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PRV PROPERTY AND THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

We consider the a.s. asymptotic behaviour of a solution of the stochastic differential equation (SDE) $dX(t) = g(X(t))dt + \sigma(X(t))dW(t)$, with $X(0) \equiv b > 0$, where $g(\cdot)$ and $\sigma(\cdot)$ are positive continuous functions and $W(\cdot)$ is the standard Wiener process. By applying the theory of PRV and PMPV functions, we find the conditions on $g(\cdot)$ and $\sigma(\cdot)$, under which $X(\cdot)$ resp. $\varphi(X(\cdot))$ may be approximated a.s. on $\{X(t) \to \infty\}$ by $\mu(\cdot)$ resp. $\varphi(\mu(\cdot))$, where $\mu(\cdot)$ is a solution of the deterministic differential equation $d\mu(t) = g(\mu(t))dt$ with $\mu(0) = b$, and $\varphi(\cdot)$ is a strictly increasing function. Moreover, we consider the asymptotic behaviour of generalized renewal processes connected with this SDE.

1. Introduction

Gikhman and Skorohod [8], §17, and later Keller et al. [11] considered the asymptotic behaviour, as $t \to \infty$, of a solution $X(\cdot) = (X(t), t \geq 0)$ of the stochastic differential equation (SDE)

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t),\ t \geq 0, \quad X(0) \equiv b > 0. \tag{1.1}$$

Here, $W(\cdot)$ is the standard Wiener process and $X(\cdot)$ denotes the Itô-solution of the SDE (1.1). One of the basic assumptions in the above works was that both $\sigma(\cdot) = (\sigma(x), -\infty < x < \infty)$ and $g(\cdot) = (g(x), -\infty < x < \infty)$ are positive functions, and the authors were only interested in situations, in which the event $\{\lim_{t\to\infty} X(t) = \infty\}$ occurs with positive probability and such that infinity will not be reached in finite time.

Gikhman and Skorohod [8], §17, and Keller et al. [11] gave the conditions, under which the asymptotics of $X(\cdot)$ is determined by a nonrandom function. In this paper, we reconsider this problem under the same basic conditions.

Denote, by $\mu(\cdot) = (\mu(t), t \geq 0)$, a solution of the deterministic differential equation corresponding to (1.1) for $\sigma(\cdot) \equiv 0$, i.e.

$$d\mu(t) = g(\mu(t))dt,\ t \geq 0, \quad \mu(0) = b. \tag{1.2}$$

We assume that the function $g(\cdot)$ is such that the solution $\mu(\cdot)$ exists, is unique, tends to ∞, as $t \to \infty$, and that infinity will not be reached in finite time. An interesting question then is under which conditions it holds that

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = 1 \quad \text{a.s.} \quad \text{on } \left\{\lim_{t\to\infty} X(t) = \infty\right\} \tag{1.3}$$

or, more generally, that

$$\lim_{t\to\infty} \frac{\varphi(X(t))}{\varphi(\mu(t))} = 1 \quad \text{a.s.} \quad \text{on } \left\{\lim_{t\to\infty} X(t) = \infty\right\}, \tag{1.4}$$

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for a given function $(\varphi(x), -\infty < x < \infty)$. Here, "a.s." stands for "almost surely".

The methods used in Gikhman and Skorohod [8], §17, Theorem 4, and in Keller et al. [11] are similar and consist of two main steps. First, they study the process

$$Y(t) = G(X(t)),\ t \geq 0, \tag{1.5}$$

where

$$G(t) = \int_1^t \frac{ds}{g(s)}\,,\ t \geq 0, \tag{1.6}$$

and prove that, under some conditions (see Section 2 below),

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.s.} \quad \text{on} \quad \Big\{\lim_{t\to\infty} X(t) = \infty\Big\}. \tag{1.7}$$

Note that $G(\cdot) = (G(t), t \geq 1)$ is the inverse function of $\mu(\cdot)$ $(G(\cdot) = \mu^{-1}(\cdot))$ if $g(\cdot)$ is positive and continuous. In the second step, relation (1.7) is used to prove (1.3).

For the second step, Gikhman and Skorohod [8], §17, Theorem 4, assume that, for some $C > 0$,

$$\lim_{\varepsilon\to 0} \sup_{z>C} \sup_{|\frac{z}{u}-1|\leq\varepsilon} \left|\frac{\mu(z)}{\mu(u)} - 1\right| = 0. \tag{1.8}$$

We will explain in Section 5, Corollary 5.1, that, under condition (1.8), the function $\mu(\cdot)$ preserves the equivalence of functions (see Definition 5.4), so that (1.7) implies in this case that

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = \lim_{t\to\infty} \frac{\mu(G(X(t)))}{\mu(t)} = 1 \quad \text{a.s.} \quad \text{on} \quad \Big\{\lim_{t\to\infty} X(t) = \infty\Big\},$$

that is, relation (1.3) holds. Note that Gikhman and Skorohod [8] use another reasoning; the general idea above, however, simplifies the proof considerably.

Condition (1.8) is formulated in terms of the function $\mu(\cdot)$, i.e. in terms of the solution of Eq. (1.2). It is more natural, however, to give conditions in terms of the functions $g(\cdot)$ and $G(\cdot)$.

Keller et al. [11], Theorem 5, use another set of conditions and prove the implication (1.7) $\Longrightarrow$ (1.4) for $\varphi(x) = \log x$, $x > 0$. Their conditions are expressed in terms of both $\mu(\cdot)$ and $g(\cdot)$. Again the disadvantage therein is the part involving the function $\mu(\cdot)$.

Our goal in this paper is to consider the conditions for implications (1.7) $\Longrightarrow$ (1.3) and (1.7) $\Longrightarrow$ (1.4) expressed in terms of the functions $g(\cdot)$ and $G(\cdot)$. For doing so, we must find the conditions, under which the functions $\mu(\cdot)$ and $\varphi(\mu(\cdot))$ preserve the equivalence of functions. To achieve this goal, we follow the general approach developed in Buldygin et al. [3]–[5]. This approach allows for solving the following general problem: Find the conditions on a given function, under which its inverse or quasi-inverse function preserves the equivalence of functions. Note that the approach in [3]–[5] works not only for continuous and increasing functions (like the ones considered in this paper), but also for discontinuous and/or non-monotone functions, where the so-called asymptotic quasi-inverse functions substitute the inverse functions.

Further in this paper, we study the asymptotic behaviour of generalized renewal processes related to the solution of SDE (1.1).

PRV and PMPV functions play an important role in this paper. For example, one of the key results is that PRV functions, and only they, preserve the equivalence of functions and sequences (see Buldygin et al. [3]). For the further studies of PRV functions and their applications, confer, e.g., Korenblyum [13], Stadtmüller and Trautner [16], Yakymiv [17], Klesov et al. [12], and Buldygin et al. [3]–[5].

Korenblyum [13] as well as Stadtmüller and Trautner [16] considered nondecreasing PRV functions and studied their properties in order to obtain analogs of the Tauberian theorems for Laplace transforms. In particular, Stadtmüller and Trautner [16] proved that the Tauberian theorem for the Laplace transform of a nondecreasing positive function f is valid if and only if f is a PRV function. A substantial progress has been achieved by Yakymiv [17], who investigated the multivariate PRV-property, but his results are of particular interest in the one-dimensional case, too. Note that PRV functions are called *weakly oscillating* in Yakymiv [17].

In Klesov et al. [12], the relationship between the strong law of large numbers for sequences of random variables and its counterpart for renewal processes constructed from these sequences has been studied. Some results of Klesov et al. [12] have been extended in Buldygin et al. [3]–[5], where the PRV property was studied in more details.

PMPV functions were introduced and studied in Buldygin et al. [3]–[5]. One of the key results is that inverse or quasi-inverse functions of PMPV functions preserve the equivalence of functions.

This paper makes use of results from Buldygin et al. [6]–[7] and is organized as follows. In Sections 2–4, we formulate and discuss the main results of the paper. In Section 5, we introduce the necessary definitions and give some auxiliary results on PRV and PMPV functions . The main problems of this paper are closely connected with the problem of finding out when differentiable functions satisfy PRV or PMPV conditions. In Section 6, we discuss these questions in some details. The proofs of some of our main results are sketched in Section 7.

2. Asymptotic Behaviour of the Solution of a Stochastic Differential Equation

General statements. Consider functions $g(\cdot)$ and $\sigma(\cdot)$ satisfying the following conditions:

(C1) $g(\cdot)$ is continuous and positive on $(0,\infty)$ and $\sigma(\cdot)$ is continuous and positive on $(-\infty,\infty)$ and such that (1.1) has a.s. a unique and continuous solution $X(\cdot)$ with arbitrary initial condition, as well as (1.2) has a unique and continuous solution with arbitrary positive initial condition.

Remark 2.1. Problem (1.1) has a.s. a unique and continuous solution $X(\cdot)$ with arbitrary initial condition, as well as problem (1.2) has a unique and continuous solution with arbitrary positive initial condition, if, for example, the functions $g(\cdot)$ and $\sigma(\cdot)$ satisfy the following conditions:

a) for some K and for all $x \in (-\infty,\infty)$,

$$|g(x)| + |\sigma(x)| \le K(1 + |x|);$$

b) for each $C > 0$, there exists L_C such that, for $|x| \le C$ and $|y| \le C$,

$$|g(x) - g(y)| + |\sigma(x) - \sigma(y)| \le L_C|x - y|$$

(see Gikhman and Skorohod [8], §15).

Now our goal is to find the conditions on $g(\cdot)$, $G(\cdot)$, and $\sigma(\cdot)$, under which relation (1.3) holds. To do so, we first consider the following general statement which describes the extra conditions for relation (1.7) to imply or being equivalent to (1.3).

Theorem 2.1. *Assume condition* (C1) *and*

$$\lim_{t\to\infty} G(t) = \int_1^\infty \frac{du}{g(u)} = \infty. \tag{2.1}$$

Let $g(\cdot)$ *and* $G(\cdot)$ *(see* (1.6)*) be such that*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g(u)G(u)} > 0 \qquad \text{for all } c > 1. \tag{2.2}$$

Then,

1) *if* (1.7) *holds, then also* (1.3) *holds true;*
2) *if*

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \int_t^{ct} \frac{du}{g(u)G(u)} = 0, \tag{2.3}$$

then (1.7) *and* (1.3) *are equivalent.*

Recall that $G(\cdot) = (G(t), t \geq 1)$ is the inverse function of $\mu(\cdot)$ ($G(\cdot) = \mu^{-1}(\cdot)$). By condition (2.1), $\mu(t) \to \infty$ as $t \to \infty$. Moreover, (2.1) excludes the possibility of explosions (that is, the solution does not reach infinity in finite time). Note that the function $g(u) = u, u > 0$, satisfies (2.1), but does not satisfy condition (2.2).

Remark 2.2. In view of Definition 5.2 below, condition (2.2) means that the function $G(\cdot)$ is a PMPV function. Observe that, by Theorem 5.2, the function $\mu(\cdot)$ preserves, under condition (2.1), the equivalence of functions (see Definition 5.4) if and only if (2.2) holds.

Condition (2.3) means that the function $G(\cdot)$ is a PRV function (see Definition 5.1) and, by Theorem 5.1, this condition is equivalent to the condition that $G(\cdot)$ preserves the equivalence of functions. The set of conditions (2.1), (2.2) and (2.3) means that both $G(\cdot)$ and $\mu(\cdot)$ preserve the equivalence of functions.

Next, we consider some sufficient conditions for (2.2) (Proposition 2.1) and (2.3) (Proposition 2.2), which can be expressed in terms of the function $g(\cdot)$, and thus are more suitable for practical use. For more details see Section 6.

Proposition 2.1. *Let* $g(\cdot)$ *be a positive and continuous function such that* (2.1) *holds. Assume that at least one of the following conditions holds:*

(i) $\limsup_{t\to\infty} g(t)G(t)/t < \infty$;
(ii) $g(\cdot)$ *is eventually nonincreasing;*
(iii) *there exists* $\alpha < 1$ *such that* $0 < \inf_{s\geq 1} g(s)s^{-\alpha}, \sup_{s\geq 1} g(s)s^{-\alpha} < \infty$*;*
(iv) $g^*(c) < c$ *for all* $c > 1$*, with* $g^*(c) = \limsup_{t\to\infty} g(ct)/g(t)$*;*
(v) $g(\cdot)$ *is an RV function with index* $\alpha < 1$ *(see Section* 5*).*

Then, $g(\cdot)$ *satisfies condition* (2.2).

Remark 2.3. Under (2.1), condition (i) of Proposition 2.1 is equivalent to (2.2), if the function $g(\cdot)$ is eventually nondecreasing.

Remark 2.4. Substituting $t \to G(t)$, we get from (1.3):

$$\lim_{t\to\infty} \frac{X(G(t))}{t} = 1 \quad \text{a.s.} \quad \text{on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\}.$$

This means that, under the conditions of Theorem 2.1, if (1.7) holds, then

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \lim_{t\to\infty} \frac{X(G(t))}{t} = 1 \quad \text{a.s.} \quad \text{on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\},$$

that is, $G(\cdot)$ is an asymptotic inverse function for the process $X(\cdot)$ a.s. on the set $\{\lim_{t\to\infty} X(t)=\infty\}$.

Remark 2.5. Condition (i) of Proposition 2.1 does not hold for any regularly varying function $g(\cdot)$ of index 1, that is, for functions $g(t)=t\ell(t)$, where $\ell(\cdot)$ is slowly varying. This is due to a result of Parameswaran [14] which proves that

$$\lim_{t\to\infty} \ell(t)\int_1^t \frac{ds}{s\ell(s)}=\infty.$$

Proposition 2.2. *Let $g(\cdot)$ be a positive and continuous function such that* (2.1) *holds. Assume that at least one of the following conditions holds*:

(i) $\liminf_{t\to\infty} g(t)G(t)/t>0$;
(ii) $g(\cdot)$ *is eventually nondecreasing*;
(iii) $\int_{0+}^1 dc/g^*(c)>0$, *with* $g^*(c)=\limsup_{t\to\infty} g(ct)/g(t)$;
(iv) *the set* $\{c\in(0,1]: g^*(c)<\infty\}$ *has positive Lebesgue measure*;
(v) *at least one of conditions* (iii), (iv), *or* (v) *of Proposition* 2.1 *holds.*

Then $g(\cdot)$ satisfies condition (2.3).

Remark 2.6. Under (2.1), condition (i) of Proposition 2.2 is equivalent to (2.3), if the function $g(\cdot)$ is eventually nonincreasing.

Asymptotic Behaviour of the Solution of a Stochastic Differential Equation under the Gikhman–Skorohod conditions. Our next result (Theorem 2.2) contains the sufficient conditions for relation (1.3) to hold. The following one is a condition from Gikhman and Skorohod [8], §17.

(GS) Let $g(\cdot)$ and $\sigma(\cdot)$ be positive and continuous functions such that (1.1) has a.s. a unique and continuous solution $X(\cdot)$ with arbitrary initial condition and with $\lim_{t\to\infty} X(t)=\infty$ a.s., as well as (1.2) has a unique and continuous solution with arbitrary positive initial condition. Let $\sigma(\cdot)/g(\cdot)$ be bounded and let $g'(x)$ exist for all $x>0$ with $g'(x)\to 0$ as $x\to\infty$.

Remark 2.7. Under (GS), relation (1.7) holds true a.s., that is,

$$\lim_{t\to\infty}\frac{G(X(t))}{t}=1 \quad \text{a.s.}$$

(see Gikhman and Skorohod [8], §17, Theorem 4 and Remark 1).

Remark 2.8. Problem (1.1) has a.s. a unique and continuous solution $X(\cdot)$ with arbitrary initial condition and with $\lim_{t\to\infty} X(t)=\infty$ a.s., as well as problem (1.2) has a unique and continuous solution with arbitrary positive initial condition, if, for example, conditions a), b) of Remark 2.1 hold and the functions $g(\cdot)$ and $\sigma(\cdot)$ satisfy the following condition:

c) for all $x\in(-\infty,\infty)$,

$$\int_{-\infty}^x \exp\left\{-\int_0^z \frac{2g(u)}{\sigma^2(u)}du\right\}dz=\infty \quad\text{and}\quad \int_x^\infty \exp\left\{-\int_0^z \frac{2g(u)}{\sigma^2(u)}du\right\}dz<\infty$$

(see Gikhman and Skorohod [8], §16, Theorem 1).

Together with condition (GS), we will consider

(C2) condition (2.1) holds and condition (2.2) or at least one of conditions (i)–(v) of Proposition 2.1 holds.

Theorem 2.2. *Assume conditions* (C2) *and* (GS). *Then relation* (1.3) *follows a.s., that is,*

$$\lim_{t\to\infty}\frac{X(t)}{\mu(t)}=1 \quad a.s..$$

Observe that Theorem 2.2 complements Theorem 4 in Gikhman and Skorohod [8], §17.

Example 2.1. (see Gikhman and Skorohod [8], §17, Corollary 1). Assume (GS) with $g(x)=Cx^{\beta}$ for $x>0$, where $0\le\beta<1$ and $C>0$. Then (2.1) and (v) of Proposition 2.1 hold. Thus, by Theorem 2.2, we have

$$\lim_{t\to\infty}\frac{X(t)}{\left(C(1-\beta)t\right)^{1/(1-\beta)}}=1 \quad \text{a.s.}$$

for all $b>0$, since $\mu(t)\sim\left(C(1-\beta)t\right)^{1/(1-\beta)}$ as $t\to\infty$, with any $b>0$.

Observe that, by Remark 2.4, we cannot use Theorem 2.2 with $\varphi(x)\equiv x$ and $g(x)\sim Cx$ as $x\to\infty$.

Asymptotic Behaviour of the Solution of a Stochastic Differential Equation under the Keller–Kersting–Rösler conditions. Here, we discuss the conditions of Keller et al. [11]. For $t>0$, put

$$h(t)=\frac{g'(t)\sigma^2(t)}{2g^2(t)}\,,\quad \psi(t)=\int_1^t\frac{\sigma^2(u)}{g^3(u)}du\,.$$

First, consider the following general condition:

(A0) $g(\cdot)$ is continuous and positive on $(0,\infty)$, $\sigma(\cdot)$ is continuous and positive on $(-\infty,\infty)$, and $g(\cdot)$ and $\sigma(\cdot)$ are such that (1.1) has a.s. a unique and continuous solution with arbitrary initial condition and $\lim_{t\to\infty}X(t)=\infty$ with positive probability, as well as (1.2) has a unique and continuously differentiable solution with arbitrary positive initial condition.

The following five conditions have been used in Keller et al. [11]:

(A1) $g:(0,\infty)\to(0,\infty)$ is strictly positive and twice continuously differentiable such that $\int_1^\infty(g(u))^{-1}du=\infty$.
(A2) $h(t)\to 0$ as $t\to\infty$.
(A3) $\sigma:(0,\infty)\to(0,\infty)$ is strictly positive and continuously differentiable such that $\int_0^\infty(tg(\mu(t)))^{-2}\sigma^2(\mu(t))dt<\infty$.
(A4) The functions $g(\cdot)$, $g'(\cdot)$, $\sigma^2(\mu(\cdot))/g^2(\mu(\cdot))$ and $h(\mu(\cdot))$ are eventually concave or convex. If $\psi(\infty)=\infty$, we require the same behaviour for the function $h(\psi^{-1}(\cdot))$.
(A5) There is a constant $C>0$ such that $\log\mu(2t)\le C\log\mu(t)$ for large t. Furthermore, the function $e^{-(\cdot)}g(e^{(\cdot)})$ together with its derivative is eventually concave or convex.

Remark 2.9. Under the above conditions, the following two statements follow (see Theorem 1 and Theorem 5 in Keller et al. [11]).

I) Under (A0)–(A4), relation (1.7) holds true.
II) Under (A0)–(A5), relation (1.4) holds true with $\varphi(t)=\log t, t>0$.

Theorem 2.3. *Assume conditions* (C2) *and* (A0)–(A4). *Then relation* (1.3) *follows.*

3. φ-Asymptotic Behaviour of the Solution of a Stochastic Differential Equation

General statements. Consider a function $\varphi(\cdot)$ satisfying the following condition:

(C3) $\varphi(\cdot) = (\varphi(x), x > 0)$ is an eventually positive and continuously differentiable function, strictly increasing to infinity as $x \to \infty$.

Put

$$G^{(\varphi)}(\cdot) = G(\varphi^{-1}(\cdot)), \quad g^{(\varphi)}(\cdot) = g(\varphi^{-1}(\cdot))\varphi'(\varphi^{-1}(\cdot)),$$

where $G(\cdot)$ is as in (1.6), the function $\varphi^{-1}(\cdot)$ is inverse to $\varphi(\cdot)$, and $\varphi'(\cdot)$ is the first derivative of $\varphi(\cdot)$.

Observe that $(G^{(\varphi)}(t), t \ge t_b)$ is the inverse function of $\varphi(\mu(\cdot))$, where $\mu(\cdot)$ is the solution of problem (1.2), with $t_b = \varphi(b)$.

For example, if $\varphi(\cdot) = \log(\cdot)$, then $G^{(\log)}(\cdot) = G(e^{(\cdot)})$ and $g^{(\log)}(\cdot) = e^{-(\cdot)}g(e^{(\cdot)})$.

If $\varphi(x) \equiv x$, then $G^{(\varphi)}(\cdot) = G(\cdot)$ and $g^{(\varphi)}(\cdot) = g(\cdot)$.

Now our main goal is to find the conditions, under which relation (1.4) holds. To do so, we first consider the following general statement which describes the extra conditions for relation (1.7) to imply or being equivalent to (1.4).

Theorem 3.1. *Assume* (C1), (C3)*, and (2.1). Let $g(\cdot)$ and $\varphi(\cdot)$ be such that*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)} = \liminf_{t\to\infty} \int_{\varphi^{-1}(t)}^{\varphi^{-1}(ct)} \frac{du}{g(u)G(u)} > 0 \qquad \text{for all } c > 1. \tag{3.1}$$

Then

1) *if* (1.7) *holds*, (1.4) *holds also true*;
2) *if*

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \int_t^{ct} \frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)} = \lim_{c\downarrow 1} \limsup_{t\to\infty} \int_{\varphi^{-1}(t)}^{\varphi^{-1}(ct)} \frac{du}{g(u)G(u)} = 0, \tag{3.2}$$

then (1.7) *and* (1.4) *are equivalent.*

Remark 3.1. In view of Definition 5.2 below, condition (3.2) means that the function $G^{(\varphi)}(\cdot)$ is a PMPV function. Observe that, by Theorem 5.2, the function $\varphi(\mu(\cdot))$ preserves, under condition (2.1), the equivalence of functions (see Definition 5.3) if and only if (3.2) holds.

Condition (3.1) means that the function $G^{(\varphi)}(\cdot)$ is a PRV function (see Definition 5.1) and, by Theorem 5.1, the function $G^{(\varphi)}(\cdot)$ preserves, under condition (2.1) the equivalence of functions if and only if (3.1) holds.

Next, we consider some sufficient conditions for both (3.1) (Proposition 3.1) and (3.2) (Proposition 3.2) which can be expressed in terms of the functions $g(\cdot)$, $G(\cdot)$, and $\varphi(\cdot)$ and thus are more suitable for practical use. For more details, see Section 6.

Proposition 3.1. *Let $g(\cdot)$ be a positive and continuous function on* $(0,\infty)$ *such that* (2.1) *holds, and let $\varphi(\cdot)$ satisfy* (C2)*. Assume that at least one of the following conditions holds:*

(i) $\limsup_{t\to\infty} g^{(\varphi)}(t)G^{(\varphi)}(t)/t = \limsup_{t\to\infty} g(t)G(t)\varphi'(t)/\varphi(t) < \infty$;
(ii) $g(\cdot)\varphi'(\cdot)$ *is eventually nonincreasing*;
(iii) *there exists* $\alpha < 1$ *such that* $0 < \inf_{t\ge 1} g^{(\varphi)}(t)t^{-\alpha}, \sup_{t\ge 1} g^{(\varphi)}(t)t^{-\alpha} < \infty$;
(iv) $(g^{(\varphi)})^*(c) < c$ *for all* $c > 1$, *with* $(g^{(\varphi)})^*(c) = \limsup_{t\to\infty} g^{(\varphi)}(ct)/g^{(\varphi)}(t)$;
(v) $g^{(\varphi)}(\cdot)$ *is an RV function with index* $\alpha < 1$ (*see Section* 5).

Then, condition (3.1) *holds true.*

Remark 3.2. Under (2.1), condition (i) of Proposition 3.1 is equivalent to (3.1), if the function $g(\cdot)$ is eventually nondecreasing.

Remark 3.3. Condition (i) of Proposition 3.1 does not hold for any regularly varying function $g^{(\varphi)}(\cdot)$ of index 1, that is, for functions $g^{(\varphi)}(\cdot)$ such that $g^{(\varphi)}(t) = t\ell(t)$, where $\ell(\cdot)$ is slowly varying (see Remark 2.5).

Proposition 3.2. *Let $g(\cdot)$ be a positive and continuous function on $(0,\infty)$ such that* (2.1) *holds, and let $\varphi(\cdot)$ satisfy* (C2). *Assume that at least one of the following conditions holds:*

(i) $\liminf_{t\to\infty} g^{(\varphi)}(t)G^{(\varphi)}(t)/t = \liminf_{t\to\infty} g(t)G(t)\varphi'(t)/\varphi(t) > 0$;
(ii) *$g(\cdot)\varphi'(\cdot)$ is eventually nondecreasing;*
(iii) $\int_{0+}^1 dc/\left(g^{(\varphi)}\right)^*(c) > 0$, *with* $\left(g^{(\varphi)}\right)^*(c) = \limsup_{t\to\infty} g^{(\varphi)}(ct)/g^{(\varphi)}(t)$;
(iv) *the set* $\{c \in (0,1] : \left(g^{(\varphi)}\right)^*(c) < \infty\}$ *has positive Lebesgue measure*;
(v) *at least one of conditions* (iii), (iv), *or* (v) *of Proposition* 3.1 *holds.*

Then, $g^{(\varphi)}(\cdot)$ satisfies condition (3.2).

Remark 3.4. Under (2.1), condition (i) of Proposition 3.2 is equivalent to (3.2), if the function $g(\cdot)$ is eventually nonincreasing.

Example 3.1. Let $g(x) = \varphi(x) = x, x > 0$. Clearly condition (2.1) holds, but condition (3.2) does not, since, for all $c > 1$,

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)} = \liminf_{t\to\infty} \int_t^{ct} \frac{du}{u\log u} \le \liminf_{t\to\infty} \frac{c-1}{\log t} = 0.$$

Next, if $g(x) = x, x > 0$, and $\varphi(x) = \log x, x > 0$, then

$$\frac{g^{(\varphi)}(t)G^{(\varphi)}(t)}{t} = 1, \quad t > 0.$$

Thus, by Propositions 3.1 and 3.2, conditions (2.1), (3.1), and (3.2) hold.

φ-Asymptotic Behaviour of the Solution of a Stochastic Differential Equation under the Gikhman–Skorohod conditions. The following Theorem 3.2 provides some conditions, under which relation (1.4) holds true, and thus generalizes Theorem 4 in Gikhman and Skorohod [8], §17.

Theorem 3.2. *Assume conditions* (C2), (C3)*, and* (GS). *Then relation* (1.4) *follows a.s., that is,*

$$\lim_{t\to\infty} \frac{\varphi(X(t))}{\varphi(\mu(t))} = 1 \quad a.s. \tag{3.3}$$

Remark 3.5. If (1.4) (or (3.3)) holds for a given function $\varphi(\cdot)$, then, by Theorem 5.1 below, this relation also holds true for the function $f(\varphi(\cdot))$, where $f(\cdot)$ is a PRV function (cf. Definition 5.1).

Example 3.2. Assume (GS) with $g(x) = Cx/(\log(x+1)^\gamma$ for $x > 0$, where $\gamma > 0$ and $C > 0$. Put $\varphi(x) = (\log(x+1))^{1+\gamma}$ for $x > 0$. Then $g^{(\varphi)}(t) \sim C(1+\gamma)$ as $t \to \infty$, that is, (2.1) and (v) of Proposition 3.1 hold. Thus, by Theorem 3.2, we have

$$\lim_{t\to\infty} \frac{(\log X(t))^{1+\gamma}}{C(1+\gamma)t} = 1 \quad \text{a.s.}$$

for all $b > 0$, since $\varphi(\mu(t)) \sim (C(1+\gamma)t)$ as $t \to \infty$.

φ-Asymptotic Behaviour of the Solution of a Stochastic Differential Equation under the Keller–Kersting–Rösler conditions. The following result provides the conditions, under which relation (1.4) holds true.

Theorem 3.3. *Assume conditions* (C2), (C3)*, and* (A0)–(A4). *Then relation* (1.4) *follows.*

Observe that Theorem 3.3 generalizes statement i) of Theorem 5 in Keller et al. [11].

Example 3.3. (see Gikhman and Skorohod [8], §17, Corollary 2). Assume (A0)–(A4) with $g(x) = Cx$ for $x > 0$, where $C > 0$. Put $\varphi(x) = \log x$ for $x > 0$. Then $g^{(\varphi)}(t) \sim C$ as $t \to \infty$, that is, (2.1) and (v) of Proposition 3.1 hold. Thus, by Theorem 3.3, we have

$$\lim_{t\to\infty} \frac{\log X(t)}{Ct} = 1 \quad \text{a.s.}$$

for all $b > 0$, since $\varphi(\mu(t)) \sim Ct$ as $t \to \infty$.

4. Asymptotic Behaviour of Generalized Renewal Processes

As above, we assume that both the functions $g(\cdot)$ and $\sigma(\cdot)$ are positive and continuous and such that problem (1.1) has a.s. a unique continuous solution $X(\cdot)$, as well as problem (1.2) has a unique continuous solution $\mu(\cdot)$.

Let us consider the following three *generalized renewal processes* for the process $X(\cdot)$:

$$F(s) = \inf\{t \geq 0 : X(t) = s\}, \quad s > 1,$$

i.e. the *first* time when the stochastic process $X(\cdot)$ crosses the level s,

$$L(s) = \sup\{t \geq 0 : X(t) = s\}, \quad s > 1,$$

the *last* time when the process $X(\cdot)$ crosses the level s, and

$$T(s) = \operatorname{meas}(\{t \geq 0 \colon X(t) \leq s\}) = \int_0^\infty I(X(t) \leq s)dt, \quad s > 1,$$

the *total time* spent by the process $X(\cdot)$ in $(-\infty, s]$, where "meas" denotes the Lebesgue measure.

The next results describe the asymptotic behaviour of these generalized renewal processes.

Theorem 4.1. *Assume conditions* (C1) *and* (1.7). *Then*

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = \lim_{t\to\infty} \frac{T(t)}{G(t)} = \lim_{t\to\infty} \frac{L(t)}{G(t)} = 1 \quad \textit{a.s. on} \ \left\{\lim_{t\to\infty} X(t) = \infty\right\}, \tag{4.1}$$

and

$$\lim_{t\to\infty} \frac{F(\mu(t))}{t} = \lim_{t\to\infty} \frac{T(\mu(t))}{t} = \lim_{t\to\infty} \frac{L(\mu(t))}{t} = 1 \quad \textit{a.s. on} \ \left\{\lim_{t\to\infty} X(t) = \infty\right\}; \tag{4.2}$$

Moreover, if conditions (C1), (C2)*, and* (1.7) *are satisfied, then*

$$\lim_{t\to\infty} \frac{\mu(F(t))}{t} = \lim_{t\to\infty} \frac{\mu(T(t))}{t} = \lim_{t\to\infty} \frac{\mu(L(t))}{t} = 1 \quad \textit{a.s. on} \ \{\lim_{t\to\infty} X(t) = \infty\}. \tag{4.3}$$

Theorem 4.2. *Under condition* (GS), *we have*

$$\lim_{t\to\infty}\frac{F(t)}{G(t)}=\lim_{t\to\infty}\frac{T(t)}{G(t)}=\lim_{t\to\infty}\frac{L(t)}{G(t)}=1\quad a.s.,$$

and

$$\lim_{t\to\infty}\frac{F(\mu(t))}{t}=\lim_{t\to\infty}\frac{T(\mu(t))}{t}=\lim_{t\to\infty}\frac{L(\mu(t))}{t}=1\quad a.s.$$

Moreover, if conditions (GS) *and* (C2) *are satisfied, then*

$$\lim_{t\to\infty}\frac{\mu(F(t))}{t}=\lim_{t\to\infty}\frac{\mu(T(t))}{t}=\lim_{t\to\infty}\frac{\mu(L(t))}{t}=1\quad a.s.$$

Theorem 4.3. *Under conditions* (A0)–(A4), *we have that relations* (4.1) *and* (4.2) *hold true. Moreover, if* (A0)–(A4) *and* (C2) *are satisfied, then relation* (4.3) *follows.*

5. Properties of PRV and PMPV Functions

Let $\mathbf{R}$ be the set of real numbers, and let $\mathbf{R}_+$ be the set of nonnegative real numbers. Also let $\mathbb{F}=\mathbb{F}(\mathbf{R}_+)$ be the space of real-valued functions $f(\cdot)=(f(t),t\geq 0)$, and $\mathbb{F}_+=\bigcup_{A>0}\{f(\cdot)\in\mathbb{F}:f(t)>0,t\in[A,\infty)\}$. Thus, $f(\cdot)\in\mathbb{F}_+$ if and only if $f(\cdot)$ is eventually positive.

Let $\mathbb{F}^{(\infty)}$ be the space of functions $f(\cdot)\in\mathbb{F}_+$ such that

(i) $\sup_{0\leq t\leq T}f(t)<\infty \quad \forall\, T>0$;
(ii) $\limsup_{t\to\infty}f(t)=\infty$.

Further, let $\mathbb{F}^{\infty}$ be the space of functions $f(\cdot)\in\mathbb{F}^{(\infty)}$ such that $\lim_{t\to\infty}f(t)=\infty$. We also use the subspaces $\mathbb{C}^{(\infty)}$ and $\mathbb{C}^{\infty}$ of continuous functions in $\mathbb{F}^{(\infty)}$ and $\mathbb{F}^{\infty}$, respectively.

Finally, the space $\mathbb{C}^{\infty}_{\mathrm{inc}}$ contains all functions $f(\cdot)\in\mathbb{C}^{\infty}$, which are strictly increasing for large t.

For a given $f(\cdot)\in\mathbb{F}_+$, we make use of the *upper* and *lower limit functions*

$$f^*(c)=\limsup_{t\to\infty}\frac{f(ct)}{f(t)}\,,\quad\text{and}\quad f_*(c)=\liminf_{t\to\infty}\frac{f(ct)}{f(t)},\qquad c>0,$$

which take values in $[0,\infty]$.

RV and ORV Functions. Recall that a measurable function $f(\cdot)\in\mathbb{F}_+$ is called *regularly varying* (RV) if $f_*(c)=f^*(c)=\varkappa(c)\in\mathbf{R}_+$ for all $c>0$ (see Karamata [9]). In particular, if $\varkappa(c)=1$ for all $c>0$, then the function $f(\cdot)$ is called *slowly varying* (SV). For any RV function $f(\cdot)$, $\varkappa(c)=c^{\alpha}$, $c>0$, for some number $\alpha\in\mathbf{R}$ which is called the *index* of the function $f(\cdot)$. Moreover, $f(t)=t^{\alpha}\ell(t)$, $t>0$, where $\ell(\cdot)$ is a slowly varying function.

A measurable function $f(\cdot)\in\mathbb{F}_+$ is called *O-regularly varying* (ORV) if $f^*(c)<\infty$ for all $c>0$ (see Avakumović [1] and Karamata [10]). It is obvious that any RV function is an ORV function. The theory of RV functions and later extensions and generalizations turned out to be fruitful in various fields of mathematics (cf. Seneta [15] and Bingham et al. [2] for excellent surveys on this topic and for the history of the theory and applications).

PRV Functions. For any RV function $f(\cdot)$, we have $f^*(c)\to 1$ as $c\to 1$. In order to generalize this property to a wider class of functions, we introduce the following notion (see Buldygin et al. [3]).

Definition 5.1. A measurable function $f(\cdot) \in \mathbb{F}_+$ is called *pseudo-regularly varying* (PRV) if

$$\limsup_{c \to 1} f^*(c) = 1. \tag{5.1}$$

Any PRV function is ORV, but not vice versa. Moreover, any RV function is PRV, but not vice versa. Corresponding examples have been given in Buldygin et al. [3].

PRV functions and their applications have been studied by Korenblyum [13], Stadtmüller and Trautner [16], Yakymiv [17], Klesov et al. [12], and Buldygin et al. [3]–[5].

Lemma 5.1. (Buldygin et al. [3]) *Assume* $f(\cdot) \in \mathbb{F}_+$. *Then,*

1) *condition* (3.1) *is equivalent to any of the following four conditions*:

(1) (*i*) $\liminf_{c \to 1} f_*(c) = 1$,

(2) (*ii*) $\lim_{c \to 1} \limsup_{t \to \infty} \left| \frac{f(ct)}{f(t)} - 1 \right| = 0$,

(3) (*iii*) $\lim_{c \downarrow 1} f^*(c) = \lim_{c \downarrow 1} f_*(c) = 1$,

(4) (*iv*) $\lim_{c \uparrow 1} f^*(c) = \lim_{c \uparrow 1} f_*(c) = 1$;

2) *condition* (3.1) *holds if and only if the upper limit function* $f^*(\cdot)$ (*or the lower limit function* $f_*(\cdot)$) *is continuous at the point* $c = 1$, *that is, either*

$$\lim_{c \to 1} f^*(c) = 1$$

or $\lim_{c \to 1} f_*(c) = 1$;

3) *if* $f(\cdot)$ *is a function with a nondecreasing upper limit function* $f^*(\cdot)$, *then condition* (3.1) *holds if and only if* $\lim_{c \downarrow 1} f^*(c) = 1$ *or* $\lim_{c \uparrow 1} f_*(c) = 1$; *moreover, under these conditions*, $f^*(\cdot)$ *is continuous at every point* $c \in (0, \infty)$.

PMPV Functions. Next we define further classes of functions playing an important role in the context of this paper (see also Buldygin et al. [3], [4]).

Definition 5.2. A measurable function $f(\cdot) \in \mathbb{F}_+$ is called *pseudo-monotone of positive variation* (PMPV) if

$$f_*(c) > 1 \qquad \text{for all } c > 1. \tag{5.2}$$

Note that any slowly varying function $f(\cdot)$ *cannot* be a PMPV function. On the other hand, any RV function of positive index as well as any *quickly* increasing monotone function, for example $f(t) = e^t,\ t \geq 0$, is PMPV.

Functions Preserving Asymptotic Equivalence. In this subsection, functions $u(\cdot)$ and $v(\cdot)$ are nonnegative and eventually positive.

Two functions $u(\cdot)$ and $v(\cdot)$ are called (*asymptotically*) *equivalent* if $u(t) \sim v(t)$ as $t \to \infty$, that is, $\lim_{t \to \infty} u(t)/v(t) = 1$. The equivalence of functions is denoted by $u(\cdot) \sim v(\cdot)$.

Definition 5.3. A function $f(\cdot)$ *preserves the equivalence of functions* if

$$f(u(t))/f(v(t)) \to 1 \quad \text{as } t \to \infty$$

for all nonnegative functions $u(\cdot)$ and $v(\cdot)$ such that $u(\cdot) \sim v(\cdot)$ and $\lim_{t \to \infty} u(t) = \lim_{t \to \infty} v(t) = \infty$.

One of the most important properties of PRV functions is that they and only they preserve the equivalence of functions.

Theorem 5.1. (*Buldygin et al.* [3]) *Assume* $f(\cdot) \in \mathbb{F}_+$. *Then the function* $f(\cdot)$ *preserves the equivalence of functions if and only if condition* (5.1) *holds.*

In view of Theorem 5.1 and Lemma 5.1, the following statement holds true.

Corollary 5.1. *Under condition* (1.7), *the function* $\mu(\cdot)$ (*see Section 1*) *preserves the equivalence of functions.*

Quasi-inverse Functions. First, we recall the definition of a *quasi-inverse function* which will be useful for our considerations below (cf. Buldygin et al. [3] and [4]).

Definition 5.4. Let $f(\cdot) \in \mathbb{F}^{(\infty)}$. A function $f^{(-1)}(\cdot) \in \mathbb{F}^{\infty}$ is called a *quasi-inverse function* for $f(\cdot)$ if $f(f^{(-1)}(s))) = s$ for all large s.

For any $f(\cdot) \in \mathbb{C}^{(\infty)}$, a quasi-inverse function exists, but may not be unique. If $f(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{inc}}$, then its *inverse function* $f^{-1}(\cdot)$ exists, that is, $f(f^{-1}(s)) = s$ and $f^{-1}(f(t)) = t$ for all sufficiently large s and t.

Example 5.1. Let $x(\cdot) \in \mathbb{C}^{(\infty)}$. Put

$$x_1^{(-1)}(s) = \inf\{t \geq 0 : x(t) = s\},$$

for $s \geq s_0 = x(0)$, and $x_1^{(-1)}(s) = 0$, for $0 \leq s < s_0$, if $s_0 > 0$. The function $x_1^{(-1)}(\cdot)$ is a quasi-inverse function for $x(\cdot)$. If $x(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{inc}}$, then $x_1^{(-1)}(\cdot) = x^{-1}(\cdot)$. □

Example 5.2. Let $x(\cdot) \in \mathbb{C}^{\infty}$. Put

$$x_2^{(-1)}(s) = \sup\{t \geq 0 : x(t) = s\},$$

for $s \geq s_0 = x(0)$, and $x_2^{(-1)}(s) = 0$, for $0 \leq s < s_0$, if $s_0 > 0$. The function $x_2^{(-1)}(\cdot)$ is a quasi-inverse function for $x(\cdot)$. Observe that $x_1^{(-1)}(s) \leq x_2^{(-1)}(s)$, $s > 0$, and, in general, $x_1^{(-1)}(\cdot) \neq x_2^{(-1)}(\cdot)$. If $x(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{inc}}$, then $x_2^{(-1)}(\cdot) = x_1^{(-1)}(\cdot) = x^{-1}(\cdot)$. □

Quasi-inverse Functions Preserving the Equivalence of Functions. Next we discuss the conditions, under which quasi-inverse functions preserve the equivalence of functions.

Theorem 5.2. (*Buldygin et al.* [3]) *Assume* $f(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{inc}}$. *Then, its inverse function* $f^{-1}(\cdot)$ *preserves the equivalence of functions if and only if condition* (5.2) *holds.*

Theorem 5.3. (*Buldygin et al.* [3]) *Assume* $f(\cdot) \in \mathbb{C}^{\infty}_{\mathrm{inc}}$, *and let* $f(\cdot)$ *satisfy condition* (3.2). *If, for some function* $x(\cdot) \in \mathbb{F}^{\infty}$,

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \qquad \text{with some } a \in (0,\infty), \tag{5.3}$$

then, for any quasi-inverse function $x^{(-1)}(\cdot)$ *of* $x(\cdot)$, *we have*

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{f^{-1}(s/a)} = 1, \tag{5.4}$$

where $f^{-1}(\cdot)$ *is the inverse function of* $f(\cdot)$.

6. Conditions for Differentiable Functions to be PRV or PMPV

The solutions of the main problems in this paper are closely connected with the question of when the differentiable functions satisfy PRV or PMPV conditions. In this section, we will discuss this question. For the proofs of all statements of this section, confer Buldygin et al. [6].

In the sequel, the following five conditions on a function $f(\cdot)$ and its derivative $f'(\cdot)$ will play a certain role:

(D) $f(\cdot) \in \mathbb{F}^\infty$, and there exists $t_0 = t_0(f(\cdot)) > 0$ such that $f(\cdot)$ is positive and continuously differentiable for all $t \ge t_0$;

(DM) condition (D) holds and $f'(t) \ge 0$ for all $t \ge t_0$;

(DM+) condition (D) holds and $f'(t) > 0$ for all $t \ge t_0$;

(DM1) condition (DM+) holds and $f'(\cdot)$ is nonincreasing for all $t \ge t_0$;

(DM2) condition (DM+) holds and $f'(\cdot)$ is nondecreasing for all $t \ge t_0$.

For a function $f(\cdot)$ satisfying condition (D), the integral representation

$$f(t) = f(t_0)\exp\left\{\int_{t_0}^{t} \frac{f'(u)}{f(u)}du\right\} \tag{6.1}$$

holds for any $t > t_0$.

Conditions for Differentiable Functions to be PRV. The following statement is immediate from Definition 5.1 in combination with (6.1).

Lemma 6.1. *Assume condition* (*D*). *Then $f(\cdot)$ is a PRV function if and only if*

$$\lim_{c\to 1}\limsup_{t\to\infty}\int_t^{ct}\frac{f'(u)}{f(u)}du = 0.$$

On applying Lemma 6.1 in combination with Lemma 5.1, we get the following result.

Lemma 6.2. *Assume condition* (*DM*). *Then $f(\cdot)$ is a PRV function if and only if*

$$\lim_{c\downarrow 1}\limsup_{t\to\infty}\int_t^{ct}\frac{f'(u)}{f(u)}du = 0.$$

Let us consider some corollaries of the above lemmas.

Corollary 6.1. *Assume condition* (*D*).

1) *If*

$$\limsup_{t\to\infty}\frac{t|f'(t)|}{f(t)} < \infty,$$

then $f(\cdot)$ is a PRV function.

2) *If $f(\cdot)$ is a PRV function, then*

$$\liminf_{t\to\infty}\frac{tf'(t)}{f(t)} < \infty.$$

3) *If condition* (*DM*) *holds and*

$$\limsup_{t\to\infty}\frac{tf'(t)}{f(t)} < \infty, \tag{6.2}$$

then $f(\cdot)$ is a PRV function.

4) *If condition* (*DM*1) *holds, then $f(\cdot)$ is a PRV function.*

Remark 6.1. If condition (D) holds and $\limsup_{t\to\infty} t|f'(t)| < \infty$, then $f^*(c) = 1$ for all $c > 0$. This means that $f(\cdot)$ is an SV function, and hence it is a PRV function. Thus, we can confine ourselves to the case where $\limsup_{t\to\infty} t|f'(t)| = \infty$.

Corollary 6.2. *Assume condition* $(DM2)$. *Then* $f(\cdot)$ *is a PRV function if and only if* (6.2) *holds true.*

Proof of Corollary 6.2. Its sufficiency follows from Corollary 6.1.

Now, let $f(\cdot)$ be a PRV function. By condition (DM2), we have, for any $c > 1$,

$$\frac{1}{c-1}\limsup_{t\to\infty}\int_t^{ct}\frac{f'(u)}{f(u)}du \ge \limsup_{t\to\infty}\frac{tf'(t)}{f(ct)}$$
$$\ge \limsup_{t\to\infty}\frac{tf'(t)}{f(t)}\liminf_{t\to\infty}\frac{f(t)}{f(ct)} = \frac{1}{f^*(c)}\limsup_{t\to\infty}\frac{tf'(t)}{f(t)}.$$

Hence,

$$\limsup_{t\to\infty}\frac{tf'(t)}{f(t)} \le \frac{f^*(c)}{(c-1)}\limsup_{t\to\infty}\int_t^{ct}\frac{f'(u)}{f(u)}du,$$

since any PRV function is an ORV function (see Buldygin et al. [3], Remark 2.1), that is, $f^*(c) < \infty$ for all $c > 0$. So, (6.2) follows from Lemma 6.2. □

The integral in the following statement means the Lebesgue integral.

Corollary 6.3. *Assume condition* $(DM+)$. *If*

$$\int_{0+}^1 (f')_*(c)dc > 0, \tag{6.3}$$

then $f(\cdot)$ *is a PRV function.*

By applying Corollary 6.3, we get the following result.

Corollary 6.4. *Assume condition* $(DM+)$. *If the set* $\{c \in (0,1] : (f')_*(c) > 0\}$ *has positive Lebesgue measure, then* $f(\cdot)$ *is a PRV function. In particular, this condition holds if* $f'(\cdot)$ *is an ORV function.*

Conditions for Differentiable Functions to be PMPV. The following statement is also immediate from Definition 5.2 in combination with (6.1).

Lemma 6.3. *Assume condition* (D). *Then* $f(\cdot)$ *is a PMPV function if and only if*

$$\liminf_{t\to\infty}\int_t^{ct}\frac{f'(u)}{f(u)}du > 0 \quad \textit{for all } c > 1.$$

Next, we consider some corollaries of Lemma 6.3.

Corollary 6.5. *Assume condition* (DM).

1) *If*

$$\liminf_{t\to\infty}\frac{tf'(t)}{f(t)} > 0, \tag{6.4}$$

then $f(\cdot)$ *is a PMPV function.*

2) *If* $f(\cdot)$ *is a PMPV function, then*

$$\limsup_{t\to\infty}\frac{tf'(t)}{f(t)} > 0.$$

3) *If* $f(\cdot)$ *is a PMPV function, then*

$$\limsup_{t\to\infty} tf'(t) = \infty.$$

4) *If condition* $(DM2)$ *holds, then* $f(\cdot)$ *is a PMPV function.*

Corollary 6.6. *Assume condition* $(DM1)$*. Then* $f(\cdot)$ *is a PMPV function if and only if* (6.4) *holds true.*

The following result gives a condition in terms of the function $(f')_*(\cdot)$.

Lemma 6.4. *Assume condition* $(DM+)$*. If*

$$c(f')_*(c) > 1 \quad \text{for all } c > 1, \tag{6.5}$$

then $f(\cdot)$ *is a PMPV function.*

7. Proofs of Some Main Results

In this section, we consider only the proofs of some results from Section 2. For further details, we refer to Buldygin et al. [6], [7].

Proof of Theorem 2.1. By conditions (2.1) and (2.2) and by Lemma 6.3, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$, we have that $G(\cdot)$ is a PMPV function, that is, it satisfies (5.2). Moreover, $G(\cdot) \in \mathbb{C}^\infty_{\text{inc}}$. Hence, by Theorem 5.2, the function $\mu(\cdot) = G^{-1}(\cdot)$ preserves the equivalence of functions (see Definition 5.4). Therefore, in view of (1.7),

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = \lim_{t\to\infty} \frac{\mu(G(X(t)))}{\mu(t)} = 1 \quad \text{a.s.} \quad \text{on} \quad \left\{\lim_{t\to\infty} X(t) = \infty\right\},$$

that is, relation (1.3) holds. Thus, statement 1) is proved.

By conditions (2.1) and (2.3) and by Lemma 6.2, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$, we have that $G(\cdot)$ is a PRV function (see Definition 5.1). Hence, by Theorem 5.1, the function $G(\cdot) = \mu^{-1}(\cdot)$ preserves the equivalence of functions. Therefore, in view of (1.3),

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \lim_{t\to\infty} \frac{G(X(t))}{G(\mu(t))} = 1 \quad \text{a.s.} \quad \text{on} \quad \left\{\lim_{t\to\infty} X(t) = \infty\right\}.$$

that is, relation (1.7) holds. Thus, statement 2) follows from the last implication in combination with 1). □

Proof of Proposition 2.1. Condition (2.2) follows from

a) (2.1) and (i) (or (ii)), in view of Corollary 6.5, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$.
b) (iii), since (i) (and even (2.1)) follows from (iii).
c) (2.1) and (iv), in view of Lemma 6.4, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$, since $(1/g(\cdot))_* = 1/g^*(\cdot)$.
d) (2.1) and (v), since (iv) follows from (v).

□

Proof of Remark 2.3. It follows from Corollary 6.6 with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$. □

Proof of Proposition 2.2. Condition (2.3) follows from

a) (2.1) and (i) (or (ii)), in view of Corollary 6.1, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$.
b) (2.1) and (iii), in view of Corollary 6.3, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$, since $(1/g)_*(c) = 1/g^*(c)$ for all $c > 0$.
c) (2.1) and (iv), since (iii) follows from (iv).
d) (2.1) and (v), since (iv) follows from (v).

□

Proof of Remark 2.6. It follows from Corollary 6.2, with $f(\cdot) = G(\cdot)$ and $f'(\cdot) = 1/g(\cdot)$. □

Proofs of Theorems 2.2 *and* 2.3. They follow from Remarks 2.7 and 2.9, respectively, in combination with Theorem 2.1 and Proposition 2.1. □

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