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NON-AXISYMMETRIC NEOCLASSICAL TRANSPORT FROM MIS-ALIGNMENT OF EQUIPOTENTIAL AND MAGNETIC SURFACES

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Coefficients of neoclassical transport in tokamaks and stellarators with non-axisymmetric equipotential surfaces mis-aligned with magnetic flux surfaces are derived for the $1/\nu$ regime. In the general case, they are given by integral expressions including field line integrals similar to those defining the effective helical ripple [V.V. Nemov et al. // Phys. Plasmas. 1999, v. 6, p. 4622]. For small mis-alignments in a tokamak with a simplified geometry, they are reduced to simple analytical expressions.

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INTRODUCTION

In the standard neoclassical theory, the equilibrium electrostatic potential is assumed to be constant within magnetic flux surfaces, i.e. its equipotential surfaces are aligned with magnetic surfaces. An exception is for strong plasma rotation in tokamaks where the centrifugal force induces an axisymmetric miss-alignment – a dependence of the potential on the poloidal angle [1]. In stellarators, a small non-axisymmetric potential miss-alignment always exists due to finite ion orbit widths, which may affect neoclassical transport of high-Z impurities (see, e.g. [2, 3]) but produces only a correction for the transport of bulk plasma components. In tokamaks with external resonant magnetic perturbations (RMPs), non-axisymmetric miss-alignment of equipotential surfaces and magnetic surfaces causes in resonant layers around rational magnetic surfaces a significant radial transport of electrons which may exceed the anomalous transport [4]. In Ref. [4], this transport has been obtained in a quasilinear limit for the straight cylinder tokamak model ignoring the toroid city and, thus, also ignoring the effects associated with particle trapping.

In the present work, we derive neoclassical transport coefficients for general toroidal magnetic and electrostatic field geometry staying, however, limited to a specific transport regime – the $1/\nu$ regime – valid for low plasma collisionality and low banana precession frequency, which should not exceed the collisional detrapping frequency. Normally, this is fulfilled for the electron component being the main driver for particle transport in tokamaks because the ion particle flux adjusts itself to the electron flux by viscous momentum transport so that the overall flux is ambipolar. Combined with the studies of mechanisms leading to the mis-alignment of equipotential surfaces, the result will be used for investigations of the increased particle transport caused by RMPs known as "density pump-out".

The derivation here is based on the results of Ref. [5], where the effective helical ripple modulation has been derived. Those results are generalized here for the case where equipotential surfaces are not necessarily aligned with magnetic surfaces. This causes some re-definitions but generally leaves the main steps of Ref. [5] intact.

1. BASIC EQUATIONS

We start with the stationary drift-kinetic equation

$$\boldsymbol{v}_g \cdot \nabla f = \hat{L}_c f \qquad (1)$$

in guiding center variables $\boldsymbol{z} = (\boldsymbol{R}, J_\perp, \phi, H)$, where $\boldsymbol{R}$ is the guiding center position, $J_\perp \approx m_\alpha v_\perp^2/(2\omega_{c\alpha})$ is the perpendicular adiabatic invariant containing the species-specific mass m_α, the perpendicular particle velocity $v_\perp$ and the species-specific cyclotron frequency $\omega_{c\alpha}$. Further,ϕ is the gyrophase and $H = m_\alpha v^2/2 + e_\alpha \Phi$ is the total energy containing the electric charge e_α. Here, $\hat{L}_c$ is the linearized Coulomb collision operator, and the guiding center velocity is given by

$$\boldsymbol{v}_g = \frac{v_\parallel \boldsymbol{B}^\star}{B_\parallel^\star}, \qquad (2)$$

$$\boldsymbol{B}^\star = \nabla \times \left(\boldsymbol{A} + \frac{v_\parallel}{\omega_{c\alpha}} \boldsymbol{B}\right), \quad B_\parallel^\star = \frac{\boldsymbol{B}^\star \cdot \boldsymbol{B}}{B}, \qquad (3)$$

where the curl is computed assuming that

$$v_\parallel(\boldsymbol{R}, J_\perp, H) = \sigma\left(\frac{2}{m_\alpha}\left(H - e_\alpha \Phi(\boldsymbol{R}) - J_\perp \omega_{c\alpha}(\boldsymbol{R})\right)\right), \sigma = \pm 1, \qquad (4)$$

i.e. to be a function of the coordinates $\boldsymbol{R}$ and constant invariants of motion $J_\perp$ and H. The Jacobian of the phase space coordinates $\boldsymbol{z}$ is

$$J = \left|\frac{\partial(\boldsymbol{r},\boldsymbol{p})}{\partial(\boldsymbol{z})}\right| = \frac{e_\alpha B_\parallel^\star}{c|v_\parallel|}. \quad (5)$$

Further, the flux surface averaged normal particle flux density is defined as

$$\Gamma = \frac{1}{S}\int \mathrm{d}\boldsymbol{S}\cdot\int d^3\,p\boldsymbol{v}_g f, \quad (6)$$

where f is the distribution function and S is the flux surface area. For any distribution function, which, within the flux surface, depends only on integrals of motion but not on the coordinates (in particular, for a local Boltzmann distribution function), it can be shown that there is no flux surface averaged radial particle or total energy flux. This can be checked directly for the surface $r(\boldsymbol{R}) = r_0$ as follows

$$\Gamma = \frac{2\pi}{S}\int_{\mathrm{r=r_0}}\mathrm{d}\boldsymbol{S}\times$$
$$\times\sum_{\sigma=\pm1}\int_{\mathrm{e_\alpha\Phi}}^{\infty} dH\int_0^{(H-e_\alpha\Phi)/\omega_{c\alpha}} dJ_\perp J\boldsymbol{v}_g f =$$
$$= \frac{2\pi e_\alpha}{cS}\sum_{\sigma=\pm1}\int_{[e_\alpha\Phi]_{\min}}^{\infty} dH\int_0^{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\max}} dJ_\perp\times$$
$$\times\int_{r=r_0,v_\parallel^2>0} d\boldsymbol{S}\cdot f\,\nabla\times\left(\frac{|v_\parallel|}{\omega_{c\alpha}}\boldsymbol{B}\right) = 0, \quad (7)$$

where the surface integral of the curl is zero for both, passing particles, which occupy the whole flux surface area, and for trapped particles existing in the regions $v_\parallel^2 > 0$. In these regions, the integral is reduced via the Stokes theorem to the region boundaries where the sub-integrand is zero due to $v_\parallel = 0$. Similarly, one can check that the total energy flux, which contains an extra factor H in the sub-integrand, is also zero.

Particle flux density (6) can be written via the species-specific fluid velocity $\boldsymbol{V}_\alpha$ as

$$\Gamma = \frac{1}{S}\int dS\,n_\alpha\boldsymbol{V}_\alpha\cdot\nabla r = \frac{\langle n_\alpha\boldsymbol{V}_\alpha\cdot\nabla r\rangle}{\langle|\nabla r|\rangle}, \quad (8)$$

where the neoclassical flux surface average is defined in flux variables $\boldsymbol{x} = (r,\vartheta,\varphi)$ as

$$\langle a\rangle = \left(\int_{-\pi}^{\pi} d\vartheta\int_{-\pi}^{\pi} d\varphi\sqrt{g}\right)^{-1}\int_{-\pi}^{\pi} d\vartheta\int_{-\pi}^{\pi} d\varphi\sqrt{g}\,a. \quad (9)$$

Using Newcomb's theorem, this expression can also be written in terms of field line integrals in field aligned variables $\boldsymbol{x}_0 = (r,\vartheta_0,\varphi_0)$ as

$$\langle a\rangle = \lim_{L_\vartheta\to\infty}\left(\int_0^{L_\vartheta}\frac{d\vartheta_0}{B^\vartheta}\right)^{-1}\int_0^{L_\vartheta}\frac{dL_\vartheta}{B^\vartheta}a, \quad (10)$$

where these variables are defined via the safety factor q as

$$\vartheta_0 = \vartheta, \qquad \varphi_0 = \varphi - q\vartheta \quad (11)$$

such that φ_0 labels the field lines. These variables have the same $\sqrt{g}$, B_φ and B^ϑ as the usual ones, while $B^{\varphi_0} = 0$. Since $\nabla\cdot\boldsymbol{B} = 0$, the product $\sqrt{g}B^\vartheta$ is constant on flux surfaces so that $\sqrt{g}$ can be replaced by $1/B^\vartheta$ in (9), thus arriving at (10).

2. LMFP PARTICLE FLUX DENSITY

Within the local neoclassical ansatz, the drift kinetic equation is solved by a series expansion over the radial drift velocity up to linear order using the ansatz $f = f_0 + \delta f$. The equilibrium distribution function is given by a local Boltzmann distribution

$$f_0(r,H) = \frac{\bar{n}_\alpha(r)}{(2\pi m_\alpha T_\alpha(r))}\exp\left(\frac{e_\alpha\bar{\Phi}(r)-H}{T_\alpha(r)}\right), \quad (12)$$

where the parameter $\bar{\Phi}$ is chosen so that the neoclassically averaged density $\bar{n}_\alpha$ is the same as the density computed from (12). By the above arguments, the fluxes are solely produced by the perturbation $\delta f = \delta f(r,\vartheta_0,\varphi_0,H,J_\perp)$. In the long mean free path (LMFP) regime, which is of interest here, the fluxes are produced by the leading order of δf in the collision frequency (its bounce averaged part) which is constant along the field lines, $\delta f = \delta f(r,\varphi_0,H,J_\perp)$. Obviously, in the passing particle domain, the leading order δf is independent of φ_0 as well and contributes to the fluxes only in the trapped particle domain. Expressing the particle flux density (7) via field line integrals (10), where the field aligned coordinates $\boldsymbol{x}_0$ are constructed from Boozer coordinates with a re-defined flux surface label such that $\langle|\nabla r|\rangle = 1$, and performing similar transformations to those in Ref.[5] this contribution is in the general form

$$\Gamma = -4\pi\int_{[e_\alpha\Phi]_{\min}}^{\infty} dH\int_{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\min}}^{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\max}} dJ_\perp$$
$$\times\lim_{L_\vartheta\to\infty}\left(\int_0^{L_\vartheta}\mathrm{d}\vartheta_0\sqrt{g}\right)^{-1}\sum_{k=1}^{K}\frac{\partial\delta f_k}{\partial J_\perp}\frac{\partial H_k}{\partial\varphi_0}, \quad (13)$$

where
$$H_k = \frac{m_\alpha^3 c}{3e_\alpha}\int_{\vartheta_k^-}^{\vartheta_k^+} d\vartheta_0\frac{|v_\parallel|^3}{B^\vartheta}. \quad (14)$$

Here, the integration is along the field line $0 < \vartheta_0 < L_\vartheta$, $\varphi_0 = \mathrm{const}$, and the index k enumerates the segments of this line where $v_\parallel^2 > 0$, with $\vartheta_k^\pm$ being the segment ends or turning points. Equation (13) does not assume that magnetic field and electrostatic potential modulations within the flux surface are small. It allows various classes of trapped particles including particles blocked by non-axisymmetric perturbations of both, the magnetic field and the electrostatic potential, which is most easily achieved even for small perturbations in a tokamak with perturbed toroidal symmetry near the extrema of the main axisymmetric magnetic field. If we ignore in those tokamaks the contributions of blocked particles and consider only the bounce averaged radial drift of usual bananas dominant at small perturbation amplitudes, eq. (13) simplifies to

$$\Gamma = -4\pi\int_{[e_\alpha\Phi]_{\min}}^{\infty} dH\int_{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\min}}^{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\max}} dJ_\perp\times$$
$$\times\left(\int_{-\pi}^{\pi}\mathrm{d}\vartheta_0\sqrt{g_0}\right)^{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0\frac{\partial\delta f_k}{\partial J_\perp}\frac{\partial H_k}{\partial\varphi_0}, \quad (15)$$

where $k = 1$ denotes the main class of banana-trapped particles and g_0 corresponds to the unperturbed magnetic field. The heat flux density is obtained similarly by adding the average kinetic energy $H - e_\alpha\bar{\Phi}$ in the sub-integrand. Note that the above expressions are valid for the general case of the LMFP regime, not necessarily the $1/\nu$ regime.

3. SOLUTION OF THE BOUNCE AVERAGED EQUATION

In the $1/\nu$ regime, the perturbation δf satisfies the linear bounce averaged equation. Using the Lorentz model for the collision integral,

$$\hat{L}_c\delta f = 2m_\alpha \nu_d v_\parallel \frac{\partial}{\partial J_\perp}\frac{J_\perp v_\parallel}{\omega_{c\alpha}}\frac{\partial \delta f}{\partial J_\perp}, \tag{16}$$

where ν_d is the deflection frequency, this equation is

$$2m_\alpha^2\nu_d\sqrt{g}B^\vartheta \frac{\partial}{\partial J_\perp}J_\perp \mathrm{I_k}\frac{\partial \delta f_k}{\partial J_\perp} = \frac{\partial}{\partial J_\perp}\frac{\partial H_k}{\partial \varphi_0}\frac{\partial \mathrm{f}_0}{\partial r},$$
$$I_k = \int_{\vartheta_k^-}^{\vartheta_k^+} d\vartheta_0 \frac{|v_\parallel|}{B^\vartheta}. \tag{17}$$

It corresponds to a particular class of trapped particles with a number of classes tending to infinity with a field line length L_ϑ. Neighbouring classes bounded by the same boundary layer $J_\perp = \text{const}$, where simultaneously $v_\parallel^2 = 0$, and which have a minimum along the field line (for the aligned equipotential surfaces this corresponds to local maxima of B) are coupled together via boundary conditions (detailed discussion can be found in Ref. [5]). As shown in [5], all those boundary conditions are satisfied by the following first integral of the kinetic equation over $J_\perp$,

$$\frac{\partial \delta f_k}{\partial J_\perp} = \frac{1}{2m_\alpha^2\nu_d\sqrt{g}B^\vartheta J_\perp I_k}\frac{\partial H_k}{\partial \varphi_0}\frac{\partial \mathrm{f}_0}{\partial r}. \tag{18}$$

Here, the derivative of the local Boltzmann distribution (12) is given as

$$\frac{\partial f_0}{\partial r} = \left(A_1 + \frac{H - e_\alpha\bar{\Phi}}{T_\alpha}A_2\right)f_0, \tag{19}$$

where the thermodynamic forces are

$$A_1 = \frac{1}{\bar{n}_\alpha}\frac{\partial \bar{n}_\alpha}{\partial r} + \frac{e_\alpha}{T_\alpha}\frac{\partial \bar{\Phi}}{\partial r} - \frac{3}{2T_\alpha}\frac{\partial T_\alpha}{\partial r},$$
$$A_2 = \frac{1}{T_\alpha}\frac{\partial T_\alpha}{\partial r}. \tag{20}$$

Thus, particle flux (13) for the general case takes the form

$$\Gamma = -\frac{2\pi}{m_\alpha^2\sqrt{g}B^\vartheta}\int_{[e_\alpha\Phi]_{min}}^{\infty} dH \frac{f_0}{\nu_d}\times$$
$$\times\left(A_1 + \frac{H - e_\alpha\bar{\Phi}}{T_\alpha}A_2\right)\int_{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\min}}^{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\max}}\frac{dJ_\perp}{J_\perp}\times$$
$$\times \lim_{L_\vartheta\to\infty}\left(\int_0^{L_\vartheta} \mathrm{d}\vartheta_0\sqrt{g}\right)^{-1}\sum_{k=1}^{K}\frac{1}{\mathrm{I_k}}\left(\frac{\partial H_k}{\partial \varphi_0}\right)^2. \tag{21}$$

This expression generalises the formula for the $1/\nu$ particle flux density of Ref. [5] for the case of non-aligned equipotential surfaces and reduces to that formula in case of alignment. In the latter case, the dependence of the quantities H_k and I_k on the kinetic energy can be factorized, which allows to factorize the integral over energy. Thus, all geometrical information is contained in the integral over the normalized perpendicular invariant $J_\perp/v^2$ which can be expressed in terms of the factor $\varepsilon_{\mathrm{eff}}^{3/2}$ which formally defines effective helical ripple modulation $\varepsilon_{\mathrm{eff}}$.

Below, we are interested in the case of weak potential perturbations in a tokamak, where one can ignore additional trapped particle classes and where the energy integral can be factorized as well. Expression (15) for this case gives

$$\Gamma = -\frac{2\pi}{m_\alpha^2\sqrt{g}B^\vartheta}\left(\int_{-\pi}^{\pi}d\vartheta_0\sqrt{g_0}\right)^{-1}\int_{[e_\alpha\Phi]_{\min}}^{\infty}dH\times$$
$$\times\left(A_1 + \frac{H - e_\alpha\bar{\Phi}}{T_\alpha}A_2\right)\frac{f_0}{\nu_d}\times$$
$$\times\int_{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\min}}^{[(H-e_\alpha\Phi)/\omega_{c\alpha}]_{\max}}\frac{dJ_\perp}{J_\perp}\frac{1}{2\pi}\int_{-\pi}^{\pi}d\varphi_0\frac{1}{\mathrm{I_k}}\left(\frac{\partial H_k}{\partial \varphi_0}\right)^2. \tag{22}$$

4. SMALL ELECTROSTATIC PERTURBATIONS AND AXISYMMETRIC MAGNETIC FIELD

In the case of small electrostatic perturbations and an axisymmetric magnetic field, we can simplify

$$\frac{\partial H_k}{\partial \varphi_0} \approx -m_\alpha^2 c\, v\int_{\vartheta_k^-}^{\vartheta_k^+}\frac{d\vartheta_0}{B^\vartheta}(1-\eta B)^{1/2}\frac{\partial \Phi}{\partial \varphi_0}, \tag{22}$$

$$I_k \approx v\int_{\vartheta_k^-}^{\vartheta_k^+}\frac{d\vartheta_0}{B^\vartheta}(1-\eta B)^{1/2}, \tag{23}$$

where v is the particle velocity module and$\eta = 2e_\alpha J_\perp/(m_\alpha^2 c v^2) = v_\perp^2/(v^2 B)$. Presenting the potential in the form

$$\Phi(r,\vartheta,\varphi) = \mathrm{Re}\sum_{\mathrm{n}=0}^{\infty}\Phi_\mathrm{n}(\mathrm{r},\vartheta)\mathrm{e}^{\mathrm{in}\varphi} \tag{24}$$

and computing the integral over φ_0, equation (21) takes the form

$$\Gamma = -\frac{\bar{n}_\alpha c^2}{2\sqrt{\pi}\sqrt{g}B^\vartheta}\left(\int_{-\pi}^{\pi}d\vartheta_0\sqrt{g_0}\right)^{-1}\int_0^\infty dz\times$$
$$\times\sqrt{z}(A_1 + zA_2)\frac{e^{-z}}{\nu_d}\int_{1/B_{\max}}^{1/B_{\min}}\frac{d\eta}{\eta}\times$$
$$\times\left(\int_{\vartheta_k^-}^{\vartheta_k^+}\frac{d\vartheta_0}{B^\vartheta}(1-\eta B)^{1/2}\right)^{-1}\sum_{n=0}^{\infty}n^2\times$$
$$\times\left|\int_{\vartheta_k^-}^{\vartheta_k^+}\frac{d\vartheta_0}{B^\vartheta}(1-\eta B)^{1/2}\,\Phi_n e^{inq\vartheta}\right|^2, \tag{25}$$

where $z = m_\alpha^2 v^2/(2T_\alpha)$. Note that the normalized heat flux Q/T_α is given by expression , (25) with an extra factor z in the sub-integrand.

Thus, we obtain a matrix of Onsager-symmetric transport coefficients D_{jk} determined via thermodynamic forces as

$$\Gamma = -\bar{n}_\alpha(D_{11}A_1 + D_{12}A_2),$$
$$Q = -\bar{n}_\alpha T_\alpha(D_{21}A_1 + D_{22}A_2). \qquad (26)$$

If we approximate the collision as $\nu_d \approx \nu_{dT} z^{-3/2}$, where ν_{dT} is the collision frequency for particles with energy T_α, then transport coefficients are related by

$$D_{12} = D_{21} = 3D_{11}, \quad D_{22} = 12D_{11}. \qquad (27)$$

The simplest expression for D_{11} is obtained for near-resonant potential perturbations, $\Phi = \Phi_n(r)\,cos(m\vartheta + n\varphi)$ with $|m + nq| \ll 1$, in a large aspect ratio circular tokamak with $\sqrt{g_0} \approx rR_0$, and $B \approx B_0(1 - \varepsilon_t\,cos(\vartheta))$, where R_0 and B_0 are values at the magnetic axis and $\varepsilon_t = r/R_0$. In this case, we arrive at

$$D_{11} \approx \frac{16\sqrt{2}}{9\pi^{3/2}}\left(\frac{cm|\Phi_n|}{rB_0}\right)^2 \frac{\varepsilon_t^{3/2}}{\nu_{dT}}.$$

CONCLUSIONS

We have derived an integral formula for non-axisymmetric neoclassical transport fluxes in the $1/\nu$ regime in general toroidal devices with non-aligned equipotential and magnetic surfaces. The formula considers all possible trapped particle classes allowing for both, magnetic and electrostatic trapping. In the case of aligned surfaces, it reduces to the result of Ref. [5] defining the effective helical ripple. The formula can be evaluated using the field line integration technique similar to the one realized in the code NEO [5].

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АКСІАЛЬНО-НЕСИМЕТРИЧНИЙ НЕОКЛАСИЧНИЙ ПЕРЕНОС ВНАСЛІДОК НЕСПІВПАДАННЯ ЕКВІПОТЕНЦІАЛЬНИХ ТА МАГНІТНИХ ПОВЕРХОНЬ

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Виведено коефіцієнти неокласичного переносу в токамаках та стелараторах з неспівпадаючими еквіпотенціальними та магнітними поверхнями для режиму $1/\nu$. У загальному випадку вони даються інтегральним виразом з інтегралами вздовж силових ліній, які подібні до тих, що визначають ефективну гвинтову модуляцію в роботі [V.V. Nemov et al. // Physics of Plasmas. 1999, v. 6, p. 4622]. Для малих неспівпадінь у токамаку зі спрощеною геометрією вони зводяться до простих аналітичних виразів.