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INVESTIGATION OF THE ASYMPTOTICS OF A RENEWAL MATRIX

A semi-Markov process with finite state space and continuous time without finiteness condition of a mean stay time of this process in every fixed state is considered. The asymptotic behaviour of the renewal matrix at infinity is established under condition that the distribution tail of the stay time of the semi-Markov process in every fixed state is a regularly varying function at infinity with exponent -1.

Introducton

The classical renewal theorems give the asymptotic behaviour of the renewal function associated with a distribution F at infinity (see [1, §12.1]). These theorems are valid when the mean of the distribution F is equal to infinity. In the case of a regularly varying distribution function tail $1-F$ at infinity with exponent $-\alpha$, $0 \le \alpha \le 1$, Erickson defined more precisely the behaviour of the renewal function at infinity (see [2, Theorems 1,5]). An analogy of the renewal function for a semi-Markov process with finite state space is the so-called renewal matrix associated with a semi-Markov matrix P. If the matrix $P(\infty)$ is irreducible and a mean stay time of the semi-Markov process in every fixed state is finite, then the asymptotic behaviour of the renewal matrix is well known (see [1, §12.6]). V.S. Korolyuk and A.F. Turbin have investigated a limit properties of the renewal matrix too (see, [3, Chapter 4]). In this paper, a semi-Markov process with finite state space without finiteness condition of a mean stay time of this process in every fixed state is considered. The asymptotics of the renewal matrix is found under condition that the distribution tail of this stay time is a regularly varying function at infinity with exponent $\alpha = -1$.

The asymptotic behaviour of the renewal matrix

Let $X(t)$, $t \ge 0$, be a semi-Markov process with finite state space $\{1, 2, \dots, m\}$ and continuous time. Define

$$\tau_1 = \tau = \inf\{t > 0 \;:\; X(t) \ne X(0)\}, \quad \dots,$$

$$\tau_n = \inf\{t > \tau_{n-1} \;:\; X(t) \ne X(\tau_{n-1})\}, \quad n \ge 2.$$

A sequence of the random quantities $X(0)$, $X(\tau_1)$, ... , $X(\tau_n)$, ... creates the so-called imbedded Markov chain in $X(t)$ with the transition probabilities

$$p_{ij} = P\{X(\tau) = j \mid X(0) = i\}, \quad i, j = \overline{1, m}.$$

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The matrix $P = ||p_{ij}||_{i,j=1}^m$ is irreducible. Therefore, a unique stationary distribution p_1, $p_2, \ldots, p_m$ exists such that

$$p_i \geq 0, \quad \sum_{i=1}^m p_i = 1, \quad p_j = \sum_{i=1}^m p_i p_{ij}, \quad j = \overline{1,m}.$$

Denote

$$F_{ij}(t) = P\{\tau \leq t,\ X(\tau) = j \mid X(0) = i\},$$
$$F_i(t) = \sum_{j=1}^m F_{ij}(t) = P\{\tau \leq t \mid X(0) = i\}, \quad i = \overline{1,m}.$$

Assume that the mean stay time of the semi-Markov process $X(t)$ in every fixed state is infinite, that is,

$$M_i\tau = M\{\tau \mid X(0) = i\} = \int_0^\infty x\, dF_i(x) = \int_0^\infty (1 - F_i(x))\, dx = +\infty,$$

$i = \overline{1,m}$.

Consider the Markov renewal equation (see [3, p. 38])

$$q_i(t) = b_i(t) + \sum_{j=1}^m \int_0^t F_{ij}\{dx\} q_j(t-x), \quad i = \overline{1,m}, \quad t \geq 0, \tag{1}$$

where the matrix $F(t) = ||F_{ij}(t)||_{i,j=1}^m$ is semi-Markov.

The solution of Eq. (1) is (see [3, p. 40])

$$q_i(t) = \sum_{j=1}^m \int_0^t U_{ij}\{dx\} b_j(t-x) =$$
$$= \sum_{j=1}^m \int_0^1 U_{ij}\{tdx\} b_j(t(1-x)), \quad i = \overline{1,m}, \tag{2}$$

where $U_{ij}\{[0,t]\} = U_{ij}(t)$, $i,j = \overline{1,m}$, are the elements of the renewal matrix $U(t)$ which is an analogy of the renewal function, that is,

$$U(t) = \sum_{n=0}^\infty F^{n*}(t), \tag{3}$$

where $F^{0*}(t) = E \equiv ||\delta_{ij}||_{i,j=1}^m$ is the unit matrix,

$$F^{1*}(t) = F(t),$$
$$F^{(n+1)*}(t) = \int_0^t F^{n*}(t-u) F\{du\}, \quad n \in N.$$

Theorem 1. *Let*

$$1 - F_i(t) \sim a_i \frac{L(t)}{t}, \quad i = \overline{1,m}, \quad \text{as} \quad t \to \infty, \tag{4}$$

where L is a slowly varying function at infinity, and $a_1, \ldots, a_m$ are some nonnegative constants such that $\sum_{i=1}^m a_i p_i > 0$. Then

$$\lim_{t\to\infty} \frac{1}{t} L_1(t) U_{ij}(t) = \frac{p_j}{\sum_{i=1}^m a_i p_i}, \quad i,j = \overline{1,m}, \tag{5}$$

where

$$L_1(t) = \int_{1/d}^t \frac{L(x)}{x} dx, \quad d > 0.$$

Remark 1. Since $\sum_{i=1}^m p_i = 1$, the condition $\sum_{i=1}^m a_i p_i > 0$ excludes a possibility, when $a_i = 0$, $i = \overline{1,m}$.

Proof. Find the asymptotics of the renewal measure $U(tx) = ||U_{ij}(tx)||_{i,j=1}^m$, $x \in [0,1]$, as $t \to \infty$. Denote, by

$$\widehat{U}(\lambda) = \int_0^\infty e^{-\lambda x} U\{dx\}, \quad \lambda > 0,$$

the Laplace transform of a measure U and, by

$$\widehat{F}(\lambda) = \int_0^\infty e^{-\lambda x} F\{dx\}, \quad \lambda > 0,$$

the Laplace transform of a distribution F.

From (3), using properties of the Laplace transformation (see [4, p. 500]), we have

$$\widehat{U}(\lambda) = \sum_{n=0}^\infty \left(\widehat{F}(\lambda)\right)^n.$$

Count the norm of the matrix $\widehat{F}(\lambda)$:

$$||\widehat{F}(\lambda)|| = \sup_{1\le i\le m} \sum_{j=1}^m |\widehat{F}_{ij}(\lambda)| = \sup_{1\le i\le m} \sum_{j=1}^m \int_0^\infty e^{-\lambda x} F_{ij}\{dx\} =$$

$$= \sup_{1\le i\le m} \int_0^\infty e^{-\lambda x} F_i\{dx\}.$$

Integrating by parts, we obtain

$$\int_0^\infty e^{-\lambda x} F_i\{dx\} = e^{-\lambda x} F_i(x)\Big|_0^\infty + \lambda \int_0^\infty F_i(x) e^{-\lambda x} dx = \int_0^\infty F_i\left(\frac{t}{\lambda}\right) e^{-t} dt.$$

By the geometrical matter of an integral, we get

$$\int_0^\infty F_i\left(\frac{t}{\lambda}\right) e^{-t} dt < \int_0^\infty e^{-t} dt, \quad i = \overline{1,m}.$$

Therefore

$$||\widehat{F}(\lambda)|| = \sup_{1\le i\le m} \int_0^\infty F_i\left(\frac{t}{\lambda}\right) e^{-t} dt < 1.$$

Thus, by Theorem 5 in [5, p. 216], we obtain

$$\sum_{n=0}^\infty \left(\widehat{F}(\lambda)\right)^n = [E - \widehat{F}(\lambda)]^{-1}.$$

Hence,

$$\widehat{U}(\lambda) = [E - \widehat{F}(\lambda)]^{-1}. \tag{6}$$

For a fixed λ, consider the matrix sequence $\left\{\widehat{F}\left(\frac{\lambda}{t}\right)\right\}$ as $t \to \infty$. It is a monotonous nondecreasing matrix sequence with nonnegative elements, for which

$$\widehat{F}\left(\frac{\lambda}{t}\right) \to P \quad \text{for} \quad t \to \infty,$$

where $P = ||p_{ij}||_{i,j=1}^m$ is the transition matrix of the imbedded Markov chain. Since P is an irreducible matrix, we obtain (see [6, p. 599])

$$c_t \left(E - \widehat{F}\left(\frac{\lambda}{t}\right)\right)^{-1} \to ||1 \cdot p_j||_{i,j=1}^m \quad \text{as} \quad t \to \infty, \tag{7}$$

where $c_t = 1 - \left(p, \widehat{F}\left(\frac{\lambda}{t}\right) \cdot 1\right)$, $p = (p_1, \dots, p_m)$, $1 = (1, \dots, 1)$.

It follows from Theorem 1 in [7, p. 30] that, for all $\lambda > 0$,

$$1 - \widehat{F}_i\left(\frac{\lambda}{t}\right) \sim \frac{\lambda a_i}{t} \cdot L_1\left(\frac{t}{\lambda}\right), \quad i = \overline{1,m}, \quad \text{as} \quad t \to \infty.$$

Consequently,

$$c_t = 1 - \left(p, \widehat{F}\left(\frac{\lambda}{t}\right) \cdot 1\right) = \sum_{i=1}^{m} p_i \left(1 - \widehat{F}_i\left(\frac{\lambda}{t}\right)\right) \sim$$
$$\sim \sum_{i=1}^{m} p_i \frac{\lambda a_i}{t} \cdot L_1\left(\frac{t}{\lambda}\right) \quad \text{as} \quad t \to \infty.$$

Since L_1 is a slowly varying function at infinity (see [8, p. 220]), we get

$$c_t \sim \frac{\lambda}{t} L_1(t) \sum_{i=1}^{m} a_i p_i \quad \text{as} \quad t \to \infty. \tag{8}$$

Thus, in view of (7) and (8), relation (7) yields

$$\frac{\lambda}{t} L_1(t) \sum_{i=1}^{m} a_i p_i \int_0^\infty e^{-\frac{\lambda}{t}x} U\{dx\} \to ||1 \cdot p_j||_{i,j=1}^{m} \quad \text{as} \quad t \to \infty,$$

that is,

$$\frac{1}{t} L_1(t) \int_0^\infty e^{-\lambda u} U\{tdu\} \to \frac{1}{\lambda \sum_{i=1}^{m} a_i p_i} \cdot ||1 \cdot p_j||_{i,j=1}^{m} \quad \text{as} \quad t \to \infty$$

Whence we get

$$\frac{1}{t} L_1(t) \int_0^\infty e^{-\lambda u} U_{ij}\{tdu\} \to \frac{p_j}{\lambda \sum_{i=1}^{m} a_i p_i}, \quad i,j = \overline{1,m}, \quad \text{as} \quad t \to \infty.$$

By the generalized continuity theorem (see [4, c. 499]), the quantity

$$\frac{p_j}{\lambda \sum_{i=1}^{m} a_i p_i}$$

is a Laplace transform of some measure $p_j \mu_1$ such that

$$\int_0^\infty e^{-\lambda u} \mu_1\{du\} = \frac{1}{\lambda \sum_{i=1}^{m} a_i p_i}, \quad \lambda > 0, \tag{9}$$

and

$$\frac{1}{t} L_1(t) U_{ij}\{tdu\} \to p_j \mu_1\{du\}, \quad i,j = \overline{1,m}, \quad \text{as} \quad t \to \infty \tag{10}$$

for every bounded continuity interval of the measure μ_1.

Since

$$\int_0^\infty e^{-\lambda u} du = \frac{1}{\lambda},$$

we see from (9) that μ_1 is the Lebesgue measure multiplied by a constant. Therefore, the interval $(0,1)$ is a continuity interval of the measure μ_1. Thus, we get from (10) that

$$\lim_{t \to \infty} \frac{1}{t} L_1(t) U_{ij}\{(0,t)\} = p_j \mu_1\{(0,1)\} = \frac{p_j}{\sum_{i=1}^{m} a_i p_i}, \quad i,j = \overline{1,m}.$$

Note that $U_{ij}\{(0,t)\} = U_{ij}(t-0) - U_{ij}(0)$, $i,j = \overline{1,m}$. From (3), we obtain $U_{ij}(0) = \delta_{ij}$, $i,j = \overline{1,m}$. Since $L_1(t) \to \infty$ as $t \to \infty$, we count, by the L'Hospital rule,

$$\lim_{t\to\infty} \frac{L_1(t)}{t} = \lim_{t\to\infty} \frac{\int_{1/d}^{t} \dfrac{L(x)}{x} dx}{t} = \lim_{t\to\infty} \frac{L(t)}{t} = 0,$$

$i = \overline{1,m}$. Consequently,

$$\lim_{t\to\infty} \frac{1}{t} L_1(t) U_{ij}\{(0,t)\} = \lim_{t\to\infty} \frac{1}{t} L_1(t)[U_{ij}(t-0) - \delta_{ij}] =$$

$$= \lim_{t\to\infty} \frac{1}{t} L_1(t) U_{ij}(t) = \frac{p_j}{\sum_{i=1}^{m} a_i p_i}, \quad i,j = \overline{1,m}.$$

Theorem 1 is proved.

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