1. *Ильюшин А.А., Победря Б.Е.* Основы математической теории термовязкоупругости.– М.: Наука, 1970.– 280 с.
2. *Колтунов М.А.* Ползучесть и релаксация.– М.: Высшая школа, 1976.– 276 с.
3. *Вольмир А.С.* Нелинейная динамика пластинок и оболочек. М.: Наука, 1972.–432 с.
4. *Бадалов Ф.Б., Эшматов Х., Юсупов М.* О некоторых методах решения систем интегро-дифференциальных уравнений, встречающихся в задачах вязкоупругости // Прикладная математика и механика.– 1987. –Т.51.– № 5. –С.867–871.
5. *Эшматов Х.* Интегральный метод математического моделирования задач динамики вязкоупругих систем. Автореферет дисс. д-ра. тех. наук, Киев, 1991.

УДК 72.25. 72.25.

A. Korostil, Ju.Korostil

QUANTUM DYNAMICS OF A TWO-LEVEL SYSTEM UNDER EXTERNAL FIELD

We present exact analytic solutions for non-linear quantum dynamics of a two-level system (TLS) subject to a periodic-in-time external field. In constructing the exactly solvable models, we use a approach where the form of external perturbation is chosen to preserve an integrability constraint, which yields a single non-linear differential equation for the ac-field. A solution to this equation is expressed in terms of Jacobi elliptic functions with three independent parameters that allows one to choose the frequency, average value, and amplitude of the time-dependent field at will. This form of the ac-drive is especially relevant to the problem of dynamics of TLS charge defects that cause dielectric losses in superconducting qubits.

1.   Introduction

The problem of a periodically-driven two-level system (TLS) appears in many physical contexts including magnetism, superconductivity, structural glasses and quantum information theory [1-7]. The interest in this old problem has been revived recently due to advances in the field of quantum computing (see, e.*g.,* [8-12] and references therein). First of all, a qubit itself is a two-level system and the question of its evolution under an external time-dependent perturbation is obviously of interest. Also, the physical mechanism that currently limits coherence particularly in superconducting qubits is believed to be due to other types of unwanted TLSs within the qubit, \whose charge dynamics under a periodic-in-time electric field gives rise to dielectric losses directly probed in experiment. [13,14]. In what follows, we mostly apply our solution to the latter charge TLS model, but the general methods and some particular results of this work evidently can be

applied to a much broader range of problems.

One of the key metrics of a superconducting qubit is the quality factor, which is defined as a ratio of the real and imaginary parts of the dielectric response function, $\varepsilon(\omega)$, evaluated at the resonant frequency of the corresponding LC-circuit, $Q = \mathrm{Re}\,\varepsilon(\omega_r)/\mathrm{Im}\,\varepsilon(\omega_r)$. Very high values of the quality factor are required for the qubit to be operational. However, existing experiments consistently show significant dielectric losses that occur in an amorphous dielectric (e.g.., in Al_2O_3) used as a barrier in the Josephson junctions. It is believed that the losses are primarily due to the presence of charge two-level system defects in the barrier and/or the contact interfaces, which respond to an AC electric field in the LC-resonator. It is still unclear what the physical origin of these defects is, but an early work of Phillips [13] as well as very recent comprehensive density functional theory studies point to the OH-rotor defects as a very likely source of the dielectric losses. The determination of the physical origin and the properties of the TLSs responsible for the dielectric loss is investigated in the presented work.

The usual theoretical approach to calculating the quality factor and more generally the full dielectric response function, $\varepsilon(\omega)$, involves a formal mapping of charge dynamics in a double-well potential onto the problem of "spin" dynamics in an AC field, described by the "spin" Hamiltonian

$$H(t) = b(t) \cdot \sigma / 2, \quad b(t) = 2(\Delta_t, 0, \varepsilon + d_{TLS} \cdot E(t))$$

where σ denotes the Pauli matrices and $b(t)$ is an effective "magnetic field" that drives TLSs, with ε, Δ_t and d_{TLS} being the TLS energy splitting, the tunneling amplitude between its two states, and the TLS dielectric moment correspondingly and $E(t)$ is the AC electric field. A linear analysis within the canonical TLS predicts that the dielectric function due to identical TLSs is peaked at the frequency, $v = \sqrt{\Delta_t^2 + \varepsilon^2}$. Ad-hoc inclusion of T_1 and T_2 relaxation processes and the assumption about random distribution of TLS energy-splitting and tunneling (typically assumed to be uniform and long-uniform correspondingly) lead to the quality factor $Q \quad \sqrt{1 + (E_0/E_c)^x}$, with $x \quad 2$, E_0 being the amplitude of an applied AC electric field and E_C is a critical value of the amplitude which also encodes the information on the strength of the relaxation processes (see, e.g., [5]). Both formulas are used widely in interpreting experimental data and probing energetic of the relevant TLS defects.

While this linear analysis is a fine approximation to describe a majority of regimes currently studied experimentally, the existing experiments are certainly

capable and some do access non-linear regimes as well, where the energy of the applied electric field is comparable or larger than the relevant TLS energies. Hence, this non-perturbative regime is of clear experimental and theoretical interest. More importantly studies of поп1inear dynamics may provide another effective means to probe the properties of TLS.
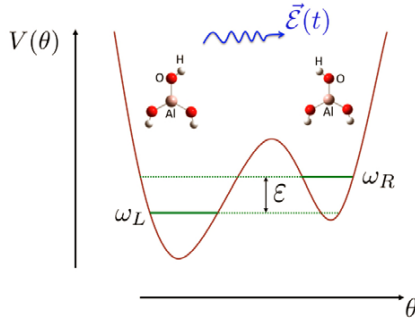

FIG. 1: Schematic representation of an OH-rotor two-level system in an Al_2O_3 oxide. [16.17]. Here, the role of the generalized variable is assigned to the angle θ defined as an angle between the OH-bond and an axis perpendicular to the vertical AlO bond. At low enough temperatures, the phase space an isolated rotor is reduced to the two-states corresponding to the minima of the double-well potential! $V(\theta)$. Application of external ac-field parametrically coupled to the rotor's dipole moment induces oscillations between the two minima.

The mathematical formulation of the non-linear TLS dynamics problem studied in this paper is deceptively simple. We will solve the Schrödinger equation for a spinor wave-function

$$i\partial_t \Psi = \frac{1}{2} b(t) \cdot \sigma \, \Psi, \quad \Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

that describes a half-integer spin subject to a periodic in time magnetic field of the form, $b(t0 = 2(\Delta_t, 0, f(t))$, where Δ_t is a constant describing the coupling between the two states and the function $f(t) = f(t+T)$, describes the time dependent perturbation. Despite the simplicity of the formulation, the problem is generally unsolvable in analytic form for most cases of practical interest. The origin of this surprising fact can be understood if we introduce a new function $R(t) = \psi_+(t)/\psi_-(t)$, which reduces the matrix Schrödinger equation to the Riccatti equation

$$\partial_{(-it)} R = 2 fR + \Delta_t (1 - R^2).$$

It is a non-linear differential equation that has known analytic solution in a very

limited number of cases (not that case of a monochromatic perturbation is not one of them).

Therefore, to solve for TLS dynamics driven by a specific nonequilibrium field is equivalent to generating a particular solution to the Ricatti equation corresponding to the perturbation. Clearly this is a challenging mathematical task and this observation partially explains the current deficit of exact mathematical results. The difficulties in obtaining exact solutions have led to the emergence of several perturbative approaches, used in particular to characterize relaxation and dephasing rates in qubits as a function of driving amplitude. These analyses provide very useful physical insights and correctly describe the physics if the time-dependent perturbation is weak, but it is also clear that there exist non-linear effects beyond perturbation theory and it is desirable to have exact results to access this qualitatively different physics.

The mathematical approach that we use to obtain exact results is to exactly solvable Hamiltonians of specific form relevant to the problem of interest. A key observation in our analysis is that finding a Hamiltonian corresponding to a given solution is much easier than solving the Schrödinger equation with a given Hamiltonian. In some generalized sense, the two procedures are related to one another much like differentiation relates to integration. To see this, it is useful to consider the evolution operator, or the S-matrix, which relates the initial state at $t = 0$ to a final state at $t > 0$ as follows, $\Psi(t) = S(t)\Psi(0)$. In the absence of relaxation process the time-evolution is unitary and it satisfies the Schrödinger equation,

$$i\partial_t S(t) = H(t)S(t) .$$

If we choose an arbitrary S-matrix,

$$S = \exp\left(-\frac{1}{2}\Phi(t)\cdot\sigma \right) \in SU_2 ,$$

we can immediately reconstruct the corresponding Hamiltonian that gives rise to such evolution as follows $H(t) = i\partial_t S(t)S^+(t)$. Using this method, one can generate an infinite number of exact non-equilibrium solutions and explicit models. These solutions may be of importance to physics of NMR, to the question of physical implementation of gate operations on a qubit as well as of some mathematical interest. Nevertheless without additional constraints such analyses would generally produce Hamiltonians of little importance to the problem of dynamics of TLS charge defects.

A very useful insight that allows us to constructively narrow down the range of relevant dynamical systems comes from the mathematically related problem of far-from-equilibrium superconductivity. It is well-known that the reduced BCS Hamiltonian is algebraically equivalent to an interacting XY-spin model in an effective "inhomogeneous" magnetic field in the z-direction, whose profile is dictated by the bare single particle-energy dispersion. Far from equilibrium, dynamics of a given Anderson pseudospin is determined by an effective time -

dependent se1f-consistent fie1d of other pseudo-spins that it interacts with. In many cases (determined by specific initial conditions), this BCS se1f-consistency constraint dynamically se1ects a specific order-parameter, such that the dynamics of essentially infinite number of spins is equiva1ent to the dynamics of few spins on1y.

For specia1 sets of initia1 conditions, these spins move in unison and therefore the se1f-consistent "magnetic fie1d" (or superconducting order parameter in the 1anguage of BCS theory) is periodic in time. The reduced BCS model is integrable and there exists a very elegant prescription for constructing exact non-equilibrium solutions to it. These solutions contain, in particular, exact spin dynamics in a periodic time-dependent field that can be expressed in terms of elliptic functions. In this paper, we generalize such anomalous soliton solutions to encompass a wider range of time dependencies relevant to the problem of TLS dynamics, which is of our primary interest.

2. General framework for constructing exact solutions

In this paper, we derive a family of exact solutions for the non-dissipative TLS dynamics subject to an external ac-field. The main ingredient of our approach is a special ansatz for the TLS's dynamics that corresponds to periodic-in-time but non-monochromatic external fields. Before proceeding to the specific ansatz, let us first introduce a general algebraic framework of exact solutions. We are interested in solving the non-equilibirum Schrödinger equation for the spinor

$$i\partial_t \Psi(t) = H(t)\Psi(t), \quad \Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad (1)$$

where the Hamiltonian is $H(t) = (1/2)b(t)\cdot\sigma$. As mentioned in the introduction, instead of solving Eq. (1) for the wave-function, we can consider the Schrödinger equation for the evolution operator that relates the initial and final states, $\Psi(t) = S(t)\Psi(0)$. This equation for the S-matrix has the form identical to Eq. (1)

$$i\partial_t S(t) = H(t)S(t), \quad S(0) = 1 \quad (2)$$

but now it is an equation for the matrix function $S(t)$, which belongs to the two-dimensional representation of the *SU(2)* group, while the Hamiltonian expressed in terms of *SU(2)*$_2$ generators belongs to the two-dimensional representation of the *su*(2*) algebra.

Note that the form of Eq. (2) is such that it may be generalized to an arbitrary spin or equivalently to an arbitrarydimensional representation of *SU(2)* or it can be viewed as an equation of motion in the abstract group such that

$$H_{abs}(t) = b(t)\cdot J_{abs} \in su(2), \quad S_{abs}(t) = \exp(-i\Phi(t)\cdot J_{abs}) \in SU(2),$$

where J_{abs} are the corresponding generators. Therefore, a solution of the problem in a particular representation, i.e., an explicit form of $\Phi(t)$, immediately gives the corresponding solutions in all other representations (e.g., a two-Ievel system dy-

namics uniquely determines a "d-level system" dynamics in the same field). This TLS problem that we are interested in corresponds to the two-dimensional generators $J_\alpha^2 = (1/2)\sigma_\alpha$ with σ_α ($\alpha = (x, y, z)$) being the familiar Pauli matrices.

The problem of determining the solution, $\Phi(t)$, from the magnetic field time-dependence $b(t)$ is a complicated one, but the inverse problem is almost trivial. Indeed, if we select a specific S-matrix (defined uniquely by the choice of a specific function, $\Phi(t)$, the Hamiltonian will read

$$H(t) = i\partial_t S(t) S^+(t),\tag{3}$$

where

$$S(t) = \exp\left(\frac{i}{2}\Phi(t)\cdot\sigma\right).\tag{4}$$

Using the algebraic identities for the Pauli matrices, we obtain the corresponding magnetic field

$$b(t) = \dot\Phi\, n + \sin\Phi\, \dot n + (1-\cos\Phi)[n\times n],\tag{5}$$

where $\Phi(t) = |\Phi(t)| n(t)$, with $|n(t)| \equiv 1$. Note that one can generate exactly-solvable models by simply picking an arbitrary $\Phi(t)$ dependence and using Eq. (3) to find the corresponding Hamiltonian. However, without guidance or luck, such an analysis would generally produce complicated non-equilibrium fields that have little to do with an underlying physical problem. Let us however mention here that this procedure may be of interest to quantum computing an general, because the time-evolution governed by an S-matrix can be viewed as a "gate operation" on the spin (if the TLS/spin corresponds to a qubit rather than to a defect within a qubit). By picking "trajectories," $\Phi(t)$, on the algebra that start in the origin, *i.e.* $\Phi(0) = 0$, but end at a specific point at a time T, one can immediately determine the non-equilibrium magnetic pulse, $b(t)$, or a class of such pulses, that will give rise to a desired gate operator $G = S(t) = \exp\left(-\frac{i}{2}\Phi(t)\cdot\sigma\right)$.

Let us note here that the function, $\Phi(t)$, contains complete information about the solution to the original problem, Eq. (1), including the overall quantum phase accumulated by the wave-function during the time evolution (as we shall see below, this phase is of particular interest to the problem of dielectric response of TLSs in superconducting qubits). An interesting question is whether and how purely quantum phase can be restires from a solution of the corresponding classic Bloch equations that are usually considered in this context. Let us recall that a classical mapping can be achieved by introducing the average magnetic moment,

$$m(t) = \Psi^+(t)\frac{\sigma}{2}\Psi(t).\tag{6}$$

Therefore $m^2(t) \equiv 1/4$ and the classical equations of motion for the spin moment follow from

$$\partial_t m(t) = \frac{1}{2} \Psi^+(t) [H(t), \sigma] \Psi(t)$$

and yield the familiar result

$$\partial_t m(t) = b(t) \times m(t) \qquad (7)$$

Let us recall that these Bloch equations are a saddle point of quantum spin dynamics, much in the same way that Newton's equations of motion governed by the force, $[-\nabla V(r)]$, represent a saddle point of the action describing a quantum particles in the potential, $V(r)$, and therefore do not contain direct information about quantum intformation and tunneling effects. Similarly, Eqs. (7) do not directly contain the quantum phase and to determine it one has to go back to the Schrödinger equation. Another more abstract way to see this is by noticing that. Eqs. (7) describe the motion on a two-dimensional (Bloch) sphere, $m(t) = S^2$, while the original quantum problem Eq. (2) describes motion on a three-dimensional sphere since $S_{abs}(t) \in SU(2) \quad S^3$.

Now let us recall that there exists the Hopf fibration such that $SU(2)/U(1) = S^2$, which summarizes the fact that classical equations, namely Eqs. (7), represent quantum motion modulo the U(1) phase dynamics. Fortunately, this phase dynamics can generally be restored from exact dependence of the $m(t)$ solution, albeit in a non-local way. To see this, we can write the magnetization in terms of the 5-matrix as follows

$$m(t) = \frac{1}{2} \Psi^+(0) [S^+(t) \sigma S(t)] \Psi(0),$$

where $\Psi(0)$ and the corresponding $m(0) = \Psi^+(0)(\sigma/2)\Psi(0)$ are initial conditions for the wave-function and Bloch magnetization, correspondingly. Using again the well-known identities for the Pauli matrices, we find the evolution matrix for the Bloch equations, as follows $m_\alpha(t) = R_{\alpha\beta}(t) m_\beta(0)$, as follows

$$m_\alpha(t) = \delta_{\alpha\beta} \cos \Phi + n_\alpha n_\beta (1 - \cos \phi) - \varepsilon_{\alpha\beta\gamma} n_\gamma \sin \Phi \qquad (8)$$

This three-dimensional matrix describes a rotation, $R(t) \in SO(3)$, and can be represented equivalently as

$$R(t) = \exp[-\Phi(t) \cdot L], \quad L = \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix}, \qquad (9)$$

where $L \in s0(3) \quad so(2)$ belong to the three-dimensional vector representation of the $so(2)$ algebra. They are related to the "usual" spin 1 representation (where J_3^z is diagonal) via simple linear transform.

Therefore, we see that if we known an arbitrary solution to the Bloch equation, $m(t)$ we can at least in principle restore the function, $\Phi(t)$, (see, Eqs. (9) and (4)), which uniquely determines the entire quantum solution. It also suggests that if we choose an arbitrary dynamics function on a sphere we may be able to restore the quantum Hamiltonian that would give rise to it, via mappings $m(t) \to R(t) \to S(t) \to H$. However, the second step in this chain of transforms involves effectively calculating a logarithm of the rotation matrix, which due to a complicated "analytic" structure of this matrix-logarithm function requires a careful calculation non-local in time.

The sequent Sections are devoted to constructing exactly solvable periodic-in-time Hamiltonians based on a specific anzats for the classical Bloch "magnetization", $m(t)$. It further involves a restoration of the corresponding quantum $U(1)$ phase via a straightforward integration. More specifically, we reverse ' the following Hamiltonian

$$H = \Delta_t \sigma_x + f(t)\sigma_z, \tag{10}$$

where $f(t) = f(t+T))$ is a periodic function, with an arbitrary period, T_f. Our solution below also allows tuning of the average splitting, $\varepsilon = <f(t)>_{T_f}$, and the AC field amplitude, $A_f = \sqrt{<|f(t)-\varepsilon|^2>_T}$. As mentioned in the introduction, this problem is of great importance to the physical problem of externally-driven TLS dynamics in superconducting qubits (with Δ_t corresponding to tunneling between the wells, ε to a splitting of energy levels in a double-well potential, and T_f and A_f being the period and the amplitude of the AC-electric field correspondingly).

Our "guess" for the relevant ansatz for the Bloch "magnetization," $m(t)$, is based on a set of formal solutions discovered in the related problem of quenched dynamics of fermionic superfluids [19-21,24,25]. Formally, the quenched dynamics of each individual Cooper pair is described by the Bogoliubov-de Gennes Hamiltonian, which is essentially a spin Hamiltonian that reduces to (10) after the unitary transformation $\sigma_x \to \sigma_z$ and $\sigma_z \to -\sigma_x$. with Δ_t corresponding to a single particle energy level and $f(t)$ to the superfluid order parameter.

A realization of each particular form of the superfluid order parameter dynamics in a steady state can be unambiguously determined by the initial conditions using the exact integrability of BCS model. Note that a self-consistency condition for the order parameter provides a limitation on the set of functions for which the corresponding problem is integrable and for some initial conditions periodicin-time self-consistent dynamics, $f(t)$, can be realized. While in ourTLS problem, there is no natural selfconsistency constraint, such insights and constraints from the BCS problem help us narrow down the range of possible ansatze to restore reasonable physical Hamiltonians, which are also exactly

solvable by construction. In what follows, we generalize the solution analysis of the paper [16] and find a general soliton configuration, characterized by three independent parameters, which we denote as $\Delta_{\pm}$ and Δ_a. For the physical problem of interest, this conveniently implies that some, generally speaking, non-trivial combination of these parameters will determine the arbitrary frequency, amplitude, and the dc-component of the field. Due to the periodicity, we can generally represent the AC-perturbation as a Fourier series

$$f(t) = \varepsilon + A_f \sum_{n=1}^{\infty} f_n \cos(n\omega_f t) \ . \tag{11}$$

Note that for certain specific choices of the parameters $\Delta_{\pm,a}$ the leading coefficient $f_1 \quad f_n$ ($n = (2,3,...)$ and one recovers the limit of a monochromatic AC-field, albeit in the regime of weak driving $A_f f_1 \quad \max\{\Delta_1, \varepsilon\}$. Therefore, our non-linear analysis contains the standard linear response results as a simple special case.

3. Non-dissipative dynamics of the ac-drived TLS

Further we provide the details on the derivation of the exact solution for the TLS dynamics. We devote the special attention to the analysis of the $U(1)$ phase of the \wave function. We also elucidate the relations between the parameters of our solution and the amplitude, phase and the dc-component of the external field, which may be useful for experimental applications of our theory.

We now focus on the Schrödinger equation for the half-integer spin in the magnetic field, $b(t) = 2(\Delta_t, 0, f(t))$. When written in terms of spinor components, it has the form

$$i\dot{\psi}_+ = \Delta_t \psi_- + f(t)\psi_+,$$
$$i\dot{\psi}_- = \Delta_t \psi_+ - f(t)\psi_- . \tag{12}$$

The corresponding Bloch equation is

$$\dot{m}(t) = 2(\Delta_t, 0, f(t) \times m(t)) . \tag{13}$$

Let us now make the following anzats for its exact solution [25]:

$$m_x = D - Cf^2, \ m_y = B\dot{f}, \ m_z = Af(t) + F . \tag{14}$$

From two of the Eqs. (13) we find $A = 2\Delta_t B$ and $B = C$. Thus among five parameters in (14) only three are independent: P, B and D. The equation for the external field, $f(t)$, can be obtained from (14) using the condition $m^2 = 1/4$. This resulting equation for the function $f(t)$ acquires the form

$$\dot{f}^2 = -f^4 - 4c_2 f^2 + 8c_1 f - 4c_3 \tag{15}$$

where coefficients c_j are given by some combinations of parameters B, D and F (see Eqs. (30) below). Equation (15) can be cast to a more symmetric form, using another set of parameters Δ_a and $\Delta_\pm$, which are chosen to be positive and are related to coefficients c_j as

$$c_1 = -\frac{\Delta_a}{4}(\Delta_+^2 - \Delta_-^2),\ c_2 = -\frac{1}{4}(\Delta_+^2 + \Delta_-^2 + 2\Delta_a^2),$$

$$c_3 = -\frac{1}{4}(\Delta_+^2 - \Delta_a^2)(\Delta_a^2 - 2\Delta_-^2). \tag{16}$$

Without loss of generality and to be more specific we also assume $\Delta_+ \geq \Delta_-$ for the remainder of this paper, while Δ_a can be assigned an arbitrary value. By virtue of expressions (16) equation (15) now reads

$$\dot{f}^2 = 4\left[(f - \Delta_a)^2 - \Delta_-^2\right]\left[\Delta_+^2 - (f + \Delta_a^2\right], \tag{17}$$

Below we will make several transformations that allow us to reduce (17) to an equation for the Weierstrass elliptic function. Firstly, let us introduce a function, $y(t)$,

$$f(t) = \Delta_+\left[\frac{2}{y(t)} - 1\right] - \Delta_a, \tag{18}$$

which satisfies the following equation

$$\left(\frac{dy}{dx}\right)^2 = 4(y - a_+)(y - a_-)(y - 1),\ x = \frac{\Delta_+ t}{\sqrt{a_+ a_-}} \tag{19}$$

where $a_\pm = 2\Delta_+ / (\Delta_+ + 2\Delta_a \pm \Delta_-)$. Now, Eq. (19) can be easily reduced to a well-known equation for the Weierstrass elliptic function by rescaling the parameters via the transformation

$$y(x) = Z(x) + \frac{a_+ + a_- + 1}{3} \tag{20}$$

so that

$$\left(\frac{dZ}{dx}\right)^2 = 4(Z - e_1)(Z - e_2)(Z - e_3), \tag{21}$$

where parameters e_j satisfy the following conditions $e_1 > e_2 > e_3$ and $e_1 + e_2 + e_3 = 0$. Coefficients e_j are determined by the parameters Δ_a and $\Delta_\pm$. The specific expressions for the coefficients e_j, however, depend on the relative values of the initially introduced set of parameters. Solution of the equation (21) is

$$Z(x) = \wp(x + x'),\ \omega' = \frac{K(\kappa')}{\sqrt{e_1 - e_2}}, \tag{22}$$

where $\wp(x)$ is a Weierstrass elliptic unction, K is a complete elliptic integral of the first kind and $\kappa' = \sqrt{(e_1 - e_2)(e_1 - e_3)}$. Function $Z(x)$ is a doubly-periodic function with the period along the physical time axis determined by, $l = 2\omega$, where $\omega = \sqrt{1 - \kappa'^2}$ is a modulus of elliptic functions. Combining (22) with Eqs. (20) and (18) allows us to express $f(t)$ in terms of elliptic functions. Expression for $f(t)$ can be compactly written in terms of Jacobï elliptic functions. Just as it is the case for the parameters e_j, the particular form of the resulting expression depends on the relation between Δ_a and $\Delta_\pm$.

All cases considered here are summarized by the following compact expression for the function, $f(t)$, \written in terms of Jacobi elliptic function as following

$$f(t) = \Delta_+ \frac{\eta_+ sn^2(z,\kappa) - 1}{\eta_- sn^2(z,\kappa) + 1} - \Delta_a , \qquad (23)$$

where variable z is

$$T_f = \frac{4K(\kappa)}{\sqrt{[(\Delta_+ - 2\Delta_a)^2 - \Delta_-^2](e_1 - e_3)}}, \quad A_f = \frac{\Delta_+}{2}\left(\frac{\eta_+ + \eta_-}{\eta_- + 1}\right). \qquad (26)$$

Lastly, the average value of the function $f(t)$ over its period is

$$< f(t) > = \frac{\Delta_+ \eta_+}{\eta_-}\left[1 - \frac{\eta_+ + \eta_-}{\eta_+ K(\kappa)}\prod(-\eta_-,\kappa)\right] - \Delta_a \equiv \varepsilon \qquad (27)$$

with $K(\kappa)$ and $\prod(\eta,\kappa)$ being a complete elliptic integral of the first and third kind correspondingly. As we have already mentioned, quantity (27) describes the dc-component of the external field. One can view Eqs. (26, 27) as the definition of yet another set of parameters A_f, $\omega_f = 2\pi/T_f$ and $\varepsilon = < f(t) >$, which allows us to cast external field $f(t)$ into the form given by (11). The dependence of the parameters of the external field, $f(t)$, on the ratio Δ_-/Δ_+ allows to determine the limits of strong and weak ac-driving. In particular, we can see that the regime of the strong ac-driving should be achieved for moderate values of Δ_a and $\Delta_-/\Delta_+ \quad 0.2$.

Expressions (23,,24,25) constitute our main results. Quite generally, our solution represeпts the superposition of monochromatic waves with frequencies integer шmultiples of $\omega_f = 2\pi/T_f$, The solution (23) can be reduced to the monochrошatic wave with frequency $2\Delta_+$ when $\Delta_a = 0$ and $\Delta_- \quad \Delta_+$.

1. *U. Weiss*, Qцaпtцm dissipative systems (World Scientific, Singapore, 2008), 3rd Ed.
2. *A. J. Leggett et al.*, Rev. Mod. Phys. **59**, 1 (1987).
3. *P. W. Anderson, B. Halperin and C. Varma*, Phil. Mag. **25**, 1 (1972).

4. *R. F. Akhmet'yanov,∗ V. O. Ponomarev, O. A. Ponomarev, and E. S. Shikhovtseva,* Theor. Math. Phys., **149**, N1 (2006).
5. *C. C. Yu and P. W. Anderson,* Phys. Rev. B **29**, 6165 (1984).
6. *S. Hunklinger and A. K. Raychaudhur.i* Atorphous Solids:Low-Temperature Properties, edited by W.A. Philips (Springer, Berlin, 1981).
7. *V. Lubchenko and P. G. Wolynes, J. Chem*. Phys. **119** (17) 9088 (2002).
8. *M. A. Nielsen, I. L. Chuang*, Qиantum Computations and Quantum Information (Cambridge Univ. Press, Саtbridge, 2002).
9. *B. D. Gerardot and P. Ohberg*, Science **326**, 1489 (2009).
10. *J. E. Mooij, T. P. Orlando, L. Levitov, L. Tian, C.H., van der. Wal and S. Lloyd*, Science, **285**, 1036 (1999).
11. *I. Chiorescu, У. Nakamura, C. J. P. M. Harmans, and J. E. Mooij,* Science 299, 1869 (2003).
12. *J. M. Martinis et al.,* Phys. Rev. Lett. **95**, 210503 (2005).
13. *W. A. Phillips,* J. of Low Temp. Phys. **11**, 757 (1973).
14. *H. Paik and K. D. Osborn*, Appl. Phys. Lett. **96**, 072505 (2010) .
15. *E.A. Yuzbashyan, M. Dzero*, Phys. Rev. Lett. **96** 230404 (2006).
16. *E.A. Yuzbashyan*, Phys. Rev. B **78** 184507 (2008).

УДК 358.4 : 656.7
Р.В. Хращевский, к.воен.н., доцент, НУОУ, м. Киев

ОЦЕНКА МЕХАНИЗМА КООРДИНАЦИИ СИСТЕМЫ ПЛАНИРОВАНИЯ

The assessment mechanism to coordinate the planning system. The possibility of well-coordinated system with large external and internal factors.

Постановка проблемы. При рассмотрении многоуровневых систем часто приходится отказываться от требования строгой глобальной оптимальности управляющих воздействий и локальных решений. Дело в том, что в практических ситуациях строгий оптимум по многим причинам оказывается нереализуемым. Чаще всего это связано с недостаточностью информации о факторах, влияющих на результаты выбранных решений или управляющих воздействий. В классических ситуациях управления и принятия решений использование алгоритмов оптимизации оправдывается в первую очередь тем фактом, что они разрешают некоторые проблемы, связанные с имеющимися в данной ситуации неопределенностями. Используя так называемый алгоритм оптимизации, можно выбрать последовательность действий, которая приводит к нужным результатам, если информация и гипотезы, на которых основан алгоритм, достаточно точны. При постановке

© Р.В. Хращевский