the function $C_a^s(y)$, $y \in \mathbb{R}$, the following equalities hold for $s > 0$:

$$C_a^s(y) = v_a^s(B-y) - a\,R_{B-y}(s+a) + D_a^s(y)\,C^*(B), \qquad y \geq 0,$$
$$C_a^s(-y) = 1 - \mathbf{E}\,e^{-s\tau^y} + \int_0^\infty \mathbf{E}\,[\,e^{-s\tau^y};\ T^y \in du\,]\,C_a^s(u), \qquad y > 0, \tag{27}$$

where

$$C^*(B) = \frac{\frac{a\lambda}{c(s)}\left(v_a^s(B) - V_a^s(B)e^{c(s)B}\right) V_a^s(B)}{r(c(s), s) + a(\lambda - c(s))\int_0^B \left(V_a^s(x) - a\,e^{-xc(s)}R_x(s+a)\right)dx},$$

$$v_a^s(x) = 1 + a\lambda \int_0^x R_u(s+a)\,du, \quad x \geq 0, \qquad v_a^s(x) = 1, \quad x < 0,$$
$$V_a^s(x) = 1 + a(\lambda - c(s)) \int_0^x e^{-uc(s)} R_u(s+a)\,du, \quad x \geq 0, \qquad V_a^s(x) = 1, \quad x < 0.$$

Proof. Since the negative jumps of the underlying process are exponentially distributed, the random variables τ_x, T_x are independent (see the first formula of (6)), and, for all $x \geq 0$, the value of the overshoot through the lower level T_x is exponentially distributed with parameter λ. Hence,

$$\mathbf{E}\,[\exp\{-s\tau_y - a\sigma_y\};\ T_x \in du\,] = D_a^s(y)\,\lambda\, e^{-\lambda u}\,du.$$

Then, according to the total probability law and the strong Markov property of the process for the function $C_a^s(y)$, $y \geq 0$, we can write the system of equations

$$\begin{aligned} C_a^s(y) &= Q_a^s(y) + D_a^s(y)\,\tilde{C}_a^s(\lambda), \qquad y \geq 0, \\ \tilde{C}_a^s(\lambda) &= 1 - m_\gamma^s + \int_0^\infty m_\gamma^s(dy)\,C_a^s(y), \end{aligned} \tag{28}$$

where

$$\tilde{C}_a^s(\lambda) = \int_0^\infty \lambda\, e^{-\lambda x} C_a^s(-x)\,dx$$

is the unknown function which will be determined later on, and

$$m_\gamma^s = \int_0^\infty \lambda\, e^{-\lambda x} \mathbf{E}\, e^{-s\tau^x}\,dx, \qquad m_\gamma^s(dy) = \int_0^\infty \lambda\, e^{-\lambda x} \mathbf{E}\,[\,e^{-s\tau^x};\ T^x \in dy\,]\,dx.$$

Substituting the right-hand side of the first equation into the second equation of the system implies

$$\tilde{C}_a^s(\lambda) = 1 - m_\gamma^s + \int_0^\infty m_\gamma^s(dy)\,Q_a^s(y) + \tilde{C}_a^s(\lambda) \int_0^\infty m_\gamma^s(dy)\,D_a^s(y)$$

which is a linear equation for the function $\tilde{C}_a^s(\lambda)$. Using expressions (18) and (23) for the functions $D_a^s(y)$ and $Q_a^s(y)$, we find from the latter equation that

$$\begin{aligned} \tilde{C}_a^s(\lambda) &= v_a^s(B) + \frac{a\lambda}{1 - \int_0^\infty m_\gamma^s(dy)\,D_a^s(y)} \times \\ &\times \left(\int_0^B m_\gamma^s(dy) \int_0^{B-y} R_u(s+a)\,du - \frac{1}{\lambda} \int_0^B m_\gamma^s(dy)\,R_{B-y}(s+a) - S_B(s+a) \right), \end{aligned} \tag{29}$$

where $S_B(s+a) = \int_0^B R_u(s+a)\,du$. Substituting the latter expression for $\tilde{C}_a^s(\lambda)$ into the first equation of system (28) yields

$$C_a^s(y) = v_a^s(B-y) - a\,R_{B-y}(s+a) + D_a^s(y)\,C^*(B), \qquad y \geq 0$$

which is the first equality of Theorem 2. Now we need to determine the constant $C^*(B) = \tilde{C}_a^s(\lambda) - v_a^s(B)$. To do this, we require the following formulae

$$\int_0^B m_\gamma^s(dy)\, R_{B-y}(s+a) = \lambda \hat{R}_B(\lambda, s+a) e^{\lambda B} - \lambda R(\lambda, s) V_a^{\,s}(B) e^{c(s)B},$$

$$\int_0^B m_\gamma^s(dy) \int_0^{B-y} R_u(s+a)\, du = S_B(s+a) +$$

$$\hat{R}_B(\lambda, s+a)\, e^{\lambda B} + \frac{R(\lambda, s)}{c(s)} \Big((\lambda - c(s)) v_a^s(B) - \lambda V_a^{\,s}(B) e^{c(s)B} \Big),$$

$$\int_0^B m_\gamma^s(dy) e^{-yc(s)} \int_0^{B-y} e^{-uc(s)} R_u(s+a)\, du = \frac{\lambda R(\lambda, s)}{\lambda - c(s)} \Big(e^{B(\lambda - c(s))} - 1 \Big) +$$

(30)

$$\frac{\lambda}{\lambda - c(s)} \left(\int_0^B e^{-uc(s)} R_u(s+a)\, du - \check{R}_B(\lambda, s+a)\, e^{(\lambda - c(s))B} \right) - \lambda R(\lambda, s) \int_0^B V_a^{\,s}(x)\, dx,$$

where

$$\check{R}_B(\lambda, s+a) = \int_0^B e^{-\lambda u} R_u(s+a)\, du, \qquad \hat{R}_B(\lambda, s+a) = \int_B^\infty e^{-\lambda u} R_u(s+a)\, du.$$

The integrals which enter the left-hand sides of the latter formulae (30) are the convolutions of known functions and are easy to be calculated by utilizing the following equalities:

$$\int_0^\infty e^{-yz} m_\gamma^s(dy) = \frac{\lambda}{\lambda - z} \left(1 - \frac{\lambda - c(s)}{z - c(s)} \frac{R(\lambda, s)}{R(z, s)} \right), \quad R(z,s)^{-1} = R(z, s+a)^{-1} + a(\lambda - z),$$

The first equality follows from (6) when $p = \lambda - z$, and the second equality follows from the definition of the function $R(z, s)$ (5). Substituting formulae (30) into (29), we obtain

$$C^*(B) = \frac{\frac{a\lambda}{c(s)} \left(v_a^s(B) - V_a^{\,s}(B) e^{c(s)B} \right) V_a^{\,s}(B)}{r(c(s), s) + a(\lambda - c(s)) \int_0^B \left(V_a^{\,s}(x) - a\, e^{-xc(s)} R_x(s+a) \right) dx},$$

where $r(c(s), s) = \frac{d}{dp} R(p, s)^{-1} \Big|_{p = c(s)}$. The second equality of the theorem follows from the total probability law and the fact that τ^y is a Markov time.

Remark 3. For a semi-continuous process with independent increments the integral transforms of the distribution of the duration of a stay inside the interval have been obtained in [11]. For a symmetric Wiener process, the distribution of the duration of stay inside the interval was obtained in [11]. This distribution is the limit distribution of the distribution of the duration of a stay of homogeneous processes and a random walk (after the re-scaling of time and space). Such a limit theorem has been proved in the same paper for a semi-continuous process with independent increments.

Bibliography

1. A.V. Skorokhod, *Random Processes with Independent Increments*, Nauka, Moscow, 1964.
2. I.I. Gikhman and A.V. Skorokhod, *The Theory of Stochastic Processes*, vol. 2, Nauka, Moscow, 1973.
3. E.A. Pec̆erskii and B.A. Rogozin, *Transformations of joint distributions of random variables connected with fluctuations of a process with independent increments*, Theor. Prob. and its Appl. **14** (1969), no. 3, 431–444.
4. V.F. Kadankov and T.V. Kadankova, *On the distribution of the first exit time from an interval and the value of overshoot through the boundaries for processes with independent increments and random walks*, Ukr. Math. J. **57** (2005), no. 10, 1359–1384.

5. A.A. Borovkov, *Stochastic Processes in Queueing Theory*, Nauka, Moscow, 1972.
6. N.S. Bratiichuk and D.V. Gusak, *Boundary Problems for Processes with Independent Increments*, Naukova Dumka, Kiev, 1990.
7. V.A. Ditkin and A.P. Prudnikov, *Operational Calculus*, Vysshaya Shkola, Moscow, 1966.
8. V.M. Shurenkov and V.N. Suprun, *On the resolvent of a process with independent increments terminating at the moment when it hits the negative real semiaxis.*, Studies in the Theory of Stochastic process, Institute of Mathematics of the Academy of Sciences of UkrSSR, Kiev (1975), 170–174.
9. V.N. Suprun, *Ruin problem and the resolvent of a terminating process with independent increments.*, Ukr. Math. J. **27** (2005), no. 1, 53–61.
10. T.V. Kadankova, *Two-boundary problems for random walk with negative jumps which have geometrical distribution*, Theor. Prob. and Math. Statist. **68** (2003), 60–71.
11. V.F. Kadankov and T.V. Kadankova, *On the distribution of duration of stay in an interval of the semi-continuous process with independent increments.*, Random Oper. and Stoch. Equ. **12** (2004), no. 4, 365–388.

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