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MODIFICATION OF THE COUPLED INTEGRAL EQUATIONS METHOD FOR CALCULATION OF THE ACCELERATING STRUCTURE CHARACTERISTICS

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In this paper we present modification of coupled integral equations method (CIEM) for calculating the characteristics of the accelerating structures. In earlier developed CIEM schemes the coupled integral equations are derived for the unknown electrical fields at interfaces that divide the adjacent volumes. In addition to the standard division of the structured waveguide by interfaces between the adjacent cells, we propose to introduce new interfaces in places where electric field has the simplest transverse structure. Moreover, the system of coupled integral equations is formulated for longitudinal electrical fields in contrast to the standard approach where the transverse electrical fields are unknowns. The final vector equations contain expansion coefficients of the longitudinal electric field at these additional interfaces. This modification makes it possible to deal with a physical quantity that plays an important role in the acceleration of particles (a longitudinal electric field), and to obtain approximate equations for the case of a slow change in the waveguide parameters.

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INTRODUCTION

The main characteristic of the slow-wave accelerating structures is the distribution of the electric field in both steady state and transient modes. This imposes certain restrictions on the methods of calculating their characteristics, manufacturing and tuning. The slow-wave accelerating structures mainly belong to the class of structured waveguides[1] – waveguides that consist of similar, but not always identical, cells (disk-loaded waveguides (DLW), chains of coupled resonators, etc.).

One of the effective approaches for calculating the characteristics of structured waveguides is the coupled integral equations method (CIEM) [1 - 5].

Based on a system of coupled integral equations, an approximate method [6] is constructed for calculating the characteristics of structured waveguides with slowly varying dimensions [7]. It is the analog of classical Eikonal and WKB methods with taking into account not only propagating waves, but also evanescent ones. The advantage of this approach is the simple physical (but not simple mathematical) interpretation of obtained equations and their solutions. This approximate method was used to study the characteristics of the simplest case of structured waveguide – a DLW with very thin diaphragms [6, 7].

Analysis of the standard method of coupled integral equations for studying the characteristics of DLWs with real geometry showed that some modifications of the standard approach can be useful.

In this paper we present such modification of coupled integral equations method for calculating the characteristics of the accelerating structures. In earlier developed CIEM schemes the coupled integral equations are derived for the unknown electrical fields at interfaces that divide the adjacent volumes. Usually these interfaces include geometrical singularities, such as sharp edges. In this case it is needed to use special basis functions.

In addition to the standard division of the structured waveguide by interfaces between the adjacent cells, we propose to introduce new interfaces in places where electric field has the simplest transverse structure. Moreover, the system of coupled integral equations is formulated for longitudinal electrical fields in contrast to the standard approach where the transverse electrical fields are unknowns. The final vector equations contain expansion coefficients of the longitudinal electric field at these additional interfaces. This modification makes it possible to deal with a physical quantity (longitudinal electric field), which plays an important role in tuning accelerator structures and particle acceleration, and to obtain approximate equations for the case of a slow change in the waveguide parameters

1. ACCELERATING STRUCTURE MODEL. BASIC EQUATIONS

Consider a segment of DLW (circular corrugated waveguide), the geometry of which is shown in Figure. The right and left ends of segment are connected to semi-infinite circular waveguides. All segment volumes are filled with dielectric ($\varepsilon = \varepsilon' + i\varepsilon''$, $\varepsilon'' > 0$). We divide the DLW into subregions each of which is a circular waveguide. Unlike earlier works [1 - 3], we divide each volume with large cross-section into two equal subvolumes (in general, they can be different). Volumes with large cross section will be numbered by the index k ($1 \le k \le N_{REZ}$), subvolumes – by k_1 and k_2 ($k_1 = k_2 = k$). A small cross-sectional volume placed to the left of a large cross-sectional volume with an index k, will be numbered by the index k' ($1' \le k' \le (N_{REZ}+1)'$).

[1] Accelerating structures on the base of waveguides with dielectric can be smooth

We will consider only axially symmetric fields with E_z, E_r, H_φ components (TM). Time dependence is $\exp(-i\omega t)$. Since we are interested in considering accelerating structures, we must remember that it will be necessary to take into account the beam loading. Therefore, we will use initial expansions that are slightly different from the standard CIEM approach and give the possibility to include current into consideration. In each cylindrical volume (with index q) we expand the electromagnetic field electromagnetic field in terms of the complete orthogonal set of transverse functions

$$
\begin{aligned}
E_z^{(q)}\left(r, z_q+\tilde{z}\right) &= \sum_m E_{z,m}^{(q)}\left(\tilde{z}\right) J_0\left(\frac{\lambda_m}{b_q} r\right), \\
E_r^{(q)}\left(r, z_q+\tilde{z}\right) &= \sum_m E_{r,m}^{(q)}\left(\tilde{z}\right) J_1\left(\frac{\lambda_m}{b_q} r\right), \\
H_\varphi^{(q)}\left(r, z_q+\tilde{z}\right) &= \sum_m H_{\varphi.m}^{(q)}\left(\tilde{z}\right) J_1\left(\frac{\lambda_m}{b_q} r\right),
\end{aligned}
\tag{1}
$$

where $0 \le \tilde{z} \le d_q$, $\operatorname{Im}\gamma_m^{(q)} > 0$, $\operatorname{Re}\gamma_m^{(q)} < 0$, $\gamma_{-m}^{(q)} = -\gamma_m^{(q)}$, $J_0(\lambda_m) = 0$.

From Maxwell equations we obtain

$$
\begin{aligned}
&\frac{d^2 E_{r,m}^{(q)}}{d\tilde{z}^2} - \gamma_m^{(q)2} E_{r,m}^{(q)} = -\gamma_m^{(q)2} \frac{1}{i\omega\varepsilon_0\varepsilon} I_{r,m}^{(q)} - \frac{1}{i\omega\varepsilon_0\varepsilon} \frac{\lambda_m}{b_k} \frac{dI_{z,m}^{(q)}}{dz}, \\
&H_{\varphi,m}^{(q)} = \frac{1}{\gamma_m^{(q)2}} \left(\frac{\lambda_m}{b_k} I_{z,m}^{(q)} + i\omega\varepsilon_0\varepsilon \frac{dE_{r,m}^{(q)}}{d\tilde{z}} \right), \\
&E_{z,m}^{(q)} = -\frac{\lambda_m}{b_k} \frac{1}{\gamma_m^{(q)2}} \frac{dE_{r,m}^{(q)}}{d\tilde{z}} + \frac{i\omega}{\varepsilon_0 c^2 \gamma_m^{(q)2}} I_{z,m}^{(q)},
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
I_{r,m}^{(k)}\left(\tilde{z}\right) &= \frac{1}{W_m^{(k)}} \int_0^{2\pi} \int_0^{b_k} j_r\left(r, \varphi, z_k+\tilde{z}\right) J_1\left(\frac{\lambda_m}{b_k} r\right) r dr d\varphi, \\
I_{z,m}^{(k)}\left(\tilde{z}\right) &= \frac{1}{W_m^{(k)}} \int_0^{2\pi} \int_0^{b_k} j_z\left(r, \varphi, z_k+\tilde{z}\right) J_0\left(\frac{\lambda_m}{b_k} r\right) r dr d\varphi,
\end{aligned}
\tag{3}
$$

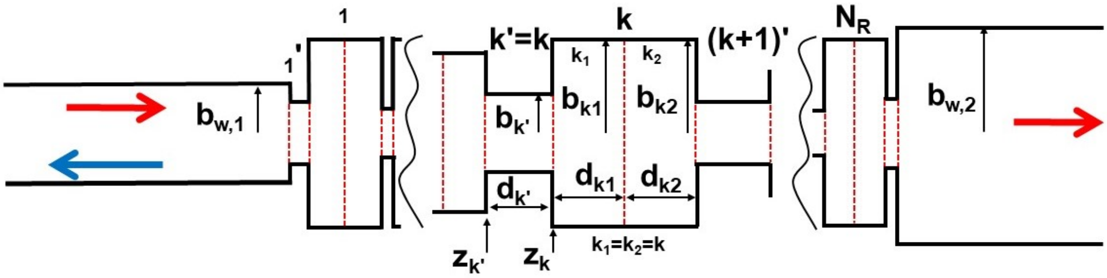
$$W_m^{(k)} = \pi b_k^2 J_1^2\left(\lambda_m\right).$$

The system of equations (2) is basic for the study electromagnetic fields in accelerating sections.

In the semi-infinite waveguides the electromagnetic field can be expanded in terms of the TM eigenmodes $\vec{\mathcal{E}}_s^{(w,p)}$, $\vec{\mathcal{H}}_s^{(w,p)}$ of a circular waveguide ($p = 1, 2$)

$$\vec{H}^{(w,p)} = \sum_s \left(G_s^{(p)} \vec{\mathcal{H}}_s^{(w,p)} + G_{-s}^{(k)} \vec{\mathcal{H}}_{-s}^{(w,p)} \right), \tag{4}$$

$$\vec{E}^{(w,p)} = \sum_s \left(G_s^{(p)} \vec{\mathcal{E}}_s^{(w,p)} + G_s^{(-p)} \vec{\mathcal{E}}_{-s}^{(w,p)} \right). \tag{5}$$



Chain of pieces of cylindrical waveguides that is connected with semi-infinite cylindrical waveguides

On the introduced interfaces we represent the electric fields as series of basis functions

$$
\begin{aligned}
E_r^{(k')}\left(r, d_{k'}\right) &= \sum_s C_s^{(k_1)} \varphi_s^{(r)}\left(r / b_{k'}\right), \\
E_r^{((k+1)')}\left(r, 0\right) &= \sum_s C_s^{(k_2)} \varphi_s^{(r)}\left(r / b_{k'+1}\right),
\end{aligned}
\tag{6}
$$

$$E_z^{(k_1)}\left(r, d_k/2\right) = E_z^{(k_2)}\left(r, 0\right) = \sum_s Q_s^{(k)} \varphi_s^{(z)}\left(r / b_k\right). \tag{7}$$

The boundary conditions for electric fields at the junctions are written as

$$
\begin{aligned}
&\sum_m E_{r,m}^{(k')}(d_{k'}) J_1\left(\frac{\lambda_m}{b_{k'}} r\right) = \sum_s C_s^{(k_1)} \varphi_s^{(r)}\left(r / b_{k'}\right), 0 \le r < b_{k'}, \\
&\sum_m E_{r,m}^{(k_1)}(0) J_1\left(\frac{\lambda_m}{b_k} r\right) = \begin{cases} \sum_s C_s^{(k_1)} \varphi_s^{(r)}\left(r / b_{k'}\right), 0 \le r < b_{k'}, \\ 0, \quad b_{k'} \le r < b_k, \end{cases}
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&\sum_m E_{z,m}^{(k_1)}(d_k/2) J_0\left(\frac{\lambda_m}{b_k} r\right) = \sum_m E_{z,m}^{(k_2)}(0) J_0\left(\frac{\lambda_m}{b_k} r\right) = \\
&= \sum_s Q_s^{(k)} \varphi_s^{(z)}\left(r / b_k\right), 0 \le r < b_k, \\
&\sum_m E_{r,m}^{(k_2)}(d_k/2) J_1\left(\frac{\lambda_m}{b_k} r\right) = \begin{cases} \sum_s C_s^{(k_2)} \varphi_s^{(r)}\left(r / b_{k'+1}\right), 0 \le r < b_{k'+1}, \\ 0, \quad b_{k'+1} \le r < b_k, \end{cases} \\
&\sum_m E_{r,m}^{(k'+1)}(0) J_1\left(\frac{\lambda_m}{b_{k'+1}} r\right) = \sum_s C_s^{(k_2)} \varphi_s^{(r)}\left(r / b_{k'+1}\right), 0 \le r < b_{k'+1}.
\end{aligned}
\tag{9}
$$

Using the completeness and orthogonality of Bessel functions $J_0\left(\frac{\lambda_m}{b} r\right)$ and $J_1\left(\frac{\lambda_m}{b} r\right)$, it is easy to find from (8),(9) coefficients of the left series. It should be noted that that the boundary conditions (9) contain also the longitudinal electric fields.

In the standard CIEM approach, the second group of boundary conditions contains, as a rule, the continuity of the tangential components of the magnetic field.

$$
\begin{aligned}
&\sum_m H_{\varphi,m}^{(k')}(d_{k'}) J_1\left(\frac{\lambda_m}{b_{k'}} r\right) = \sum_m H_{\varphi,m}^{(k_1)}(0) J_1\left(\frac{\lambda_m}{b_k} r\right), \quad 0 < r < b_{k'}, \\
&\sum_m H_{\varphi,m}^{(k_2)}(d_k/2) J_1\left(\frac{\lambda_m}{b_k} r\right) = \sum_m H_{\varphi,m}^{(k'+1)}(0) J_1\left(\frac{\lambda_m}{b_{k'+1}} r\right), \quad 0 < r < b_{k'+1}.
\end{aligned}
\tag{10}
$$

Multiplying the right and left sides of this relations by a testing function $\psi_{s'}\left(r / b_k\right)$ and integrating with respect to r from 0 to b_k, we get such equations

$$
\begin{aligned}
&\sum_m H_{\varphi,m}^{(k')}\left(d_{k'}\right) R_{s',m}^{\psi(k',k')} = \sum_m H_{\varphi,m}^{(k_1)}\left(0\right) R_{s',m}^{\psi(k',k)}, \\
&\sum_m H_{\varphi,m}^{(k_2)}\left(d_k/2\right) R_{s',m}^{\psi(k'+1,k)} = \sum_m H_{\varphi,m}^{(k'+1)}\left(0\right) R_{s',m}^{\psi(k'+1,k'+1)}.
\end{aligned}
\tag{11}
$$

In our case, it is necessary to add additional conditions for the continuity of the tangential components of the electric field at the interfaces in the middle of volumes of large cross section

$$\sum_m E_{r,m}^{(k_1)}\left(d_k/2\right)J_1\left(\frac{\lambda_m}{b_k}r\right)= \\ =\sum_m E_{r,m}^{(k_2)}\left(0\right)J_1\left(\frac{\lambda_m}{b_k}r\right)\Rightarrow E_{r,m}^{(k_1)}\left(d_k/2\right)=E_{r,m}^{(k_2)}\left(0\right). \quad (12)$$

We will consider the case when the dimensions of two semi-infinite waveguides are chosen such that only the dominant mode TM_{01} propagates, and the higher-order modes are all evanescent We will suppose that there is an incident wave that travels from $z=-\infty$ with amplitude $G_1^{(1)}=1$ ($G_s^{(1)}=0, s\geq 2$).

Using the standard CIEM technique, we obtain such system of vector equations

$$-\varepsilon T'^{(1,1')}C^{(1')}+\varepsilon T'^{(2,1')}C^{(L)}-T^{(L)}C^{(L)}=R^{(L)}+Z^{(L)},$$

$$k=2,...,N_R$$

$$\begin{cases}-T'^{(1,k')}C^{(k_2-1)}+\left(T'^{(2,k')}+T^{(2,k',k)}\right)C^{(k_1)}+T^{(1,k',k)}Q^{(k)}=\tilde{Z}^{-(k)},\\ T^{(1,k',k)}Q^{(k)}-\left(T'^{(2,k'+1)}+T^{(2,k'+1,k)}\right)C^{(k_2)}+T'^{(1,k'+1)}C^{(k_1+1)}=\tilde{Z}^{+(k)},\\ T^{r(k'+1,k)}C^{(k_2)}-T^{r(k',k)}C^{(k_1)}+T^{z(k)}Q^{(k)}=Z^{(k)},\end{cases} \quad (13)$$

$$T^{(R)}C^{(R)}-\varepsilon T'^{(2,(N_R+1)')}C^{(R)}+\varepsilon T'^{(1,(N_R+1)')}C^{((N_R)_2)}=Z^{(R)},$$

where $C_s^{(L)}$ and $C_s^{(R)}$ are the expansion coefficients of the electric field tangential components at the left and right interfaces between the DLW and the semi-infinite waveguides. $Z_{s'}$ (with different superscripts) are "current" integrals that equal zero if current is absent. $T_{s',s}$ (with different superscripts) are such matrices

$$T'^{(1,k')}_{s',s}=\frac{b_{k'}}{b_k}\sum_m\frac{2}{\gamma_m^{(k')}b_{k'}sh\left(\gamma_m^{(k')}d_{k'}\right)J_1^2\left(\lambda_m\right)}R_{s',m}^{\psi(k',k')}R_{m,s}^{\varphi,r,k',k'},$$

$$T'^{(2,k')}_{s',s}=\frac{b_{k'}}{b_k}\sum_m\frac{2ch\left(\gamma_m^{(k')}d_{k'}\right)}{\gamma_m^{(k')}b_{k'}sh\left(\gamma_m^{(k')}d_{k'}\right)J_1^2\left(\lambda_m\right)}R_{s',m}^{\psi(k',k')}R_{m,s}^{\varphi,r,k',k'},$$

$$T^{(2,k',k)}_{s',s}=\left(\frac{b_{k'}}{b_k}\right)^2\sum_m\frac{2sh\left(\gamma_m^{(k)}d_k/2\right)}{\gamma_m^{(k)}b_k ch\left(\gamma_m^{(k)}d_k/2\right)J_1^2\left(\lambda_m\right)}R_{s',m}^{\psi(k',k)}R_{m,s}^{\varphi,r(k',k)}, \quad (14)$$

$$T^{(1,k',k)}_{s',s}=\sum_m\frac{2}{\lambda_m}\frac{1}{ch\left(\gamma_m^{(k)}d_k/2\right)J_1^2\left(\lambda_m\right)}R_{s',m}^{\psi(k',k)}R_{m,s}^{\varphi,z},$$

$$T^{r(k',k)}_{m,s}=\frac{\lambda_m}{2b_k\gamma_m^{(k)}sh\left(\gamma_m^{(k)}d_k/2\right)}\frac{b_{k'}^2}{b_k^2}R_{m,s}^{\varphi,r(k',k)},$$

$$T^{z(k)}_{m,s}=R_{m,s}^{\varphi,z},$$

where $R_{m,s}^{\varphi,r(k',k)}=\int_0^1\varphi_s^{(r)}\left(x\right)J_1\left(b_{k'}\lambda_m x/b_k\right)xdx$,

$$R_{m,s}^{\varphi,z}=\int_0^1\varphi_s^{(z)}\left(x\right)J_0\left(\lambda_m x\right)xdx,$$

$$R_{s',s}^{\psi(k',k)}=\int_0^1\psi_{s'}\left(x\right)J_1\left(b_{k'}\lambda_s x/b_k\right)xdx.$$

Amplitudes of the eigen waves in the semi-infinite waveguides are determined by the expansion coefficients $C_s^{(L)}$ and $C_s^{(R)}$

$$G_{-1}^{(1)}=1+2\frac{b_{1'}^2\lambda_1}{J_1^2\left(\lambda_1\right)b_{w_1}^2\gamma_1^{(w_1)}b_{w_1}}\sum_{s'}R_{1,s'}^{w,L}C_{s'}^{(L)},$$

$$G_{-s}^{(1)}=-2\frac{b_{1'}^2\lambda_s}{J_1^2\left(\lambda_s\right)b_{w_1}^2\gamma_{-s}^{(w_1)}b_{w_1}}\sum_{s'}R_{s,s'}^{w,L}C_{s'}^{(L)},\quad s=2,3,... \quad (15)$$

$$G_s^{(2)}=-2\frac{b_{(N_R+1)'}^2\lambda_s}{J_1^2\left(\lambda_s\right)b_{w_2}^2\gamma_s^{(w_2)}b_{w_2}}\sum_{s'}R_{s,s'}^{w,R}C_{s'}^{(R)},\ s=1,2,...$$

where $\gamma_s^{(w,p)2}=\frac{\lambda_s^2}{b_{w,p}^2}-\frac{\omega^2}{c^2}$,

$$R_{m,s}^{w,L}=\int_0^1\varphi_s^{(r)}\left(x\right)J_1\left(b_{1'}\lambda_m x/b_{w_1}\right)xdx,$$

$$R_{m,s}^{w,R}=\int_0^1\varphi_s^{(r)}\left(x\right)J_1\left(b_{(N_R+1)'}\lambda_m x/b_{w_2}\right)xdx.$$

For the numerical solution of system (13), it is necessary to limit the number of basis and testing functions $\varphi_s^{(r)},\varphi_s^{(z)},\psi_s$. We will suppose that $\varphi_s^{(r)}\left(r\right)\equiv 0,\ \psi_s\left(r\right)\equiv 0,\ s>N_r$, $\varphi_s^{(z)}\left(r\right)\equiv 0,\ s>N_z$. Then we will have such sizes of defined matrices: $T'^{(1,k')},T'^{(2,k')},T^{(2,k',k)}$ are $N_r\times N_r$ matrices, $T^{(1,k',k)}$ are $N_r\times N_z$ matrices, $T^{r(k',k)}$ are a $N_z\times N_r$ matrices, $T_{m,s}^{z(k)}$ are $N_z\times N_z$ matrices.

2. INFINITIVE UNIFORM DISK LOADED WAVEGUIDE

To demonstrate the difference between the standard and the proposed approaches, consider an infinite homogeneous disk-loaded waveguide without current ($b_{k'}=a, d_{k'}=t, b_k=b,\ d_k=d$).

If we omit the presence of boundaries for the uniform segment, we obtain from (13) the equations that describe such waveguide. These difference equations in the matrix form are written as

$$\begin{cases}\left(T'^{(2)}+T^{(2)}\right)C^{(k_1)}=T'^{(1)}C^{(k_2-1)}-T^{(1)}Q^{(k)},\\ \left(T'^{(2)}+T^{(2)}\right)C^{(k_2)}=T'^{(1)}C^{(k_1+1)}+T^{(1)}Q^{(k)},\\ T^rC^{(k_2)}-T^rC^{(k_1)}+T^zQ^{(k)}=0,\end{cases} \quad (16)$$

where T (with different superscripts) are complex matrices, $C^{(k)}\in\mathbb{C}^{N_R},Q^{(k)}\in\mathbb{C}^{N_z}$ – complex vectors.

Excluding $C^{(k_2)}$ and $Q^{(k)}$ from (16), we get the standard matrix difference equation [4,5]

$$\overline{T}C^{(k_1)}=\overline{T}^{(+)}C^{(k_1+1)}+\overline{T}^{(-)}C^{(k_1-1)}. \quad (17)$$

We supposed that all matrices are invertible. The size of matrices $\overline{T},\overline{T}^{(+)},\overline{T}^{(-)}\in\mathbb{C}^{N_R\times N_R}$ is defined by the number of basis functions $\varphi_s^{(r)}\left(r/b_{k'}\right)$ in the E_r expansion (6).

The difference equation (17) is not symmetric ($T^{(+)}\neq T^{(-)}$) as it includes only vectors that describe the fields on the left side of the volumes with large cross section. These fields have a different "interaction" with right and left neighbors. The absence of symmetry makes it more difficult[2] to apply a transformation [8, 9], which gives simple method of finding Floquet coefficients and possibility to use the WKB approach [6, 7].

Eliminating $C^{(k_1)}$ and $C^{(k_2)}$ we can transform (16) into a symmetric difference equation ($-\infty < k < \infty$)

$$\tilde{T}Q^{(k)} = Q^{(k+1)} + Q^{(k-1)}, \tag{18}$$

where

$$\tilde{T} = \begin{Bmatrix} T^r \left\{ \left(T'^{(2)} + T^{(2)} \right) - T'^{(1)} \left(T'^{(2)} + T^{(2)} \right)^{-1} T'^{(1)} \right\}^{-1} \times \\ \times T'^{(1)} \left(T'^{(2)} + T^{(2)} \right)^{-1} T^{(1)} \end{Bmatrix}^{-1} \times$$
$$\times \left\{ T^z + 2T^r \left\{ \left(T'^{(2)} + T^{(2)} \right) - T'^{(1)} \left(T'^{(2)} + T^{(2)} \right)^{-1} T'^{(1)} \right\}^{-1} T^{(1)} \right\}. \tag{19}$$

The size of matrix $\tilde{T} \in \mathbb{C}^{N_z \times N_z}$ is defined by the number of basis functions $\varphi_s^{(z)}(r/b_k)$ in the E_z expansion (7). The E_r expansion (6) contains N_R basis functions $\varphi_s^{(r)}(r/b_{k'})$. Such approach gives possibility to improve the accuracy of E_r representation (to increase N_R) without increasing the size of matrix $\tilde{T}$ (N_z). It should also be noted that matrix $\tilde{T}$ is not Hermitian.

Using the transformation [6, 8]

$$\begin{aligned} Q^{(k)} &= Q^{(k,1)} + Q^{(k,2)}, \\ Q^{(k+1)} &= M^{(1)}Q^{(k,1)} + M^{(2)}Q^{(k,2)}, \end{aligned} \tag{20}$$

where

$$\left(\tilde{T}M^{(i)} - M^{(i)2} - I \right) = 0, \tag{21}$$

we get ($i = 1,2$)

$$Q^{(k+1,i)} = M^{(i)}Q^{(k,i)}. \tag{22}$$

It can be shown that in our case[3] the matrix $\tilde{T}$ is non-defective, and can be decomposed as

$$\tilde{T} = U\Theta U^{-1}, \tag{23}$$

where U is the matrix of eigen vectors U_s and $\Theta = diag(\theta_1, \theta_2, ...)$, θ_s – eigen values.

Then the solutions of quadratic matrix equations (21) are ($i = 1,2$)

$$M^{(i)} = U\Lambda^{(i)}U^{-1}, \tag{24}$$

where $\Lambda^{(i)} = diag(\lambda_1^{(i)}, \lambda_2^{(i)}, ...)$ and $\lambda_s^{(i)}$ are the solutions of the characteristic equations

$$\begin{aligned} &\lambda_s^{(i)2} - \theta_s\lambda_s^{(i)} + 1 = 0, \\ &\lambda_s^{(1)} = \theta_s/2 + \sqrt{(\theta_s/2)^2 - 1}, \\ &\lambda_s^{(2)} = \theta_s/2 - \sqrt{(\theta_s/2)^2 - 1}. \end{aligned} \tag{25}$$

The matrices $M^{(i)}$ have the same eigen vectors, therefore they are commutative. As $\lambda_s^{(1)}\lambda_s^{(2)} = 1$, the matrices $M^{(i)}$ satisfy the condition $M^{(1)}M^{(2)} = I$. We will suppose that $\left|\mathrm{Re}\left(\lambda_s^{(1)}\right)\right| < 1$ ($\left|\mathrm{Re}\left(\lambda_s^{(2)}\right)\right| > 1$).

Representing the vector $Q^{(k)}$ as the sum of two new vectors $Q^{(k,1)}$ and $Q^{(k,2)}$ we did not assume that they are individually solutions to the difference equation (18). Let us show that when $M^{(i)}$ are chosen as solutions to Eqs. (20), the vectors $Q^{(k,1)}$ and $Q^{(k,2)}$ are independent solutions to the equation (18).

If we know the radial distribution of longitudinal components of electric fields in two consecutive sections of the waveguide ($Q^{(0)}, Q^{(1)}$) then we can find vectors $Q^{(0,1)}, Q^{(0,2)}$

$$\begin{aligned} Q^{(0,1)} &= \left(M^{(2)} - M^{(1)} \right)^{-1} \left(M^{(2)}Q^{(0)} - Q^{(1)} \right), \\ Q^{(0,2)} &= -\left(M^{(2)} - M^{(1)} \right)^{-1} \left(M^{(1)}Q^{(0)} - Q^{(1)} \right). \end{aligned} \tag{26}$$

To find the solutions of equations (22) with conditions (26) and the conditions at the infinity for all values of k we have to consider the equations (22) for $k > 0$ and $k < 0$ separately.

Then the solutions of the difference matrix equations (22) with taking into account the conditions at the infinity are

$$\begin{aligned} Q^{(k,1)} &= M^{(1)k}Q^{(0,1)}, \quad k \geq 0, \\ Q^{(k,2)} &= M^{(2)k}Q^{(0,2)}, \quad k \leq 1. \end{aligned} \tag{27}$$

Vectors $Q^{(0)}$ and $Q^{(1)}$ we can represent as a sum of eigen vectors ($i = 0,1$)

$$Q^{(i)} = \sum_s A_s^{(i)}U_s. \tag{28}$$

The matrix $\tilde{T}$ is not Hermitian and the vectors U_s are not orthogonal. In this case

$$A_s^{(i)} = \sum_{s'} \left(U^{-1} \right)_{s,s'} Q_{s'}^{(i)}. \tag{29}$$

Substitution (29) into (26) gives

$$\begin{aligned} Q^{(0,1)} &= \left(M^{(2)} - M^{(1)} \right)^{-1} \sum_s \left(\lambda_s^{(2)}A_s^{(0)} - A_s^{(1)} \right) U_s = \\ &= \sum_s \frac{\left(\lambda_s^{(2)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} U_s, \\ Q^{(0,2)} &= -\left(M^{(2)} - M^{(1)} \right)^{-1} \sum_s \left(\lambda_s^{(1)}A_s^{(0)} - A_s^{(1)} \right) U_s = \\ &= -\sum_s \frac{\left(\lambda_s^{(1)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} U_s. \end{aligned} \tag{30}$$

Then the solution of the equation (18) takes the form

$$Q^{(k)} = \begin{cases} -\sum_s \frac{\lambda_s^{(2)k}\left(\lambda_s^{(1)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} U_s, \ k < 1. \\ \sum_s \begin{Bmatrix} \frac{\lambda_s^{(1)k}\left(\lambda_s^{(2)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} - \\ - \frac{\lambda_s^{(2)k}\left(\lambda_s^{(1)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} \end{Bmatrix} U_s, \ k = 0,1, \\ \sum_s \frac{\lambda_s^{(1)k}\left(\lambda_s^{(2)}A_s^{(0)} - A_s^{(1)} \right)}{\lambda_s^{(2)} - \lambda_s^{(1)}} U_s, \ k > 1. \end{cases} \tag{31}$$

For the case when $Q^{(0)} = U_m$ and $Q^{(1)} = \lambda_m^{(1)}U_m$ we have $A_s^{(0)} = \delta_{s,m}$, $A_s^{(1)} = \lambda_m^{(1)}\delta_{s,m}$ and

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[2] Matrix equations, whose solutions are necessary to construct the WKB equations, become more complicated.

[3] The infinitive uniform disk-loaded waveguide has $2N_z$ different independent solutions (waves).

$$Q^{fw(k)} = \begin{cases} 0, & k<0 \\ \lambda_m^{(1)k} U_m, & k \ge 0 \end{cases} . \qquad (32)$$

For the case $Q^{(1)} = \lambda_m^{(2)} U_m$

$$Q^{bw(k)} = \begin{cases} \lambda_m^{(2)k} U_m, & k \le 1, \\ 0, & k>1. \end{cases} \qquad (33)$$

Therefore, the vector sequences $\lambda_s^{(i)k} U_s$ can be considered as forward ($i=1$) or backward ($i=2$) eigen solutions of the equation (18).

It was shown [6], that the vector equation (22) can be transformed into a difference equations for any component of the vector $Q^{(k,i)}$. For a homogeneous waveguide these equations have the same form. Therefore, if we choose basis function that fulfill a condition $\varphi_s^{(z)}(0)=1$ (for example, $J_0\left(\frac{\lambda_s}{b} r\right)$), we can write a difference equation of the $2N_z$-order that connects the values of the electric field $E_z^{(k)} = \sum_s \left(Q_s^{(k,1)} + Q_s^{(k,2)}\right)$ at different points of the axis $r=0,\ z_k = k(d+t)+d/2$

$$\widehat{\det}\begin{pmatrix} \widehat{L}_1 & -\tilde{T}_{1,2} & \ldots & -\tilde{T}_{1,N_z} \\ -\tilde{T}_{2,1} & \widehat{L}_2 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ -\tilde{T}_{N_z,1} & -\tilde{T}_{N_z,2} & \ldots & \widehat{L}_{N_z} \end{pmatrix} E_z^{(k)} = 0, \qquad (34)$$

where the operator $\widehat{\det}$ is defined on the base of rules of common determinants

$$\widehat{\det}\begin{pmatrix} \widehat{L}_1 & -\tilde{T}_{1,2} \\ -\tilde{T}_{2,1} & \widehat{L}_2 \end{pmatrix} = \widehat{L}_1\widehat{L}_2 - \tilde{T}_{1,2}\tilde{T}_{2,1}, \qquad (35)$$

$\widehat{L}_i = \widehat{\sigma}^+ + \widehat{\sigma}^- - \tilde{T}_{i,i}$, $\quad \widehat{\sigma}^+\left(\widehat{\sigma}^+ b^{(k)} = b^{(k+1)}\right)$ and $\widehat{\sigma}^-\left(\widehat{\sigma}^- b^{(k)} = b^{(k-1)}\right)$ are shift operators. It was shown [6] that equation (34) does not have spurious solutions as it was for the equation based on a coupled cavities model [10].

3. MODIFIED VECTOR EQUATIONS

The system of vector equations (13) can be transformed to a system with only unknowns $Q^{(k)}$

$$\begin{aligned} & T^{(Q_1)}Q^{(1)} + T^{(Q_2)}Q^{(2)} = Z^{Q(1)}, \\ & k = 2,\ldots,N_{REZ}-1, \\ & T^{(k)}Q^{(k)} = T^{+(k)}Q^{(k+1)} + T^{-(k)}Q^{(k-1)} + Z^{Q(k)}, \\ & T^{(Q_{NREZ}-1)}Q^{(N_{REZ}-1)} + T^{(Q_{NREZ})}Q^{(N_{REZ})} = Z^{Q(N_{REZ})}, \end{aligned} \qquad (36)$$

where the sizes of all T matrices are $N_z \times N_z$.

There are additional equations relating $Q^{(1)}, Q^{(N_z)}$, $C_s^{(L)}, C_s^{(R)}$, from which we can calculate the reflection and transmission coefficients (see (15)). Based on system (36), a computer code has been developed. The results of studying the characteristics of inhomogeneous DLWs will be presented in subsequent papers.

System (36) is similar to that analyzed in [6] and, therefore, can be the basis for deriving the WKB equations.

CONCLUSIONS

The presented approach to the description of inhomogeneous disk-loaded waveguides can be a useful tool in studying the properties of slow wave system. Proposed modification of the coupled integral equations method makes it possible to deal directly with a longitudinal electric field and to obtain approximate equations for the case of a slow change in the waveguide parameters.

REFERENCES

1. J. Esteban, J.M. Rebollar. Characterization of corrugated waveguides by modal analysis // *IEEE Transactions on Microwave Theory and Techniques.* 1991, v. 39(6), p. 937-943
2. S. Amari, J. Bornemann, and R. Vahldieck. Accurate analysis of scattering from multiple waveguide discontinuities using the coupled integral equation technique // *J. Electromag. Waves Applicat*. 1996, v. 10, p. 1623-1644.
3. J. Bornemann, S. Amari, and R. Vahldieck. Analysis of waveguide discontinuities by the coupled-integral equations technique // *Recent Res. Devel. Microwave Theory and Techniques.* 1999, v. 1, p. 25-35.
4. S. Amari, R. Vahldieck, J. Bornemann. Analysis of propagation in periodically loaded circular waveguides // *IEE Proceedings – Microwaves Antennas and Propagation* 1999, v. 146, № 1, p. 50-54.
5. S. Amari, R. Vahldieck, J. Bornemann, and P. Leuchtmann. Spectrum of corrugated and periodically loaded waveguides from classical matrix eigenvalues // *IEEE Transactions on Microwave Theory and Techniques.* 2000, v. 48, № 3, p. 453-459.
6. M.I. Ayzatsky. Model of finite inhomogeneous cavity chain and approximate methods of its analysis// *Problems of Atomic Science and Technology*. 2021, № 3, p. 28-37; Modelling of inhomogeneous disk-loaded waveguides: matrix difference equations and WKB approximation// https://arxiv.org/abs/2010.10349, 2020.
7. M.I. Ayzatsky. Inhomogeneous travelling-wave accelerating sections and WKB approach *// Problems of Atomic Science and Technology*. 2021, № 4, p. 43-48; http://arxiv.org/abs/2103.10664, 2021.
8. M.I. Ayzatsky. Transformation of the linear difference equation into a system of the first order difference equations // *https: //arxiv.org/abs/1806.04378, 2018.*
9. M.I. Ayzatsky. A note on the WKB solutions of difference equations // https://arxiv.org/abs/1806.02196, 2018.
10. M.I. Ayzatsky. Nonlocal Equations for the Electromagnetic Field in the Inhomogeneous Accelerating Structures *// Problems of Atomic Science and Technology*. 2019, № 6, p. 71-76; Nonlocal Equations for the Electromagnetic field in the Coupled Cavity Model //https://arxiv.og/abs/1810.10382, 2018.

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МОДИФІКАЦІЯ МЕТОДУ ЗВ'ЯЗАНИХ ІНТЕГРАЛЬНИХ РІВНЯНЬ ДЛЯ РОЗРАХУНКУ ХАРАКТЕРИСТИК ПРИСКОРЮВАЛЬНОЇ СТРУКТУРИ

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Представлено модифікацію методу зв'язаних інтегральних рівнянь для розрахунку характеристик прискорювальних структур. У раніше розроблених схемах зв’язані інтегральні рівняння формулюються для невідомих електричних полів на поверхнях розділу, що ділять суміжні об’єми. На додаток до стандартного поділу структурованого хвилеводу на межі розділу між сусідніми комірками пропонуємо ввести нові інтерфейси в місцях, де електричне поле має найпростішу поперечну структуру. Крім того, система зв'язаних інтегральних рівнянь сформульована для поздовжніх електричних полів на відміну від стандартного підходу, де поперечні електричні поля невідомі. Кінцеві векторні рівняння містять коефіцієнти розкладання поздовжнього електричного поля на цих додаткових поверхнях розділу. Ця модифікація дає змогу мати справу з фізичною величиною, яка відіграє важливу роль у прискоренні частинок (поздовжнє електричне поле), та отримати наближені рівняння для випадку повільної зміни параметрів хвилеводу.

МОДИФИКАЦИЯ МЕТОДА СВЯЗАННЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ДЛЯ РАСЧЕТА ХАРАКТЕРИСТИК УСКОРЯЮЩЕЙ СТРУКТУРЫ

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Представлена модификация метода связанных интегральных уравнений для расчета характеристик ускоряющих структур. В разработанных ранее схемах связанные интегральные уравнения формулируются для неизвестных электрических полей на границах раздела, разделяющих соседние объемы. В дополнение к стандартному разделению структурированного волновода границами раздела между соседними ячейками предлагается ввести новые границы раздела в местах, где электрическое поле имеет простейшую поперечную структуру. Кроме того, система связанных интегральных уравнений формулируется для продольных электрических полей в отличие от стандартного подхода, когда поперечные электрические поля неизвестны. Окончательные векторные уравнения содержат коэффициенты разложения продольного электрического поля на этих дополнительных границах раздела. Эта модификация позволяет оперировать с физической величиной, играющей важную роль в ускорении частиц (продольным электрическим полем), и получить приближенные уравнения для случая медленного изменения параметров волновода.