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ONE VERSION OF THE CLARK REPRESENTATION THEOREM FOR ARRATIA FLOWS

The article contains the description of functionals from a family of coalescing Brownian particles. A new type of the stochastic integral is introduced and used.

Introduction

The aim of this article is to establish the Clark representation for functionals from an Arratia flow of coalescing Brownian particles [1-4]. The following description of this flow will be used. We consider the random process $\{x(u); u \in \mathbb{R}\}$ with values in $C([0;1])$ such that, for every $u_1 < \cdots < u_n$,

1) $x(u_k, \cdot)$ is the standard Wiener process starting at the point u_k,

2) $\forall t \in [0;1]$

$$x(u_1, t) \leq \cdots \leq x(u_n, t),$$

3) The distribution of $(x(u_1, \cdot), \ldots, x(u_n, \cdot))$ coincides with the distribution of the standard n-dimensional Wiener process starting at $(u_1, \ldots, u_n)$ on the set

$$\{f \in C([0;1], \mathbb{R}^n) : f_k(0) = u_k, k = 1, \ldots, n, f_1(t) < \cdots < f_n(t), t \in [0;1]\}.$$

Roughly speaking, the process x can be described as a family of Wiener particles which start from every point of $\mathbb{R}$, move independently up to the moment of the meeting, and then coalesce and move together.

The following fact is well known. If $\{w(t); t \in [0;1]\}$ is the standard Wiener process and a square-integrable random variable α is measurable with respect to w, then α can be represented as a sum

$$\alpha = E\alpha + \int_0^1 f(t)dw(t)$$

with the usual Itô stochastic integral on the right-hand side. Our aim is to establish a variant of this theorem in the case where α is measurable with respect to the Arratia flow $\{x(u,t); u \in [0;U], t \in [0;1]\}$. It follows from the description above that the Brownian motions $\{x(u, \cdot); u \in [0;U]\}$ are not jointly Gaussian. So, the original Clark theorem cannot be used in this situation. The article is divided into three parts. In the first part, the construction of the stochastic integral with respect to the Arratia flow is presented. The next part is devoted to the variants of the Clark theorem for a finite number of the Brownian motions stopped at random times. In the last part, the modification of the construction from the first part is applied to the representation of the functionals from the flow.

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1. Spatial stochastic integral with respect to the Arratia flow

Let $\{x(u); u \in \mathbb{R}\}$ be the Arratia flow, i.e. the flow of Brownian particles with coalescence described above. For $U > 0$, consider a partition π of the interval $[0; U]$, $\pi = \{u_0 = 0, \ldots, u_n = U\}$. For $k = 1, \ldots, n$, we define

$$\tau(u_k) = \inf\{1, t \in [0; 1] : x(u_k, t) = x(u_{k-1}, t)\}.$$

Note that $\tau(u_k),\ k = 1, \ldots, n$ are the stopping moments with respect to the flow

$$F_t^\pi = \sigma(x(u_k, s),\ k = 1, \ldots, n,\ s \leq t).$$

For a bounded measurable function $a : \mathbb{R} \to \mathbb{R}$, let us consider the sum

$$S_\pi = \sum_{k=1}^{n} \int_0^{\tau(u_k)} a(x(u_k, s))dx(u_k, s). \tag{1.1}$$

Our aim is to investigate the limit of S_π under

$$|\pi| = \max_{k=0,\ldots,n-1} (u_{k+1} - u_k) \to 0$$

and its properties depending on the function a and the spatial variable U.

Let us begin with the moments of S_π. It follows from the standard properties of Itô stochastic integrals that

$$ES_\pi = 0$$

and

$$ES_\pi^2 = E\sum_{k=1}^{n} \int_0^{\tau(u_k)} a^2(x(u_k, s))ds. \tag{1.2}$$

Let us denote

$$\overline{S}_\pi = \sum_{k=1}^{n} \int_0^{\tau(u_k)} a^2(x(u_k, s))ds.$$

Consider the sequence of increasing partitions $\{\pi_n; n \geq 1\}$ of the interval $[0; U]$ with $|\pi_n| \to 0, n \to \infty$.

Lemma 1.1. *There exists a limit*

$$\lim_{n\to\infty} \overline{S}_{\pi_n} \ a.s. \tag{1.3}$$

Proof. To prove the lemma, we will check two properties of the sequence $\{\overline{S}_{\pi_n};\ n \geq 1\}$:

$$\forall n \geq 1 : \overline{S}_{\pi_n} \leq \overline{S}_{\pi_{n+1}}, \tag{1.4}$$

and

$$\sup_{n\geq 1} E\overline{S}_{\pi_n} < +\infty. \tag{1.5}$$

Note that it is enough to prove (1.4) in the case where π_{n+1} contains only one additional point v_0 comparing with π_n. Suppose that $\pi_n = \{u_0 = 0, \ldots, u_n = U\}$ and $\pi_{n+1} = \{u_0 = 0, \ldots, u_k, v_0, u_{k+1}, \ldots, u_n = U\}$. Denote

$$\acute{\tau}(u_{k+1}) = \inf\{1, t \in [0; 1] : x(u_{k+1}, t) = x(v_0, t)\}.$$

Now

$$\overline{S}_{\pi_{n+1}} - \overline{S}_{\pi_n} = \int_0^{\acute{\tau}(u_{k+1})} a^2(x(u_{k+1}, s))ds +$$
$$+ \int_0^{\tau(v_0)} a^2(x(v_0, s))ds - \int_0^{\tau(u_{k+1})} a^2(x(u_{k+1}, s))ds.$$

There are two possibilities. In the first one, $\tau(v_0) < \tau(u_{k+1})$. Now $\acute{\tau}(u_{k+1}) = \tau(u_{k+1})$. So in this case

$$\overline{S}_{\pi_{n+1}} - \overline{S}_{\pi_n} = \int_0^{\tau(v_0)} a^2(x(v_0, s))ds \geq 0.$$

The next case is $\tau(v_0) \geq \tau(u_{k+1})$. This possibility can be realized only if $\tau(v_0) = \tau(u_{k+1})$. Now $\acute{\tau}(u_{k+1}) \leq \tau(v_0)$ and

$$\int_0^{\tau(u_{k+1})} a^2(x(u_{k+1}, s))ds = \int_0^{\acute{\tau}(u_{k+1})} a^2(x(u_{k+1}, s))ds + \int_{\acute{\tau}(u_{k+1})}^{\tau(u_{k+1})} a^2(x(v_0, s))ds.$$

So, in this case,

$$\begin{aligned}
&\overline{S}_{\pi_{n+1}} - \overline{S}_{\pi_n} = \\
&= \int_0^{\tau(v_0)} a^2(x(v_0, s))ds - \int_{\acute{\tau}(u_{k+1})}^{\tau(u_{k+1})} a^2(x(v_0, s))ds = \\
&= \int_0^{\tau(v_0)} a^2(x(v_0, s))ds - \int_{\acute{\tau}(u_{k+1})}^{\tau(v_0)} a^2(x(v_0, s))ds = \\
&= \int_0^{\acute{\tau}(u_{k+1})} a^2(x(v_0, s))ds \geq 0.
\end{aligned}$$

Hence, (1.4) is true. Let us estimate the expectation of $\overline{S}_{\pi_n}$. Consider two independent standard Wiener processes w_1, w_2 which start from 0 and $u > 0$, correspondingly. Denote

$$\tau = \inf\{1,\ t : w_1(t) = w_2(t)\}.$$

Then

$$E\tau = \int_0^1 \int_{-u}^{u} p_{2t}(v)dv + \int_{-u}^{u} p_2(v)dv, \tag{1.6}$$

where p_t is the density of the normal distribution with zero mean and covariance t. It follows from (1.6) that

$$E\tau \sim \frac{3u}{2\sqrt{\pi}},\ u \to 0+. \tag{1.7}$$

Consequently,

$$\overline{\lim_{n\to\infty}} E\overline{S}_{\pi_n} \leq U \frac{3}{2\sqrt{\pi}} \sup_{\mathbb{R}} a^2.$$

Now the statement of the lemma follows from (1.4) and (1.7).

Remark 1. It follows from the proof of the lemma that there exists a limit

$$\lim_{n\to\infty} E\overline{S}_{\pi_n}.$$

Lemma 1.2. *There exists a limit*

$$m(U) = L_2 - \lim_{n\to\infty} S_{\pi_n}.$$

Proof. Let the partitions π_n, π_{n+1} be the same as in the proof of the previous lemma. Then

$$\begin{aligned}
ES_{\pi_n} S_{\pi_{n+1}} =& E \sum_{j_1=1}^{n} \int_0^{\tau(u_{j_1})} a(x(u_{j_1}, s))dx(u_{j_1}, s)\cdot \\
&\cdot \Big(\sum_{j_2 \neq k+1} \int_0^{\tau(u_{j_2})} a(x(u_{j_2}, s))dx(u_{j_2}, s) + \int_0^{\tau(v_0)} a(x(v_0, s))dx(v_0, s) + \\
&+ \int_0^{\acute{\tau}(u_{k+1})} a^2(x(u_{k+1}, s))dx(u_{k+1}, s)\Big) =
\end{aligned}$$

$$=E\sum_{j\neq k+1}\int_0^{\tau(u_j)} a^2(x(u_j,s))ds+E\int_0^{\tau(u_{k+1})} a(x(u_{k+1},s))dx(u_{k+1},s)\cdot$$
$$\cdot\Big(\int_0^{\tau(v_0)} a(x(v_0,s))dx(v_0,s)+\int_0^{\acute{\tau}(u_{k+1})} a(x(u_{k+1},s))dx(u_{k+1},s)\Big)=$$
$$=E\sum_{j\neq k+1}\int_0^{\tau(u_j)} a^2(x(u_j,s))ds+E\int_0^{\tau(u_{k+1})\wedge\acute{\tau}(u_{k+1})} a^2(x(u_{k+1},s))ds+$$
$$+E\int_{\acute{\tau}(u_{k+1})\wedge\tau(v_0)}^{\tau(u_{k+1})\wedge\tau(v_0)} a^2(x(u_{k+1},s))ds=$$
$$=E\sum_{j\neq k+1}\int_0^{\tau(u_j)} a^2(x(u_j,s))ds+E\int_0^{\tau(u_{k+1})} a^2(x(u_{k+1},s))ds=$$
$$=ES^2_{\pi_n}.$$

Consequently, for all $n\le m$,

$$ES_{\pi_n}S_{\pi_m}=E\overline{S}_{\pi_n}.$$

Now the statement of the lemma follows from Remark 1.

Remark 2. Note that the limit $m(U)$ does not depend on the choice of the sequence of partitions $\{\pi_n; n\ge 1\}$.

To prove this, we need in the estimation of the rate of convergence $ES^2_{\pi_n}$ to its limit.

Lemma 1.3. *There exists a constant C such that, for every partition π of the interval $[0;U]$,*

$$|ES^2_\pi - Em^2(U)|\le C|\pi|\sup_{\mathbb{R}} a^2. \tag{1.8}$$

Proof. First consider the partitions π', π'', where π'' is obtained from π' by adding one point on the interval $[u_k, u_{k+1}]$. As was mentioned in the proof of Lemma 1,

$$\overline{S}_{\pi''}-\overline{S}_{\pi'}=\int_0^{\zeta(v_0)} a^2(x(v_0,s))ds. \tag{1.9}$$

Here,

$$\zeta(v_0)=\inf\{1; t: (x(v_0,t)-x(u_k,t))(x(v_0,t)-x(u_{k+1},t))=0\}.$$

Let us estimate $E\zeta(v_0)$. Consider the standard Wiener process $\overrightarrow{w}$ on the plane, which is starting from the point $\overrightarrow{r}$. Suppose that this point lies inside the angle with the vertex in the origin. Let the value of the angle be less than $\frac{\pi}{2}$, and let the angle lie in the part of the plane where the both coordinate are nonnegative. Define $\acute{\zeta}$ as the first exit time of $\overrightarrow{w}$ from the angle. Then, enlarging the angle up to $\frac{\pi}{2}$ and using one-dimensional expressions like (1.6), we can check that there exists $C>0$ such that

$$E\acute{\zeta}\wedge 1\le Cr_1r_2, \tag{1.10}$$

where $\overrightarrow{r}=(r_1,r_2)$.

From these remarks, we can conclude that there exists $C_1>0$ such that

$$E\zeta(v_0)\le C_1(u_{k+1}-v_0)(v_0-u_k). \tag{1.11}$$

This conclusion can be obtained if we note that $\zeta(v_0)$ is the minimum of 1 and the first exit time of the 3-dimensional Wiener process from the space angle with the value $\frac{\pi}{3}$. It follows from (1.9) and (1.11) that

$$E(\overline{S}_{\pi''}-\overline{S}_{\pi'})\le C_1\sup_{\mathbb{R}} a^2(u_{k+1}-v_0)(v_0-u_k). \tag{1.12}$$

Now let us consider the general case where π'' is obtained from π' by the adding of a few new points. Denote, by $v_1<\dots<v_m$, the new points on the interval $[u_k,u_{k+1}]$. Then

the new amount which is obtained in $E(\overline{S}_{\pi''} - \overline{S}_{\pi'})$ from these points can be estimated due to (1.12) by the sum

$$C_1 \sup_{\mathbb{R}} a^2 \sum_{j=1}^{m} (v_j - v_{j-1})(u_{k+1} - v_j),$$

where we suppose that $v_0 = u_k$. Consequently

$$E(\overline{S}_{\pi''} - \overline{S}_{\pi'}) \le C_1 \sup_{\mathbb{R}} a^2 \sum_{k=0}^{n-1} (u_{k+1} - u_k)^2 \le C_1 \sup_{\mathbb{R}} a^2 U |\pi'|.$$

This inequality leads to the existence of the limit

$$\lim_{|\pi|\to 0} E\overline{S}_{\pi}.$$

It follows from the proof of Lemma 2 that

$$Em^2(U) = \lim_{|\pi|\to 0} E\overline{S}_{\pi}.$$

It is clear now that (1.8) holds.

The independence $m(U)$ on the choice of the sequence $\{\pi_n; n \ge 1\}$ now follows in standard way.

For $U \ge 0$, we define the $\sigma-$field

$$\widetilde{F}_U = \sigma(x(u,\cdot); 0 \le u \le U).$$

Lemma 1.4. *The process $\{m(U); U \ge 0\}$ is $(\widetilde{F}_U)-$martingale.*

Proof. The measurability of $m(U)$ with respect to $\widetilde{F}_U$ is evident. Let $0 \le U_1 < U_2$. Consider the partition $|\pi|$ of $[0; U_2]$ which contains the point U_1. Then

$$E(S_\pi / \widetilde{F}_{U_1}) = \sum_{u_k \le U_1} \int_0^{\tau(u_k)} a(x(u_k,s)) dx(u_k,s) +$$
$$+ E(\sum_{u_k > U_1} \int_0^{\tau(u_k)} a(x(u_k,s)) dx(u_k,s) / \widetilde{F}_{U_1}).$$

To prove that the last summand is equal to zero, it is enough to consider the expression

$$E(\int_0^{\tau(u)} a(x(u,s)) dx(u,s) / \widetilde{F}_{U_1})$$

for $u > U_1$. Take $0 \le u_1 < \cdots < u_n = U_1$. For a bounded Borel function $f : C([0;1])^n \to \mathbb{R}$, the expectation

$$E \int_0^{\tau(u)} a(x(u,s)) dx(u,s) f(x(u_1,\cdot), \ldots, x(u_n,\cdot)) \tag{1.13}$$

can be rewritten as

$$E \int_0^{\acute{\tau}} a(w(s)) dw(s) \acute{f}(w_1, \ldots, w_n),$$

where w and $w_1, \ldots, w_n$ are independent standard Wiener processes starting from the points u and $u_1, \ldots, u_n$, correspondingly, $\acute{\tau}$ is a stopping time for $(w, w_1, \ldots, w_n), and \acute{f}$ is a bounded Borel function on $C([0;1])^n$. Denote, by Γ, the $\sigma-$field corresponding to $\acute{\tau}$. Then

$$E \int_0^{\acute{\tau}} a(w(s)) dw(s) \acute{f}(w_1, \ldots, w_n) = E \int_0^{\acute{\tau}} a(w(s)) dw(s) E(\acute{f}(w_1, \ldots, w_n)/\Gamma)$$
$$= E \int_0^{\acute{\tau}} a(w(s)) dw(s) \tilde{f}(\tilde{w}_1, \ldots, \tilde{w}_n),$$

where $\tilde{w}_k(s) = w_k(s \wedge \tilde{\tau}), k = 1, \ldots, n$, and $\tilde{f}$ is a new bounded Borel function. Due to the Clark representation theorem,

$$\tilde{f}(\tilde{w}_1, \ldots, \tilde{w}_n) = c + \sum_{k=1}^{n} \int_0^{\tilde{\tau}} \eta_k(s) dw_k(s),$$

where, for $k = 1, \ldots, n, \eta_k$ is the square-integrable random function adapted to the flow

$$\Gamma_t = \sigma(w(s), w_1(s), \ldots, w_n(s), s \le t).$$

Consequently,

$$E \int_0^{\tilde{\tau}} a(w(s))dw(s)\tilde{f}(\tilde{w}_1, \ldots, \tilde{w}_n) = E \int_0^{\tilde{\tau}} a(w(s))dw(s) \cdot$$
$$\cdot \left(c + \sum_{k=1}^{n} \int_0^{\tilde{\tau}} \eta_k(s) dw_k(s) \right) = 0.$$

Hence, (1.13) is also equal to zero. Finally,

$$E(\sum_{k > U_1} \int_0^{\tau(u_k)} a(x(u_k, s))dx(u_k, s)/\widetilde{F}_{U_1}) = 0.$$

Taking the limit where the diameter of the partition tends to 0, we get the statement of the lemma.

2. Clark representation for a finite family of coalescing Brownian motions

This section is devoted to the integral representation of the functionals from

$$x(u_1, \cdot), \ldots, x(u_n, \cdot), \qquad u_1 < u_2 < \ldots < u_n.$$

Let us start with the following simple lemma, which was already used in the previous section.

Lemma 2.1. *Let w be the standard Wiener process on $[0;1]$, and let $0 \le \tau \le 1$ be the stopping time for w. Suppose that the square-integrable random variable α is measurable with respect to $\{w(\tau \wedge t); t \in [0;1]\}$. Then α can be represented as*

$$\alpha = E\alpha + \int_0^{\tau} f(t)dw(t)$$

with the certain adapted square-integrable random function f.

Proof. Note that

$$\sigma(w(\tau \wedge t); t \in [0;1]) = \mathcal{F}_\tau,$$

where $\mathcal{F}_\tau$ is the σ-field corresponding to the stopping moment τ. Now, due to the original Clark theorem,

$$\alpha = E\alpha + \int_0^1 f(t)dw(t).$$

It remains now to apply the conditional expectation with respect to $\mathcal{F}_\tau$ to the both sides of this equality. Lemma 2.1 is proved.

Consider the following situation. Let w_1, w_2 be independent standard Wiener processes on $[0;1]$, and let τ be a stopping time with respect to its join flow of σ-fields. The processes w_1, w_2 and the random variable τ can be considered on the product of probability spaces $\Omega_1 \times \Omega_2$. Here, Ω_1 is related to w_1 and Ω_2 is related to w_2.

Lemma 2.2. *For every fixed $\omega_1 \in \Omega_1$, the random variable $\tau(\omega_1, \cdot)$ on Ω_2 is the stopping moment for w_2 on Ω_2.*

Proof. The set $\{\omega_2 : \tau(\omega_1, \omega_2) < t\}$ is the cross section of $\{\tau < t\}$ in $\Omega_1 \times \Omega_2$. Hence, its measurability with respect to $\sigma(w_2(s); s \le t)$ follows from the usual arguments of measure theory.

The previous two lemmas lead to the following result.

Theorem 2.1. *Let $w_0, w_1, \dots, w_n$ be independent standard Wiener processes on $[0;1]$ and, for every $k = 1, \dots, n$, τ_k is the stopping time for the process $(w_0, w_1, \dots, w_k)$. Suppose that the square-integrable random variable α is measurable with respect to the set $(w_0(\cdot), w_1(\tau_1 \wedge \cdot), \dots, w_n(\tau_n \wedge \cdot))$. Then α can be represented as*

$$\alpha = E\alpha + \sum_{k=0}^{n} \int_0^{\tau_k} f_k(t) dw_k(t),$$

where $\tau_0 = 1$ and f_k is adapted to the flow generated by w_k under fixed $w_j, j \ne k$.

Proof. Denote $\widetilde{w}_k(t) = w_k(\tau_k \wedge t),\ k = 1, \dots, n$. Consider the random variable $\alpha - E(\alpha/\widetilde{w}_0, \dots, \widetilde{w}_{n-1})$. It is measurable with respect to $\widetilde{w}_n$ under fixed $\widetilde{w}_0, \dots, \widetilde{w}_{n-1}$ and has zero mean. Due to the previous lemma, it can be written as

$$\alpha - E(\alpha/\widetilde{w}_0, \dots, \widetilde{w}_{n-1}) = \int_0^{\tau_n} f_n(t) dw_n(t),$$

where the random function f_n under fixed $\widetilde{w}_0, \dots, \widetilde{w}_{n-1}$ is adapted to the flow generated by w_n. Repeat the same procedure for the random variable $E(\alpha/\widetilde{w}_0, \dots, \widetilde{w}_{n-1})$. Then

$$E(\alpha/\widetilde{w}_0, \dots, \widetilde{w}_{n-1}) - E(\alpha/\widetilde{w}_0, \dots, \widetilde{w}_{n-2}) = \int_0^{\tau_{n-1}} f_{n-1}(t) dw_{n-1}(t).$$

After n steps, we get the statement of the theorem.

Remark 3. Note that the representation from the theorem has the property

$$E\alpha^2 = (E\alpha)^2 + \sum_{k=1}^{n} E \int_0^{\tau_k} f_k(t)^2 dt.$$

Consider an example of the application of Theorem 2.1.

Example 2.1. Let α be the square-integrable random variable measurable with respect to $x(u_0, \cdot), \dots, x(u_n, \cdot)$, where $u_0, \dots, u_n$ are different points. Define the random moments

$$\tau_0 = 1,\ \tau_k = \inf\{1, t : \ x(u_k, t) \in \{x(u_0, t), \dots, x(u_{k-1}, t)\}\}, k = 1, \dots, n.$$

Then α can be represented as

$$\alpha = E\alpha + \sum_{k=0}^{n} \int_0^{\tau_k} f_k(t) dx(u_k, t),$$

where, for every k, the random function f_k is measurable with respect $x(u_0, \cdot), \dots, x(u_k, \cdot)$ and (under fixed $x(u_0, \cdot), \dots, x(u_{k-1}, \cdot)$) is adapted to the flow $x(u_k, \tau_k \wedge \cdot)$. In this representation,

$$E\alpha^2 = (E\alpha)^2 + \sum_{k=0}^{n} E \int_0^{\tau_k} f_k(t)^2 dt.$$

3. Clark representation

Let α be a square-integrable random variable measurable with respect to $\{x(u, \cdot); u \in [0; U]\}$. Suppose that $\{u_n; n \ge 0\}$ is a dense set in $[0; U]$ containing 0 and U. Define the random moments $\{\tau_k; k \ge 0\}$ as in Theorem 2.1. The following analog of the Clark representation holds.

Theorem 3.1. *The random variable* α *can be represented as an infinite sum*

$$\alpha = E\alpha + \sum_{n=0}^{\infty} \int_0^{\tau_k} f_k(t)dx(u_k, t), \tag{3.1}$$

where $\{f_k\}$ *satisfy the same conditions as those in Theorem 2.1, and the series converges in the square mean. Moreover,*

$$E\alpha^2 = (E\alpha)^2 + \sum_{n=0}^{\infty} E \int_0^{\tau_k} f_k(t)dt.$$

Proof. As was mentioned in the first section, x has cadlág trajectories as a random process in $C([0;1])$. Consequently,

$$\sigma(x(u,\cdot); u \in [0;U]) = \sigma(x(u_n;\cdot); n \geq 0) = \bigvee_{n=0}^{\infty} \sigma(x(u_0,\cdot), \dots, x(u_n,\cdot)).$$

Hence, due to the Lévy theorem,

$$\alpha = L_2\text{-}\lim_{n\to\infty} E(\alpha/x(u_0,\cdot), \dots, x(u_n,\cdot)).$$

Due to Theorem 2.1,

$$E(\alpha/x(u_0,\cdot), \dots, x(u_n,\cdot)) = E\alpha + \sum_{k=0}^{n} \int_0^{\tau_k} f_k(t)dx(u_k, t),$$

where τ_k and f_k do not change with n. So, taking the limit as $n \to \infty$, we get the statement of the theorem.

Note that the sum in (3.1) is closely related to the spatial stochastic integral which was built in the first section. Really, suppose that a is a bounded measurable function on $\mathbb{R}$ and the set $\{u_n; n \geq 0\}$ is dense in $[0;U]$ with $u_0 = 0,\ u_1 = U$.

Lemma 3.1.

$$\sum_{n=0}^{\infty} \int_0^{\tau_n} a(x(u_n,t))dx(u_n,t) = m(U) + \int_0^1 a(x(0,t))dx(0,t),$$

where $m(U)$ *was defined in the first section.*

Proof. Note that, for every $n \geq 1$, the points $u_0, \dots, u_n$ if ordered in the growing order form a partition of $[0;U]$. Under $n \to \infty$, these partitions increase, and their diameters tend to zero. To prove the lemma, it remains to note that, for every $n \geq 1$, the sum $\sum_{k=0}^{n} \int_0^{\tau_k} a(x(u_k,t))dx(u_k,t)$ coincides with the sum S_π for the corresponding partition. Lemma 3.1 is proved.

Bibliography

1. Arratia, R. A., *Brownian motion on the line* (1979), PhD dissertation, Univ. Wisconsin, Madison.
2. Le Jan, Yves, Raimond, Oliver, *Flows, coalescence and noise*, The Annals of Probability **32** (2004), no. 2, 1247-1315.
3. Dorogovtsev, A.A., *One Brownian stochastic flow*, Theory of Stochastic Processes **10(26)** (2004), no. 3-4, 21-25.
4. Dorogovtsev, A.A., *Some remarks on the Wiener flow with coalescence*, Ukrainian Math. Journ. **57** (2005), no. 10, 1327-1333.

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