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TWO-STREAM NONLINEAR LANGMUIR OSCILLATIONS

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Langmuir oscillations of a large amplitude in a cold collisionless plasma are considered in this work. We passed to the conclusion that multiple streams is an inevitable feature of the Langmuir oscillations in the cold collisionless inhomogeneous plasma with resonant point even in the low-signal limit. By extending the equation of the cold plasma oscillations on the multi-stream regimes, we find two-stream modes in the homogeneous plasma, which are stationary solutions of this equation.

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INTRODUCTION

An interest to the large-amplitude waves and oscillations in a cold plasma was aroused by the pioneer work [1] and has not faded to this day. The fine result of [1] is the nonlinear Langmuir oscillations are harmonic with stable frequency. It turned out that various factors, such as plasma inhomogeneity, ion movement, thermal and relativistic effects, etc., disturb this feature. Partially, the plasma oscillations driven in the inhomogeneous plasma with the resonance point are accompanied by “bursts” of electrons from the resonance region [2, 3], which turn the one-stream electron flow to multi-stream. The development of muti-stream flow, generally, breaks the one-stream resonance. It would be interesting to construct the simple example of a multi-stream resonance flow in an inhomogeneous plasma. In the paper, we present the two-stream flow in the homogeneous plasma.

1. FORCED OSCILLATIONS

Following [4], let us consider the situation, when the plasma electrons moves only along the spatial coordinate x, and the ions are stable but the ion density n_0 varies with x. First, we must separate the electric field **E** of the plasma electric charges and the electric field $\mathbf{E_0}$ of the outer source, which drives the oscillations. Newton’s second law for the electron movement is

$$m\frac{d\mathbf{v}}{dt} = -e\,(\mathbf{E}+\mathbf{E}_0)\,, \qquad (1)$$

where m is the electron mass and e is the elementary charge. Let us consider Ampere’s circuital law,

$$\text{rot }\mathbf{H} = \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E}+\mathbf{E}_0) + \frac{4\pi}{c}\mathbf{j}, \qquad (2)$$

where **H** is the magnetic field strength and **j** is the current density of the particles of the plasma. As the ions are stable, $\mathbf{j} = -en\mathbf{v}$, where n is the electron density. In 1D geometry equation (2) allows only the following separation:

$$\frac{\partial}{\partial t}E - 4\pi e n v = 0\,, \qquad (3)$$

$(\text{rot }\mathbf{H})_x = c^{-1}\partial_t E_0$. Because of $\mathbf{E}_0$ in is considered away from the outer charges, it follows, that div $\mathbf{E}_0$=0, and E_0 doesn’t depend on x. Next, we find the equation of movement of the electron. Excluding E from Newton’s law (1), equation (3) and Gauss’s law gives:

$$\frac{\partial E}{\partial x} = 4\pi e(n_0 - n)\,, \qquad (4)$$

where n_0 is the ion density. After differentiating (1) by t once again, we obtain the dE/dt in the right part. Let us expand it through substantial derivation:

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + v\frac{\partial E}{\partial x}\,. \qquad (5)$$

Inserting into (5) the expressions of partial derivatives $\partial E/\partial t$ from (3), and $\partial E/\partial x$ from (4), we obtain, that $d_t E = -4\pi e n_0$ and (1) passes to

$$\frac{d^2 v}{dt^2} + \omega_0^2(x)\,v = -\frac{e}{m}\frac{dE_0}{dt}\,, \qquad (6)$$

where ω_0 is the electron plasma frequency:

$$\omega_0^2 = \frac{4\pi e^2}{m}n_0(x)\,. \qquad (7)$$

Just due to one-streaming, n_0 in (7) is the ion density [5]. It follows from the (6), that free oscillations of the homogeneous plasma are harmonic for any velocity value, and the frequency of the oscillations does not depend on the magnitude [1].

2. MULTI-STREAM REGIMES

The model described above has in mind that the electron velocity is a one-valued function of the coordinate, and its applicability strongly depends on the magnitude of the oscillations. The one-valued dependence breaks when two electrons with different velocities pass through the certain point. Following this, one can find the condition, when the solution is single-streamed. Under the assumption that all the electrons oscillates in phase, the magnitude of the oscillation $x_m(x)$ should be smooth enough [5]:

$$\left|\frac{dx_m}{dx}\right| < 1 . \tag{8}$$

On the other hand, the spatial variation of the phase shift $\varphi(x)$ of the oscillations of the same magnitude and frequency must obey the condition

$$x_m \left|\frac{d\varphi}{dx}\right| < 1 . \tag{9}$$

Expressions (8), (9) can serve as a requirement that the solution is single-streamed for the monochromatic oscillations.

It is worth noting, that single-stream solutions do not exist infinitely long in the case of free Langmuir oscillations of the collisionless *inhomogeneous* plasma, when, roughly, the oscillation frequency varies from point to point. Indeed, for a frequency difference $\delta\omega$ as small as desired, there is the time *t*, when the phase shift $\varphi = \delta\omega \cdot t$ will not satisfy condition (9). Therefore, the homogeneous equation (6) cannot have single-stream solutions if $\omega_0(x) \neq \text{const}$. Otherwise, the forced oscillations can be single streaming, if any dissipation mechanism is provided, which reduces the general solution of the homogeneous equation.

The expression of the steady state value of the magnitude x_m of the linear oscillator with the main frequency ω_0 driven by the force of magnitude F_m and frequency ω reads as:

$$x_m = \frac{F_m}{\omega_0^2 - \omega^2} . \tag{10}$$

Extending of (10) to the case of a plasma with the density profile smooth enough, we come to the conclusion, that the forced oscillations are counterphased on the different sides of resonant point x_1, $\omega_0(x_1) = \omega$. Then, the forced Langmuir oscillations, even of small magnitude, are not single-stream near the resonant point x_1. Accounting for dissipation modifies this result.

Let us evaluate the distance $x_1 \pm \Delta x$, where single-streaming is breaking. Making the use of (8), (10), one can find that (SI)

$$\Delta x^2 \sim \frac{F_m}{d\omega_0^2 / dx} = \frac{\varepsilon_0 E_m}{e\, dn / dx} . \tag{11}$$

For example, in a plasma column of several centimeters in diameter, with $n \sim 10^8$ cm^{-3} and $E_m \sim 10$ V/cm, the size of Δx is about of 1 mm.

When there are N flows in the same point, characterized by the densities n_k and velocities v_k, the velocity of the j-th flow obeys to the equation:

$$\frac{d^2 v_j}{dt^2} + \omega_0'^2 v_j - \sum_{k=1}^{N} \omega_k^2 (v_j - v_k) = -\frac{e}{m}\frac{dE_0}{dt} , \tag{12}$$

where

$$\omega_0'^2 = \omega_0^2 + \frac{e}{m}\frac{\partial}{\partial x} E_a , \tag{13}$$

and $\omega_k^2 = 4\pi e^2 n_k / m$. To be thorough, the ambipolar electric field E_a is included in the ω'_0 (13). The physical sense of the equation (12) is illustrated by the Fig. 1.

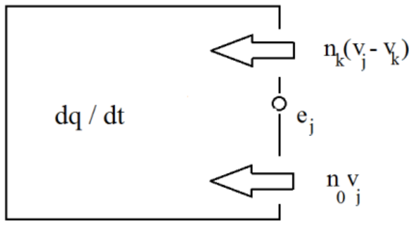

Fig. 1. Streams of ions (0) and electrons (k), passed by the electron of j-th stream in its rest frame. In the rest frame of ions its velocity is directed to the right and equals to v_j

The first term is the velocity of the changing of the acceleration of the electron. Second term is the velocity of changing of electrostatic force between the electron and the ion background. As the last is defined by the ions charge spaced, say, to the left of the electron, it changing is proportional to the ion flow $n_0 v_j$ passed in the rest frame of the electron.

Following the case of the opposite streams under the Langmuir oscillations, let us consider the simpler example of two colliding electron beams with the equal densities n_0 and velocities v_0. Let the beams initially are ion-compensated. If the beams meet in x=0, then, from the symmetry,

$$v_2(t,x) = -v_1(t, -x); \tag{14}$$

$$\omega_2(t,x) = \omega_1(t, -x). \tag{15}$$

In the dimensionless variables $\tau = \omega_0 t$, $\xi = x\omega_0 / v_0$, $\upsilon_k = v_k / v_0$, $\Omega_k = (\omega_k/\omega_0)^2$, with regard of the symmetry conditions, and dropping out the index in υ_1, Ω_1, the governing equation set (12), (14), (15) reads:

$$d_{\tau\tau}\upsilon + (1-\Omega)\upsilon - (\Omega\upsilon)\,|_{-\xi} = 0 , \tag{16}$$

$$\partial_\tau \Omega + \partial_\xi (\Omega\upsilon) = 0 , \tag{17}$$

$$\Omega(t = 0, \xi) = P(-\xi) , \tag{18}$$

$$\upsilon(t = 0, \xi) = 1 . \tag{19}$$

The equation (17) is the continuity equation for the beam electron density, and $P(\xi)$ is the Heaviside step function. In the Fig. 2 the footage of two periods of the simulation of the initial problem (16)-(19) is shown. After that, the number of streams doubles because of overturn of wave of velocity. The simulation shows that the beams quickly fly away from the point when the collision starts, and the region where the streams overlap is extending over a much greater distance, then the estimate (11). This fact is easy to understand because of the increasing negative charge in the overlap region. The direct simulation of the Langmuir resonance under the conditions of the linear inhomogeneity and the harmonic driving field E_0, performed by the PIC method, gives greatly complicated picture.

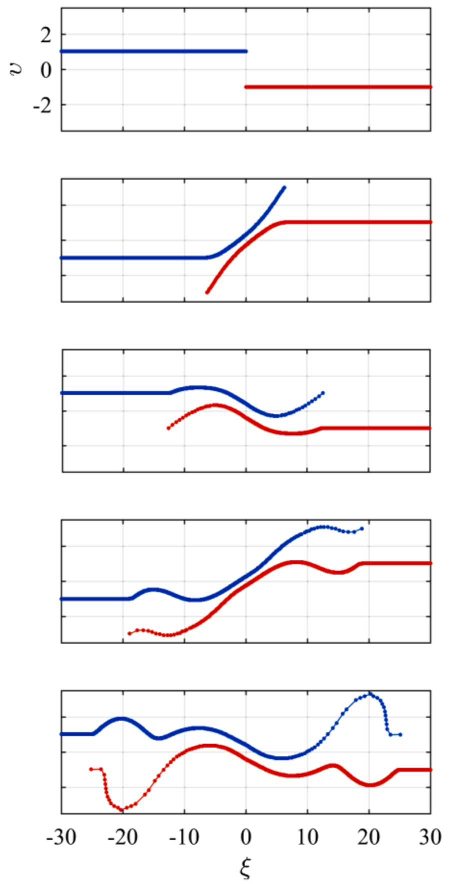

Fig. 2. The colliding beams velocity distributions. The time step is one-half of the plasma oscillation period

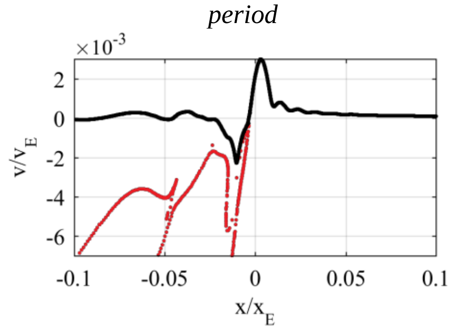

Fig. 3. The phase space of the large particles in the PIC simulation of the Langmuir resonance

An example is shown in the Fig. 3. It is seen the consequence of the beams scattering near the resonance $x/x_E=0$ (points in the interval $v/v_E \sim -2\cdot10^{-3}\ldots-7\cdot10^{-3}$). Apparently, these particles bursts were found at first numerically in [2].

3. STATIONARY FLOWS

Nevertheless, it turns out that there are simple two-stream solutions, which are, in fact, special cases of Langmuir oscillations. Let us consider two stationary opposite streams in the homogenous plasma, with densities $n_1(x) = n_2(x)$ and velocities $v_1(x) = -v_2(x) > 0$. Taking into account, that by the continuity equation $n_1v_1 = -n_2v_2 = \text{const}$, it follows from (12) that

$$\frac{d^2v_1}{dt^2} + \omega_0^2 v_1 = 2\omega_1^2 v_1 = j = \text{const}. \qquad (20)$$

Then, the dependence $v_1(x)$ is determined parametrically from the solutions of the equation (20):

$$v_1 = v_0 + v_m \cos(\omega_0 t + \phi_0), \qquad (21)$$

$$x = v_0 t + x_m \sin(\omega_0 t + \phi_0), \qquad (22)$$

where $v_0 = j/\omega_0^2$ and $x_m = v_m/\omega_0$. It follows from (21), (22) that the curve $v_1(x)$ is the positive part of the trochoid:

$$(v_1 - v_0)^2 + (\omega_0(x - v_0 t))^2 = v_m^2,\ v_1 \ge 0. \qquad (23)$$

From the (23) one can obtain the spatial period Λ and the temporal period T of the flow. Let us introduce the parameter η as follows:

$$\eta = \frac{v_0}{v_m} = \frac{j}{\omega_0^2 v_m}. \qquad (24)$$

If η≤1, then

$$\Lambda = 2x_m\left(\eta(\pi - \arccos(\eta)) + \sqrt{1-\eta^2}\right), \qquad (25)$$

$$T = 2(\pi - \arccos(\eta))/\omega_0. \qquad (26)$$

If η=1, phase curve of the stream pass to cycloid with $\Lambda=2\pi x_m$, $T=2\pi/\omega_0$. The meaning of T is the time for the electron to pass the distance Λ in the positive direction. If η>1, the streams have not any turning points. Fig. 4 shows the imaging of the flow in phase space for different values of the parameter η. If η≤1, the stream v_1 can be interpreted as short-circuited with the contrary stream v_2 at the turning points x_0, where $v(x_0)=0$. These two streams form the closed stationary flow located in space segment with size Λ. In the framework of the model, there are not reasons against solely existence of such formation in the unbounded plasma. Because of the nonzero dispersion Δv of the beams in a velocity space, the formations should to decay on the time scale $\sim\Delta v/(v_0\omega_0)$. The period of localized electron oscillations rises from $2\pi/\omega_0$ to $4\pi/\omega_0$, and their size Λ raises from $2v_m/\omega_0$ to $2\pi v_m/\omega_0$ as η changes from 0 to 1. When η>1, the electron velocity does not vanish in any point, what corresponds to two distinct flows.

The parameter η is also connected with the quotient of the number of electrons N=jT involved in the flow and the number of ions $N_0=n_0\Lambda$, contained in the segment Λ. If η<1 this quotient is equal to

$$\nu = \frac{N}{N_0} = \eta\frac{\pi - \arccos(\eta)}{\eta(\pi - \arccos(\eta)) + \sqrt{1-\eta^2}}, \qquad (27)$$

else

$$\nu = \eta. \qquad (28)$$

Due to the quasineutrality of plasma and the expression (28), ν must be equal to unity, so, only the cycloidal mode of oscillations, characterized by η=1, survives (see Fig. 4). Nevertheless, let us consider the situation when the plasma within the layer occupied by the flow has some unbalanced charge. In accordance with (27), the curves for η<1 in the Fig. 4 imply a positive charge. A negative charge corresponds to the unbounded streams (ν=η>1). If η→0, the movement of electrons occurs in the layer of size $\Lambda\to2x_m$ and tends to the sinusoidal with the frequency ω_0. Because of the number of the electrons is sufficiently less than number of ions in the layer, the movement of the electrons is mainly due to the ions electric field. As the density n_0=const, the field is linear and the oscillations are harmonic.

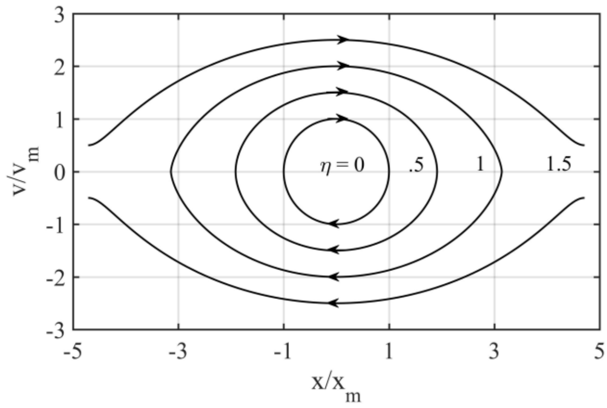

Fig. 4. Stationary flows in the phase space for different values of η (24)

The self-consistent potential does n ot tends to the infinity in the turning points whereas the beam densitydoes. The Fig. 5 illustrates spatial distributions of the dimensionless self-consistent potential $\psi = 2e\varphi/(mv_m^2)$ and dimensionless electron density $\Omega^2 = 2(\omega_1/\omega_0)^2$. The reference point of the potential is at the point where the volume charge density vanishes.

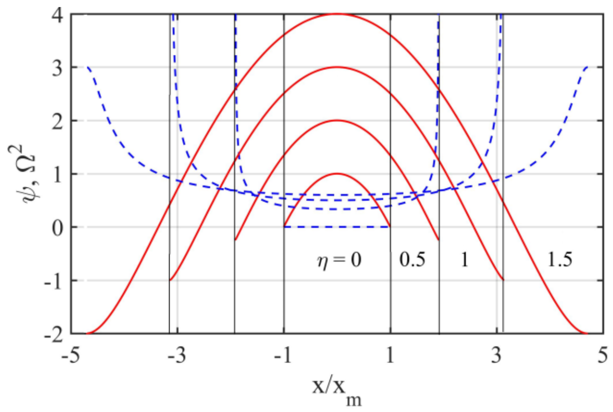

Fig. 5. Dependencies of ψ (----) and Ω^2 (——) on normalized coordinate for different values of η. Turning points are marked by solid vertical lines

As the ions also move in the same potential as electrons, the formations cannot be stable and tend to expand. The lifetime is less than the period of the electron motion by the electron to the ion mass ratio.

CONCLUSIONS

The considered examples shows, that multiple streams is the inevitable feature of the Langmuir oscillations in the cold collisionless inhomogeneous plasma with resonant point, even if the magnitude is small. By extending the equation of the cold plasma oscillations on the multi-stream regime, the two-stream modes in the homogeneous plasma were found. Their periods do not depend on the magnitude, as in the case of one-stream Langmuir oscillations, but the electron movement is not sinusoidal.

The dipole momentum of the layer containing the flow does not oscillate because of charge distribution is symmetric, in contrast to the one-stream Langmuir oscillations. Generally, the modes imply an uncompensated average plasma charge, and even if this is not the case, they tend to diffuse due to the acceleration of ions in the self-consistent potential.

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ДВОПОТОЧНІ НЕЛІНІЙНІ ЛЕНГМЮРІВСЬКІ КОЛИВАННЯ

М.О. Азарєнков, О.В. Гапон

Розглядаються ленгмюрівські коливання великої амплітуди в холодній беззіткнівній плазмі. Ми дійшли висновку, що множинні потоки є неминучою особливістю ленгмюрівських коливань у холодній беззіткнівній неоднорідній плазмі з резонансною точкою навіть за низьких амплітуд. Поширюючи рівняння коливань холодної плазми на багатопотокові режими, ми знаходимо двопотокові моди в однорідній плазмі, які є стаціонарними розв’язками цього рівняння.