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On size effect of mechanical properties of thermoelastic solids

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In this paper the principal relations of a thermoelastic solid model for the local gradient approach in thermomechanics are presented. Within the framework of such an approach the stationary stressed-strained state of a stretched isotropic thermoelastic layer is examined. On this basis using the common meaning of elasticity modules the size effects of Young's modulus, Poisson's ratio and bulk modulus are studied. The isotropic quality of elasticity modules size effects is confirmed. It is shown that in the considered model the shear modulus does not depend on the specific size (thickness) of the layer. The numerical results are presented as graphs.

Keywords: size effects, elasticity modules, local gradient approach.

Introduction. In the last decades in scientific literature considerable attention is paid to the construction of mathematical models and study of physicomechanical processes in the deformable solids taking into account the effect of local, including interface, heterogeneity. Such an interest is caused in particular by the wide use of thin films and fibres, and also nanomaterials in engineering practice [1-3]. Techniques for working on the nanoscale have become essential to electronic engineering and nanoengineered materials began to appear in consumer products. It is known, that device elements of the nanoscale size (traditionally defined as less than 100 nanometers) feature the increasing ratio of surface to bulk volume and the influence of surface energy becomes significant on the nanoscale. When studying the nanomaterial properties a special attention is paid to investigation of the mechanical properties, including dependences of the elasticity modules on a characteristic size [2, 3].

There are nonlocal [4, 5] and local gradient [6, 7] approaches to construction of locally heterogeneous models of solid mechanics. Within the framework of the first approach the dependence between stresses and deformations is adopted, in the general case, as integral correlation on spatial coordinates and it represents conditions that stresses in the considered point of a body depend on deformations both herein, and in neighboring points. On the basis of nonlocal models a wide complex of researches of the deformable solids has been performed and presented in literature, taking into account scale and damage effects, interface phenomena etc. It should be noted that this approach, as well as other phenomenological approaches, meets significant difficulties when considering the influence of the nonmechanical processes, including thermal ones on body deformation.The other mentioned approach, which allows us to take into account heterogeneity when describing the state of a physically small element, is the local gradient approach in thermomechanics [6-9]. The dependence between stresses and strain tensors within a model of the local gradient approach in the elasticity theory can be presented in a nonlocal form [6]. Within the framework of this approach the number of models are built and it is shown on the example of model problems that this approach allows us to describe deformation of bodies taking into account the interface phenomena, scale effects, including that of tensile strength, and also to investigate the influence of temperature, admixtures, environment etc on them [7-9].

The paper considers the principal relations of a thermoelastic solid model, the stationary stressed-strained state of stretched isotropic thermoelastic layer and does examine the stress-strain dependence and the proportionality coefficients (elasticity modules) dependence on layer thickness.

1. Basic relations of local gradient thermoelasticity

Local gradient approach is based on the principles of nonequilibrium thermodynamics and nonlinear mechanics [6]. According to the approach the local state parameters space is expanded by chemical potential and its gradient allows us to describe varying particle interaction conditions in the different body regions. The local gradient approach model of one-component thermoelastic body includes in the space of local state the strain tensor $\hat{e}$, temperature T, chemical potential H and its gradient $\vec{\nabla}H$. Conjugated parameters are stress tensor $\hat{\sigma}$, entropy S, density of mass ρ and vector of elastic mass replacement $\vec{\pi}_m$ respectively. For linear approach the state equations of the model of local gradient thermomechanics for an isotropic solid can be written as

$$S - S_* = a_{te}e - a_{th}\eta + a_{tt}\theta, \quad \rho - \rho_* = -a_{eh}e - a_{hh}\eta - a_{th}\theta,$$
$$\vec{\pi}_m = -a_{gg}\vec{\nabla}\eta, \quad \hat{\sigma} = 2a_e\hat{e} + \left[a_{ee}e + a_{eh}\eta - a_{te}\theta\right]\hat{I}. \qquad (1)$$

Here subscript «*» represents the value of the parameters in the initial state which is considered to be the state of load-free infinite isotropic media with material identical to the material of the body; $\theta = T - T_*$, $\eta = H - H_*$, $\vec{\nabla}\eta = \vec{\nabla}H - \vec{\nabla}H_*$, $\hat{e} = \hat{e} - 0$ are the disturbances of state parameters; a_e, a_{ee}, a_{eh}, a_{te}, a_{gg}, a_{hh}, a_{th} are constants; $e = \hat{e} : \hat{I}$, $\hat{I}$ is identity tensor.

For the model of local gradient thermomechanics the key set of equations describing the steady state of a body taking as solving functions the displacement vector $\vec{u}$, disturbances of temperature θ and chemical potential η is

$$a_e\nabla^2\vec{u} + \left(a_e + a_{ee}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right) + a_{eh}\vec{\nabla}\eta - a_{te}\vec{\nabla}\theta = 0,$$
$$\nabla^2\theta = 0, \quad a_{gg}\nabla^2\eta - a_{hh}\eta - a_{eh}e - a_{th}\theta = 0. \qquad (2)$$

If instead of $\vec{u}$ as the solving function the stress tensor $\hat{\sigma}$ is used, the set is

$$\vec{\nabla}\cdot\hat{\sigma} = 0, \quad \vec{\nabla}\times\left[\left(3a_{ee} + 2a_e\right)\hat{\sigma} - \left(a_{ee}\sigma + 2a_e a_{eh}\eta - 2a_e a_{te}\theta\right)\hat{I}\right]\times\vec{\nabla} = 0,$$

$$\nabla^2\theta = 0\,, \qquad \nabla^2\eta - \kappa_\eta^2\eta - \kappa_\sigma^2\sigma - \kappa_\theta^2\theta = 0\,, \tag{3}$$

where $\kappa_\eta^2 = \dfrac{1}{a_{gg}}\left(a_{hh} - \dfrac{3a_{eh}^2}{3a_{ee}+2a_e}\right)$, $\kappa_\sigma^2 = \dfrac{a_{eh}}{a_{gg}\left(3a_{ee}+2a_e\right)}$, $\kappa_\theta^2 = \dfrac{1}{a_{gg}}\left(a_{th} + \dfrac{3a_{eh}a_{te}}{3a_{ee}+2a_e}\right)$, $\sigma = \hat{\sigma} : \hat{I}$.

We shall use the last set to investigate the relation between mechanical load of a body and deformations in a layer loaded at infinity and at the surface.

2. Stressed-strained state of the layer

Let us consider an isotropic thermoelastic layer (domain $|x| \le l$ in the Cartesian coordinates $\{x, y, z\}$). At infinity $y \to \pm\infty$ the layer is loaded by forces of intensity $p_y \ge 0$. Suppose that at the surfaces $x = \pm l$ there are the normal to surface mechanical load p_x, the values of chemical potential η_a and temperature θ_a. The equations (3), written for nonzero components of stresses, temperature and chemical potential are

$$\frac{d\sigma_x}{dx} = 0\,, \quad \frac{d^2\sigma_y}{dx^2} = \frac{d^2\sigma_z}{dx^2}\,, \quad \frac{d^2\sigma}{dx^2} = b_m\frac{d^2\eta}{dx^2}\,,$$
$$\frac{d^2\theta}{dx^2} = 0\,, \quad \frac{d^2\eta}{dx^2} - \kappa_\eta^2\eta - \kappa_\sigma^2\sigma - \kappa_\theta^2\theta = 0\,, \tag{4}$$

where $b_m = 4a_e a_{eh}/\left(a_{ee} + 2a_e\right)$.

The boundary conditions and integral load conditions we write in the form

$$\theta\big|_{x=\pm l} = \theta_a\,, \quad \eta\big|_{x=\pm l} = \eta_a\,, \quad \vec{n}\cdot\hat{\sigma}\big|_{\pm l} = \left(\pm p_x, 0, 0\right),$$
$$\frac{1}{2l}\int_{-l}^{l}\sigma_y dx = p_y\,, \quad \int_{-l}^{l}\sigma_z dx = 0\,, \quad \int_{-l}^{l}x\sigma_y dx = 0\,, \quad \int_{-l}^{l}x\sigma_z dx = 0\,. \tag{5}$$

Here $\vec{n}$ is the external normal vector.

The problem solution is

$$\eta(x) = \eta_a + \frac{\chi}{\zeta_l}\left(\frac{\mathrm{ch}(\xi x)}{\mathrm{ch}(\xi l)} - 1\right), \quad \theta(x) = \theta_a\,, \quad \sigma_x(x) = p_x\,,$$
$$\sigma_y(x) - p_y = \sigma_z(x) = \frac{b_m}{2}\frac{\chi}{\zeta_l}\left(\frac{\mathrm{ch}(\xi x)}{\mathrm{ch}(\xi l)} - \frac{\mathrm{th}(\xi l)}{\xi l}\right). \tag{6}$$

Here $\xi^2 = \kappa_\eta^2 + b_m\kappa_\sigma^2$, $\chi = \dfrac{\kappa_\sigma^2\left(p_x + p_y\right) + \kappa_\eta^2\eta_a + \kappa_\theta^2\theta_a}{\xi^2}$, $\zeta_l = 1 - D\left(1 - \dfrac{\mathrm{th}(\xi l)}{\xi l}\right)$, $D = b_m\dfrac{\kappa_\sigma^2}{\xi^2}$,

ch and th are hyperbolical cosine and tangent.

These formulas describe the state of the layer, conditioned by the external force and temperature loading and the difference of chemical potential at the surface of the layer to compare to its values in an infinite homogeneous media. If the characteristic size of the body is considerably greater than the size of nearsurface heterogeneity region ($\xi l >> 1$) then in the free of external load body ($p_x = p_y = 0$) internal areas are practically unstressed, and heterogeneity is localized in the narrow nearsurface regions. The uniform temperature affects the size of stresses in the body not changing the picture of their distribution.

The stressed state described with formulas (6) causes the layer deformation. For determining strain components on the basis of the last equation (1) we write

$$\hat{e} = \frac{1}{2a_e}\hat{\sigma} - \frac{1}{3a_{ee}+2a_e}\left[\frac{a_{ee}}{2a_e}\sigma + a_{eh}\eta - a_{te}\theta\right]\hat{I} .$$

This relation alongside with solution (6) yields strain in the directions of the stress applying

$$e_x = \frac{1}{3a_{ee}+2a_e}\left[\frac{a_{ee}+a_e}{a_e}p_x - \frac{a_{ee}}{2a_e}p_y - a_{eh}\eta_a + a_{te}\theta_a + \right.$$
$$\left. + \frac{a_{eh}\chi}{\zeta_l}\left(1 - \frac{3a_{ee}+2a_e}{a_{ee}+2a_e}\frac{\operatorname{ch}\xi x}{\operatorname{ch}\xi l} + \frac{2a_{ee}}{a_{ee}+2a_e}\frac{\operatorname{th}\xi l}{\xi l}\right)\right],$$
$$e_y = \frac{1}{3a_{ee}+2a_e}\left[-\frac{a_{ee}}{2a_e}p_x + \frac{a_{ee}+a_e}{a_e}p_y - a_{eh}\eta_a + a_{te}\theta_a + \frac{a_{eh}\chi}{\zeta_l}\left(1 - \frac{\operatorname{th}\xi l}{\xi l}\right)\right]. \quad (7)$$

Obtained relations (7) contain the constituents of deformation caused by the external power loading (p_x, p_y) as well as nonmechanical action (η_a, θ_a). For examining the deformation caused by mechanical load only we consider

$$e_x^{ef} = \frac{1}{2l}\int_{-l}^{l}\left[e_x(x)\right] - \left[e_x(x)\right]_{p_x=0;p_y=0}dx , \qquad e_y^{ef} = \left[e_y\right] - \left[e_y\right]_{p_x=0;p_y=0} . \quad (8)$$

Here it is taken into account that strain e_x is variable and it allows to consider its integral characteristic. On the basis of (7), (8) we write expressions

$$e_x^{ef} = \frac{1}{3a_{ee}+2a_e}\left[\frac{a_{ee}+a_e}{a_e}p_x - \frac{a_{ee}}{2a_e}p_y + \left(p_x+p_y\right)\Psi\left(\xi l\right)\right],$$
$$e_y^{ef} = \frac{1}{3a_{ee}+2a_e}\left[-\frac{a_{ee}}{2a_e}p_x + \frac{a_{ee}+a_e}{a_e}p_y + \left(p_x+p_y\right)\Psi\left(\xi l\right)\right], \quad (9)$$

for deformation caused by action of p_x and p_y. Here $\Psi\left(\xi l\right) = \frac{a_{ee}+2a_e}{4a_e}\cdot\frac{1-\zeta_l\left(\xi l\right)}{\zeta_l\left(\xi l\right)}$.

3. Size effect of elasticity modules

Let us consider the layer under mechanical load in the direction of Y-axis ($p_x = 0, p_y > 0$). In this case strain components e_x^{ef} and e_y^{ef} are

$$e_x^{ef} = \frac{1}{3a_{ee}+2a_e}\left(-\frac{a_{ee}}{2a_e} + \Psi(\xi l)\right) p_y ,$$

$$e_y^{ef} = \frac{1}{3a_{ee}+2a_e}\left(\frac{a_{ee}+a_e}{a_e} + \Psi(\xi l)\right) p_y . \tag{10}$$

The ratio of the applied stress to the fractional expanding (or contracting) of the sample length due to axial tension (or compression) is Young's modulus E and the ratio of lateral strain (perpendicular to the applied stress) to the longitudinal strain (parallel to applied stress) is Poisson's ratio ν, in this case

$$e_y^{ef} = \frac{p_y}{E}, \quad e_x^{ef} = -\nu \frac{p_y}{E},$$

from (10) we get

$$E = \frac{p_y}{e_y^{ef}} = (3a_{ee} + 2a_e)\left[\frac{a_{ee}+a_e}{a_e} + \Psi(\xi l)\right]^{-1},$$

$$\nu = -\frac{e_x^{ef}}{e_y^{ef}} = \left[\frac{a_{ee}}{2a_e} - \Psi(\xi l)\right]\left[\frac{a_{ee}+a_e}{a_e} + \Psi(\xi l)\right]^{-1}. \tag{11}$$

Note that neglecting interconnection in Eq. (4) that results in $D = 0$ we get classical relations

$$E_0 = \frac{(3a_{ee} + 2a_e)a_e}{a_{ee} + a_e}, \quad \nu_0 = \frac{a_{ee}}{2(a_{ee} + a_e)}, \tag{12}$$

that coincide with expressions of E and ν through Lame constants λ, μ. Relations (11) thus describe the size effect of elasticity modules for considered state. For such effects the description of the interconnectivity of set (4) is essential because the equality to zero either of the coefficients b_m or κ_σ gives coupling parameter D turning zero. Note that uniform temperature does not influence the received relations.

To clarify whether size effect of elasticity modules presented in (11) is implied by geometry and conditions of specific problem or is a more general case let us consider another stressed state of the layer.

Let us consider the layer under mechanical load in the direction of X-axis ($p_x = 0, p_y > 0$). In this case for strains e_x^{ef} and e_y^{ef} we get from (9)

$$e_x^{ef} = \frac{1}{3a_{ee}+2a_e}\left(\frac{a_{ee}+a_e}{a_e} + \Psi(\xi l)\right) p_x ,$$

$$e_y^{ef} = \frac{1}{3a_{ee}+2a_e}\left(-\frac{a_{ee}}{2a_e}+\Psi(\xi l)\right)p_x . \qquad (13)$$

Further we obtain

$$E = \frac{p_x}{e_x^{ef}} = \left(3a_{ee}+2a_e\right)\left[\frac{a_{ee}+a_e}{a_e}+\Psi(\xi l)\right]^{-1},$$
$$\nu = -\frac{e_y^{ef}}{e_x^{ef}} = \left[\frac{a_{ee}}{2a_e}-\Psi(\xi l)\right]\left[\frac{a_{ee}+a_e}{a_e}+\Psi(\xi l)\right]^{-1} . \qquad (14)$$

Comparing (11) and (14) one can see that for different load conditions we obtaine the same type of the dependence of elasticity modules on the layer thickness. Thus the model of thermoelastic body built for local gradient approach allows us to describe the size effects of Young's modulus and Poisson's ratio. In this case the isotropic quality of the elasticity modules size effects is proved. Using formulas [10]

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)},$$

on the basis of (14) we shall write expressions for Lame constants λ, μ, shear G and bulk K modules often used alongside with Young's modulus and Poisson's ratio

$$\mu = G = a_e ,$$
$$\lambda = \left(a_{ee}-a_e\frac{a_{ee}+2a_e}{2a_e}\frac{1-\zeta_l(\xi l)}{\zeta_l(\xi l)}\right)\left[1+3\frac{a_{ee}+2a_e}{4a_e}\frac{1-\zeta_l(\xi l)}{\zeta_l(\xi l)}\right]^{-1},$$
$$K = \left(a_{ee}+\frac{2}{3}a_e\right)\left[1+3\frac{a_{ee}+2a_e}{4a_e}\frac{1-\zeta_l(\xi l)}{\zeta_l(\xi l)}\right]^{-1} .$$

In the considered model the shear module does not show the size effect.

One should note that in thick layers the values of Young's modulus and Poisson's ratio (11) tend to limits

$$\bar{E} = \left(3a_{ee}+2a_e\right)\left[\frac{a_{ee}+a_e}{a_e}+\frac{a_{ee}+2a_e}{4a_e}\frac{D}{1-D}\right]^{-1},$$
$$\bar{\nu} = \left(\frac{a_{ee}}{2a_e}-\frac{a_{ee}+2a_e}{4a_e}\frac{D}{1-D}\right)\left[\frac{a_{ee}+a_e}{a_e}+\frac{a_{ee}+2a_e}{4a_e}\frac{D}{1-D}\right]^{-1},$$

respectively. Assuming $D = 0$ we obtain relation (12) in the case of reducing the model relations to the classical ones.

Let us examine dependence of reduced Young's modulus and Poisson's ratio

$$E = E_0 \left(1 + \frac{a_e}{a_{ee} + a_e} \Psi(\xi l) \right)^{-1},$$

$$\nu = \nu_0 \left(1 - \frac{2a_e}{a_{ee}} \Psi(\xi l) \right) \Big/ \left(1 + \frac{a_e}{a_{ee} + a_e} \Psi(\xi l) \right),$$

on the body size ξl, the model constants ratio a_e / a_{ee} and the coupling parameter D.

Specified values of Young's modulus E_p (solid line, left scale) and Poisson's ratio ν_p (dashed line, right scale) for $D = 0{,}1$, $k = a_e / a_{ee} = 0{,}4; 0{,}6; 0{,}8$ (curves 1, 2, 3) decrease with body size increase. This is demonstrated in Fig. 1.

Specified values of Young's modulus E_p (solid line, left scale) and Poisson's ratio ν_p (dashed line, right scale) for $D = 0{,}1$, $\xi l = 5, 10, 20$ (curves 1, 2, 3) decrease with parameter $k = a_e / a_{ee}$ change from 0,2 to 1,0 as demonstrated in Fig. 2.

With increase of parameter D the size effects become more evident as shown in Fig. 3 for $\xi l = 5, 10, 20$ (curves 1, 2, 3), $a_e / a_{ee} = 0{,}6$.

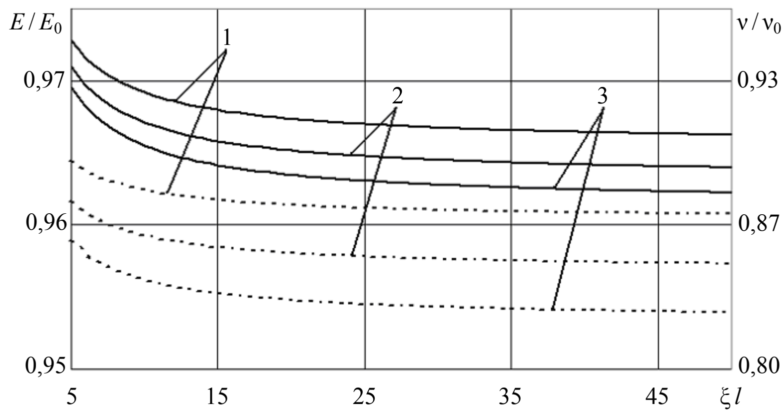

Fig. 1. Elasticity modules E, ν versus layer thickness

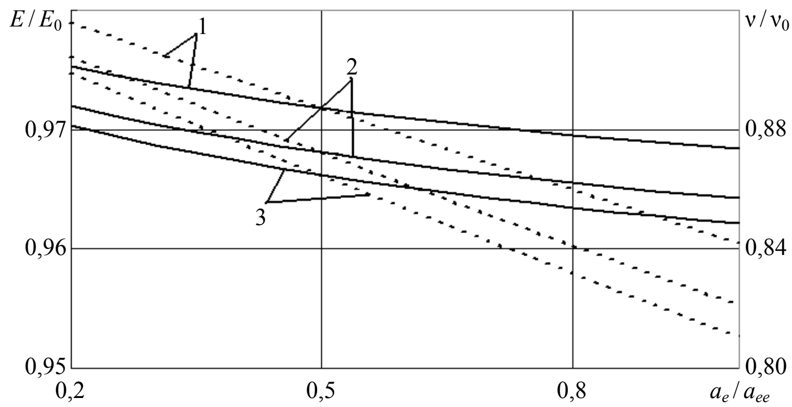

Fig. 2. Elasticity modules E, ν versus parameters a_e / a_{ee} ratio

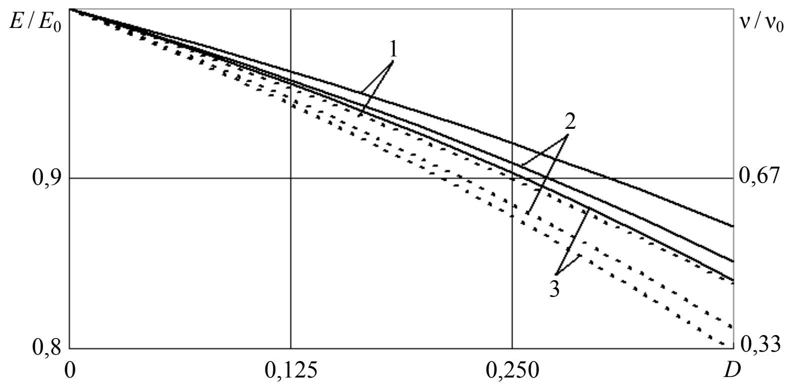

Fig. 3. Elasticity modules E, ν versus parameter D

Conclusions. Within the framework of local gradient approach in thermomechanics the dependence of the elasticity modules on the characteristic body size is investigated. The basic relations of local gradients of thermoelasticity are used to investigate the stationary state of stretched isotropic thermoelastic layer. Considering different stressed-strained states and common definition of the elasticity modules the Young's modulus and Poisson's ratio are examined. The isotropic quality of elasticity modules size effects is proved. It is shown that in the considered model the shear modulus does not depend on specific size (thickness) of the layer and the elasticity modules tend to constant value with the characteristic body size increase. The change of these parameters is considerable for the layers whose size is comparable to the characteristic size of nearsurface inhomogeneous region. The obtained relations may be used as a starting point in construction of the theory of nanomaterials mechanics.

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Про розмірний ефект механічних характеристик термопружних тіл

Тарас Нагірний, Костянтин Червінка

У даній роботі наведено основні співвідношення моделі термопружного тіла за локально градієнтного підходу у термомеханіці. У рамках такого підходу досліджено стаціонарний напружено-деформований стан розтягнутого ізотропного термопружного шару. На цій основі, використовуючи загальне означення модулів пружності, вивчено розмірні ефекти модуля Юнга, коефіцієнта Пуассона та модуля всестороннього стиску. Встановлено ізотропний характер розмірного ефекту модулів пружності. Показано, що у рамках розглядуваної моделі модуль зсуву не залежить від характерного розміру (товщини) шару. Результати числових досліджень подано у вигляді графіків.

О размерном эффекте механических характеристик термоупругих тел

Тарас Нагирный, Константин Червинка

В данной работе приведены основные соотношения модели термоупругого тела при локально градиентном подходе в термомеханике. В рамках такого подхода исследовано стационарное напряженно-деформированное состояние растянутого изотропного термоупругого слоя. На этой основе, используя общее определение модулей упругости, изучены размерные эффекты модуля Юнга, коэффициента Пуассона и модуля всестороннего сжатия. Установлен изотропный характер размерного эффекта модулей упругости. Показано, что в рамках рассматриваемой модели модуль сдвига не зависит от характерного размера (толщины) слоя. Результаты числовых исследований представлены в виде графиков.

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