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MEAN VALUE PROPERTIES FOR EIGENFUNCTIONS OF THE LAPLACIAN ON SYMMETRIC SPACES

Let $X = G/K$ be a symmetric space of the noncompact type with complex group G, L the Laplace-Beltrami operator on X. It is proved that eigenfunctions of L are characterized by vanishing of integrals over all balls in X with special radii.

A well-known theorem on ball means for the Helmholtz equation asserts that a function $f \in C(\mathbb{R}^n)$ is a solution of the equation $\Delta u + \lambda^2 u = 0$ if and only if for all $x \in \mathbb{R}^n$ and $r > 0$

$$\int\limits_{|u|\leqslant r} f(x+u)du = \left(\frac{2\pi r}{\lambda}\right)^{n/2} J_{n/2}(\lambda r) f(x),$$

where J_k is the kth-order Bessel function of the first kind. In particular, this implies that every solution of the equation

$$\Delta u + u = 0$$

has zero integrals over all balls in $\mathbb{R}^n$ with radii belonging to the zero set of $J_{n/2}$. In [1] (also see [2], Part II, Theorem 1.12), the first author proved the following.

Theorem A. *Let $f \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$, $n \geqslant 2$, and let $\{\nu_k\}_{k=1}^{\infty}$ be the sequence of all positive zeros of the function $J_{n/2}$. Then the integrals of f over all balls in $\mathbb{R}^n$ with radii $\nu_1, \nu_2, \ldots$ are equal to zero if and only if $\Delta f + f = 0$ (here the equality is understood in the sense of distributions).*

Analogues of Theorem A for rank one symmetric spaces were established by V.V.Volchkov in [3]. These results enabled him to get a description of the Pompeiu sets in terms of approximations of their indicator functions by linear combinations of indicator functions of balls with special radii in L^1 (see [1–3]). The purpose of this paper is to obtain an analogue of Theorem A for symmetric spaces of arbitrary rank.

As regards basic notions and facts from the theory of symmetric spaces, see [4]–[6]. Let $X = G/K$ be a symmetric space of the noncompact type, G being a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $o = \{K\}$ be the origin in X and denote the action of G on X by $(g, x) \to gx$ for $g \in G$, $x \in X$. The Lie algebras of G and K are respectively denoted by $\mathfrak{g}$ and $\mathfrak{k}$. The adjoint representations of $\mathfrak{g}$ and G are respectively denoted by ad and Ad. Let $\langle\ ,\ \rangle$ be the Killing form of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. The form $\langle\ ,\ \rangle$ induces a G-invariant Riemannian structure on X with the corresponding distance function $d(\cdot,\cdot)$ and the Riemannian measure dx. For $R \geqslant 0$, $y \in X$ we set

$$B_R(y) = \{x \in X : d(x, y) < R\}, \quad B_R = B_R(o).$$

Let $L^1(X)$ and $L^{1,\mathrm{loc}}(X)$ be the classes of complex-valued functions on X that are dx-integrable and locally integrable, respectively. The Laplace-Beltrami operator on X is denoted by L.

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$ and let $\mathfrak{a}\subset\mathfrak{p}$ be any maximal abelian subspace (all such subspaces have the same dimension). The dimension of $\mathfrak{a}$ is called the real rank of G and the rank of the space X. We shall write $\mathrm{rank}X=\dim\mathfrak{a}$.

Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$, $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}$ their respective complexifications. Next, let $A_\lambda\in\mathfrak{a}_{\mathbb{C}}$ be determined by $\langle H,A_\lambda\rangle=\lambda(H)$ $(H\in\mathfrak{a})$ and put

$$\langle\lambda,\mu\rangle=\langle A_\lambda,A_\mu\rangle,\quad \lambda,\,\mu\in\mathfrak{a}^*_{\mathbb{C}}.$$

For $\lambda\in\mathfrak{a}^*$ put $|\lambda|=\langle\lambda,\lambda\rangle^{1/2}$,

$$\mathfrak{g}_\lambda=\{U\in\mathfrak{g}:[H,U]=\lambda(H)U\quad\text{for}\quad\text{all}\quad H\in\mathfrak{a}\},$$

where $[\cdot,\cdot]$ is the bracket operation in $\mathfrak{g}$. If $\lambda\neq 0$ and $\mathfrak{g}_\lambda\neq\{0\}$ then λ is called a (restricted) root and $m_\lambda=\dim\mathfrak{g}_\lambda$ is called its multiplicity. The spaces $\mathfrak{g}_\lambda$ are called root subspaces. The set of restricted roots will be denoted by Σ. A point $H\in\mathfrak{a}$ is called regular if $\lambda(H)\neq 0$ for all $\lambda\in\Sigma$. The subset $\mathfrak{a}'\subset\mathfrak{a}$ of regular elements consists of the complement of finitely many hyperplanes, and its components are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+\subset\mathfrak{a}$. We call a root positive if it is positive on $\mathfrak{a}^+$. Let Σ^+ denote be the set of positive roots; for $\alpha\in\Sigma^+$ we will also use the notation $\alpha>0$ and put

$$\rho=\frac{1}{2}\sum_{\alpha>0}m_\alpha\alpha.$$

Let exp be the exponential mapping of $\mathfrak{g}$ into G. As usual we set $\mathrm{Exp}\,P=(\exp P)\,K\in X$ for each $P\in\mathfrak{p}$. Denote by J the corresponding Jacobian, i.e.,

$$\int_X f(x)dx=\int_{\mathfrak{p}} f(\mathrm{Exp}\,P)J(P)dP,\quad f\in L^1(X).$$

Define

$$d\mu(x)=(J(\mathrm{Exp}^{-1}x))^{-1/2}dx.$$

Let $\mathcal{D}'(X)$ (resp. $\mathcal{E}'(X)$) be the space of distributions (resp. distributions of compact support) on X, $\mathcal{E}'_\natural(X)$ the space of K-invariant compactly supported distributions on X, $\widetilde{T}$ the spherical transform of a distribution $T\in\mathcal{E}'_\natural(X)$. As in [7], denote by $\mathcal{E}'_{\natural\natural}(X)$ the set of all distributions $T\in\mathcal{E}'_\natural(X)$ with the following property: $T\in\mathcal{E}'_{\natural\natural}(X)$ if and only if there exists a function $\overset{\circ}{T}:[0,+\infty)\to\mathbb{C}$ such that

$$\widetilde{T}(\lambda)=\overset{\circ}{T}(|\lambda|)$$

for all $\lambda \in \mathfrak{a}^*$. It can be shown that for each $T \in \mathcal{E}'_{\natural\natural}(X)$ the function $\overset{\circ}{T}$ admits extension to $\mathbb{C}$ so that $\overset{\circ}{T}$ becomes an even entire function. For each $T \in \mathcal{E}'_{\natural\natural}(X)$ we set $\mathcal{Z}(\overset{\circ}{T}) = \{\zeta \in \mathbb{C} : \overset{\circ}{T}(\zeta) = 0\}$.

From the Paley-Wiener theorem for the spherical transform (see [5, Chap.4, Theorem 7.1]) it follows that the class $\mathcal{E}'_{\natural\natural}(X)$ is broad enough. We point out that

$$\mathcal{E}'_{\natural\natural}(X) = \mathcal{E}'_{\natural}(X) \quad \text{provided} \quad \operatorname{rank} X = 1.$$

Let $\times$ denotes the convolution on X. We recall that if $f \in \mathcal{D}'(X)$ and $T \in \mathcal{E}'(X)$ then

$$\langle f \times T, u\rangle = \left\langle T(g_2K), \left\langle f(g_1K), \int_K u(g_1kg_2K)dk \right\rangle\right\rangle, \quad u \in \mathcal{D}(X), \tag{1}$$

where $\mathcal{D}(X)$ is the space of complex-valued C^∞-functions of compact support on X. The following characterizations of the class $\mathcal{E}'_{\natural\natural}(X)$ were proved in [7], [8].

Theorem B. *If $T \in \mathcal{E}'_{\natural}(X)$ then the following assertions are equivalent.*

(i) $T \in \mathcal{E}'_{\natural\natural}(X)$.

(ii) *For each $\lambda \in \mathfrak{a}^*$ every solution $f \in C^\infty(X)$ of the equation*

$$Lf = -\left(|\lambda|^2 + |\rho|^2\right) f$$

satisfies the equality

$$f \times T = \widetilde{T}(\lambda) f.$$

Theorem C. *Let $X = G/K$ be a symmetric space of the noncompact type with complex group G, and let $T \in (\mathcal{E}'_{\natural} \cap L^1)(X)$. Then the following assertions are equivalent.*

(i) *T has the form*

$$T(x) = (J(\operatorname{Exp}^{-1}x))^{-1/2} u(d(o,x)), \quad x \in X,$$

for some function $u : [0, +\infty) \to \mathbb{C}$.

(ii) $T \in \mathcal{E}'_{\natural\natural}(X)$.

The main result of this paper is as follows.

Theorem 1. *Let $X = G/K$ be a symmetric space of the noncompact type with complex group G. Let $\{\lambda_q\}_{q=1}^\infty$ be the sequence of all positive zeros of $J_{l/2}$, where $l = \frac{1}{2}\dim X$, and let $c > 0$. Assume that $f \in L^{1,\text{loc}}(X)$ and $r_q = \lambda_q/c$, $q = 1, 2, \ldots$. Then the following items are equivalent.*

(i) *f satisfies the equation*

$$(L + |\rho|^2 + c^2) f = 0.$$

(ii) *For all* $g \in G$, $q \in \mathbb{N}$,

$$\int\limits_{B_{r_q}} f(gx)d\mu(x) = 0. \qquad (2)$$

Proof. Let $g \in G$ and let χ_q be the characteristic function of the ball B_{r_q}. Then

$$\chi_q(g^{-1}o) = \chi_q(go). \qquad (3)$$

Using Lemma 4.4 and Propositions 4.8 and 4.10 from [5, Chap. 4], one infers that

$$J^{-1/2}(\mathrm{Exp}^{-1}(go))\psi_\lambda(\mathrm{Exp}^{-1}(go)) = J^{-1/2}(\mathrm{Exp}^{-1}(g^{-1}o))\psi_{-\lambda}(\mathrm{Exp}^{-1}(g^{-1}o)),$$

where

$$\psi_\lambda(P) = \int\limits_K e^{i\langle A_\lambda, \mathrm{Ad}(k)P\rangle}dk, \quad P \in \mathfrak{p}.$$

Hence

$$J^{-1/2}(\mathrm{Exp}^{-1}(go)) = J^{-1/2}(\mathrm{Exp}^{-1}(g^{-1}o)). \qquad (4)$$

In view of (3), (4) and (1) relation (2) can be written as

$$f \times T_q = 0, \qquad (5)$$

where $T_q(x) = J^{-1/2}(\mathrm{Exp}^{-1}x)\chi_q(x)$, $x \in X$. The proof of Theorem C (see [7], [8]) shows that $T_q \in \mathcal{E}'_{\natural\natural}(X)$ and

$$\widetilde{T_q}(\lambda) = \gamma_q \mathbf{I}_{l/2}(r_q\sqrt{\langle\lambda,\lambda\rangle}), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

where $\mathbf{I}_{l/2}(z) = J_{l/2}(z)z^{-l/2}$ and the constant γ_q is independent of λ (see [2], Part I, Example 6.1). This together with Theorem B gives the implication (i)→(ii).

Let us prove that (ii)→(i). Define now $T_q \in \mathcal{E}'_{\natural\natural}(X)$ by the relation

$$\widetilde{T_q}(\lambda) = (\langle\lambda,\lambda\rangle - c^2)^{-1}\mathbf{I}_{l/2}\big(r_q\sqrt{\langle\lambda,\lambda\rangle}\big), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ q \in \mathbb{N}.$$

One therefore has $\bigcap_{q=1}^{\infty} \mathcal{Z}\big(\overset{\circ}{T}_q\big) = \varnothing$ (see [2], Part II, Lemma 1.29). In addition, equality

$$((L + |\rho|^2 + c^2)T_q)\widetilde{\ }(\lambda) = (c^2 - \langle\lambda,\lambda\rangle)\widetilde{T_q}(\lambda)$$

and the argument in the proof of the implication (i)→(ii) show that (ii) can be brought to the form

$$((L + |\rho|^2 + c^2)f) \times T_q = 0 \quad \text{for all} \quad q.$$

Appealing now to [7, Theorem 4.12] we arrive at the desired statement. □

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