https://doi.org/10.46813/2021-134-102

DIFFUSION IN MOMENTUM OF RELATIVISTIC ELECTRONS WITH A THERMAL SPREAD PASSING THROUGH AN UNDULATOR

V.V. Ognivenko
National Science Center "Kharkov Institute of Physics and Technology", Kharkiv, Ukraine
E-mail: ognivenko@kipt.kharkov.ua

The longitudinal momentum diffusion of electrons moving in a spatially periodic magnetic field of an undulator is investigated, taking into account their initial energy spread. Expressions for the coefficient are obtained and the dependences of the diffusion coefficient are determined both on the distance traveled by the electrons in the undulator and on the value of the initial energy spread of the electrons. The possibility of decreasing the wavelength in X-ray free electron lasers is discussed.

PACS: 41.60.-m, 41.60.Cr, 52.25.Gj

INTRODUCTION

The change in the mean square value of the momentum in the flow of electrons interacting by means of the electromagnetic fields they produce at the stage of spontaneous emission was studied in [1]. In this work, the radiative relaxation of the electrons at the pre-Brownian stage of the evolution of the system [2], when the electrons have the same initial energy, is considered.

Since electrons move with different velocities in real flows, it is necessary to establish a quantitative criterion that determines the possibility of neglecting the difference in the initial velocities of electrons when describing the radiation relaxation of the beam. In addition, one should take into account that the difference in the velocities of electrons makes it possible to study the diffusion of electrons in momentum space at the kinetic stage of the evolution of the system [3], when the motion of the electrons in the process of radiation relaxation becomes completely random. The description of electron momentum diffusion in such flows is also of considerable interest in connection with the development of X-ray free electron lasers (FEL) based on self-amplification of spontaneous emission by a monoenergetic ultrarelativistic electron beam moving in an undulator [4 - 10].

1. DIFFUSION COEFFICIENT

The expression for the diffusion coefficient obtained on the basis of the dynamics of individual particles motion can be written in the form [1, 2]

$$D_z = \frac{d}{2dt}\left\langle (\Delta p_z)^2 \right\rangle = \int_{t_0}^{t} dt_1 \int_{\Omega_q} F_z^{(s)}\left[X(t,q_{0s})\right]\times \times F_z^{(s)}\left[X(t_1,q_{0s})\right] f(q_{0s})\,\mathrm{v}_z(t_{0s})\,dq_{0s} \quad , \tag{1}$$

where $\Delta p_z = p_z - \langle p_z \rangle$, p is the momentum of the electron, $\mathrm{F}_z^{(s)}(x,t;x_s)$ is the pair interaction force, $X(t,q_{0s}) = \left(x^{(0)}(t),t;x_s^{(0)}(t,q_{0s})\right)$, $x=(\mathbf{r},\mathbf{p})$, $\mathbf{p}=m\gamma\mathbf{v}$, m is the mass of the electron, $\gamma = (1-\mathrm{v}^2/c^2)^{-1/2}$, c is the velocity of light, $x_s^{(0)} = \left(\mathbf{r}_s^{(0)},\mathbf{p}_s^{(0)}\right)$ are the equilibrium trajectory and momentum of *s*th electron in an undulator, $q_{0s} = (\mathrm{x}_{0s}, y_{0s}, t_{0s}, \mathbf{p}_{0s})$ are the initial coordinates and momentum of the sth electron at the time t_{0s} when it intersects the z=0 plane, $dq_{0s} = d\mathbf{p}_{0s}\,d\,\mathrm{x}_{0s}\,dy_{0s}\,dt_{0s}$; $f(q_{0s})$ is the single particle momentum distribution of electrons, Ω_q is the region of integration over the initial coordinates and momenta of emitting electrons.

We will assume that the relativistic electrons beam moves in the positive direction of the z axis in a static periodic magnetic field of the undulator

$$\mathbf{H}_u = H_0\left[\mathbf{e}_x\cos(k_u z) + \mathbf{e}_y\sin(k_u z)\right],$$

where $k_u = 2\pi/\lambda_u$, H_0 and λ_u are the amplitude and period of the magnetic field, $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the unit vectors of the Cartesian coordinate system x, y, z.

In the approximation of a small value of the undulator parameter, the expression for the diffusion coefficient has the form [11]

$$D_z = \left(\frac{e^2\,\mathrm{K}^2 k_u}{\gamma}\right)^2 \int_{t_0}^{t} dt_1 \int_{-\infty}^{\infty} dp_{z0s}\gamma_{0s}^2 w(p_{z0s})\Phi\,, \tag{2}$$

where

$$\Phi = \iiint_{\Omega_q} \frac{G(\mathbf{r},t;q_{0s})G(\mathbf{r}_1,t_1;q_{0s})}{R_*(\mathbf{r},t;q_{0s})R_*(\mathbf{r}_1,t_1;q_{0s})} n_b\,\mathrm{v}_{zs}\,dx_{0s}dy_{0s}dt_{0s}\,, \tag{3}$$

$$G(\mathbf{r},t;q_{0s}) = \left(\beta_{zs} + \frac{R_{0z}}{R_*} - \frac{\beta_{zs}}{k_{0s}^2 R_*^2} - \frac{\beta_{zs}R_{0\perp}^2}{2R_*^2\gamma_{zs}^2}\right)\sin\psi + \left(\beta_{zs} + \frac{R_{0z}}{R_*}\right)\frac{\cos\psi}{k_{0s}R_*}$$

$$\psi = \gamma_{zs}^2 k_u\left(R_{0z} + \beta_{zs}R_*\right),\ k_{0s} = \beta_{zs}\gamma_{zs}^2 k_u\,.$$

To obtain the explicit expressions for the diffusion coefficient, we use simplifying assumptions. Let us consider the spread in momenta of the electrons moving near the beam axis $x_0=y_0=0$. We will take into account the electromagnetic field of the emitting electrons moving only behind the considered (test) electron. We assume that the distance between the considered electrons is much greater than the thermal dispersal of these electrons during the process under consideration, i.e. $r' >> \xi$. Nevertheless, in this case, thermal dispersal of the electrons at a distance greater than the wavelength of undulator radiation $\lambda = \lambda_u\left(1+K^2\right)/2\gamma_{0s}^2$ is possible.

Assuming also that the beam radius is greater than the wavelength of undulator radiation in the transverse direction: $r_b >> \lambda_\perp/2\pi$, where $\lambda_\perp = \lambda_u/\beta_{zs}\gamma_{zs}$, using Eq. (2), we find

$$D_z = \frac{\pi e^4 K^4 k_u^2 n_b}{4\gamma^2 v_z} \int_0^z dz_1 \int_{-\infty}^{\infty} dp_{z0s}\gamma_{0s}^4 w(p_{z0s}) \times$$
$$\times \int_0^{\pi/2} d\theta \sin\theta \chi^4(\theta) \cos\left[\gamma_{zs}^2 k_u \xi \chi(\theta)\right] \cdot r'_{\max}(z_1, \theta), \quad (4)$$

where $\chi = 1 + \beta_{zs}\cos\theta$.

Suppose that the function w (p_{z0s}) is a Maxwellian

$$w(p_{z0s}) = \frac{1}{\sqrt{2\pi}p_{th}} \exp\left[-\frac{(p_{z0s} - p_{zm})^2}{2p_{th}^2}\right]. \quad (5)$$

We will assume that the initial thermal spread in the beam, as well as an increase in the energy spread of the electrons, due to the radiation interaction, satisfy the conditions $p_{th} << p_{zm}$ $\Delta\gamma << \gamma_m$. Substituting (5) into (4) and integrating over the longitudinal momentum using the value of the integral [12], we obtain the following expression for the diffusion coefficient [11]

$$D_z(p_z) = \frac{\pi e^4 K^4 k_u^2 \gamma_m^4 n_b}{4\gamma^2 v_z} \int_0^z dz_1$$
$$\int_0^{\pi/2} d\theta \sin\theta \chi^4(\theta) r'_{\max}(z_1, \theta) \alpha[\chi(\theta)(z - z_1)]. \quad (6)$$

where $\alpha(x) = \exp\left(-\frac{x^2}{z_c^2}\right)\cos(\eta x)$, $\eta = k_u\left(1 - \frac{p_{zm}}{p_z}\right)$,

$z_c = \frac{\sqrt{2}p_z}{p_{th}k_u}$, $r'_{\max} = \min\left(\frac{z_1}{\gamma_{zs}^2(\cos\theta + \beta_{zs})}, \frac{r_b}{\gamma_{zs}\sin\theta}\right)$.

Pre-Brownian stage. In the case $z << z_c$ from expression (6), taking into account (1), we obtain

$$\frac{d}{dz}\left\langle(\Delta p_z)^2\right\rangle = \frac{\pi e^4 K^4 \gamma_m^2 k_u^2 n_b}{\gamma^2 v_z^2} z \begin{cases} (15/16)z, & z < z_r \\ 2z_r B(z/z_r), & z > z_r \end{cases}, \quad (7)$$

where

$$B(x) = \frac{5}{3}\left(1 - \frac{1}{x}\right) + \frac{35}{16}\left(arctgx - \frac{\pi}{4}\right) - \frac{2}{x}\ln\left(\frac{1 + x^2}{2}\right) + \frac{15}{32x}.$$

Kinetic stage. In the case $z >> z_c$, the asymptotic expression for the diffusion coefficient (6) at $z > z_r$ takes the form

$$D_z(p_z, z) = \frac{5\pi^{5/2}}{16}\left(\frac{e^2 K^2 \gamma_m}{\gamma}\right)^2 \frac{n_b}{v_z} k_u^2 z_r z_c \times$$
$$\times \left[B_2\left(\frac{z}{z_r}\right) e^{-\frac{(p_z - p_{zm})^2}{2p_{th}^2}} - \frac{8z_r z_c}{5\pi^{3/2}z^2}\phi\left(\frac{p_z - p_{zm}}{\sqrt{2}p_{th}}\right)\right], \quad (8)$$

where $B_2(x) = \frac{2}{\pi}arctgx - \frac{1}{2} + \frac{22}{15\pi} - \frac{6}{5\pi x}$,

$$\phi(x) = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt.$$

For very large deviations of the momentum from the equilibrium value (x is large) and very small deviations of the momentum (x is small), we have

$$\phi(x) = \begin{cases} -\frac{1}{2x^2}\left(1 + \frac{3}{2x^2} + \frac{15}{4x^4} + \ldots\right). & x >> 1 \\ 1 - 2x^2\left(1 + \frac{2x^2}{3} + \frac{4x^4}{15} + \ldots\right) & x << 1 \end{cases} \quad (9)$$

Fig. 1 shows the graph of the function $\phi(x)$.

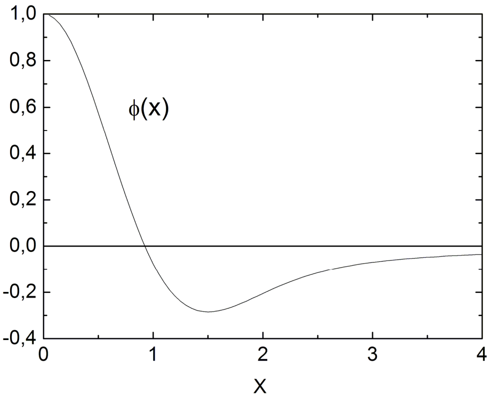


Fig. 1. Function ϕ entering the formula for the diffusion coefficient

Expression (8) at $z >> z_r$ takes the form

$$D_z = \frac{5\pi^{5/2} e^4 K^4 \gamma_m^2 n_b}{16\gamma^2 v_z} k_u^2 z_r z_c e^{-\frac{(p_z - p_{zm})^2}{2p_{th}^2}}. \quad (10)$$

Thus, at large distances, the diffusion coefficient is independent of z. Such a dependence of the diffusion coefficient on the coordinate or on time corresponds to the kinetic stage of the evolution of a system consisting of a large number of particles (see, for example, [3, 13-15]).

2. MONOENERGETIC BEAM

For self amplification of spontaneous ultrashort wavelength electromagnetic radiation, the spread in the momentum of relativistic electrons should be sufficiently small $p_{th} << p_{zm}\lambda_u/L_u$ (see, for example, [5, 16]). This inequality can be rewritten as $L_u << 2z_c$, where $L_u = \lambda_u/\rho_{1D}$ is the distance at which the intensity of electromagnetic radiation reaches its maximum value, $\rho_{1D} = (K\lambda_u/4\pi r_b)^{2/3}(I_b/I_A)^{1/3}\gamma_0^{-1}$, $I_b = \pi|e|v_{z0}r_b^2 n_b$, $I_A = mc^3/|e|$=17 kA. Therefore, at the stage of spontaneous emission ($z<L_u$), the initial energy spread of electrons can be neglected, assuming that the electrons are monoenergetic: $w(p_{zs}) = \delta(p_{zs} - p_{z0})$. In this case, using equations (2) and (3), the expression for the mean square spread of the longitudinal momentum can be written in the form

$$\left\langle(\Delta p_z)^2\right\rangle = \frac{\pi e^4 K^4 n_b}{c^2 k_u} V, \quad (11)$$

$$V = \int_0^\zeta d\zeta_1(\zeta - \zeta_1)\int_0^1 dy(\beta_{z0} + y)^3 \int_0^{x_{\max}(\zeta_1, y)} dx g^2(x), \quad (12)$$

where $g(x) = \left(1 - \frac{2}{x^2}\right)\sin x + 2\frac{\cos x}{x}$;

$x_{\max} = \psi_m(y)$ at $0 \le y < y_*(\zeta_1)$; $x_{\max} = \zeta_1$ at $y_*(\zeta_1) \le y \le 1$;

$$\psi_m(y) = \frac{\zeta_r(\beta_{z0} + y)}{\beta_{z0}\sqrt{1-y^2}}; \quad y_*(\zeta_1) = \max\left(0, \frac{\zeta_1^2 - \zeta_r^2}{\zeta_1^2 + \zeta_r^2}\right);$$

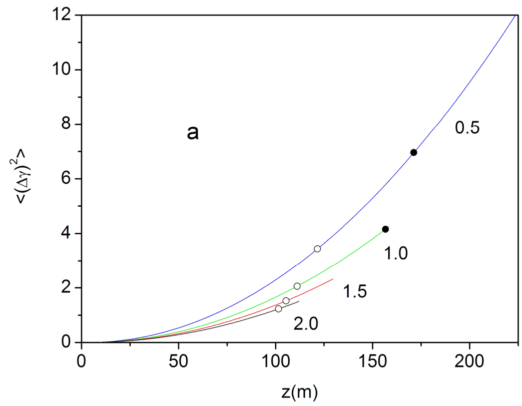
$\zeta = k_u z$, $\zeta_1 = k_u z_1$, $\zeta_r = \beta_{z0}\gamma_{z0} r_b k_u = 2\pi r_b / \lambda_\perp$.

Formula (12) can be transformed into a simpler form

$$V = \frac{15}{8}\int_0^{\zeta_2} dx\, u(x,\zeta) + 2\int_{\zeta_2}^{\zeta} dx\, u(x,\zeta)\left[1 - \frac{x^8}{\left(\zeta_r^2 + x^2\right)^4}\right], \quad (13)$$

where $u(x,\zeta) = (\zeta - x)^2 g^2(x)$, $\zeta_2 = \min(\zeta, \zeta_r)$.

Fig. 2 plots the normalized momentum spread $\left\langle(\Delta\gamma)^2\right\rangle = \left\langle(\Delta p_z)^2\right\rangle / m^2c^2$ as a function of coordinate z, calculated by formulas (11), (13) for the values H_0=1.4 kG, λ_u=4 cm, r_b=50 μm, I_b=1 kA (a), 4 kA (b), (n_b=2.6 10^{15} cm^{-3}, $1.32 \cdot 10^{16}$ cm^{-3}, respectively). The curves correspond to the radiation wavelength λ and the energy of electrons $E_b = mc^2\gamma_0$: 2 (5.76) (black), 1.5 (6.65) (red), 1.0 (8.15) (green), 0.5 Å (11.53 GeV) (blue).



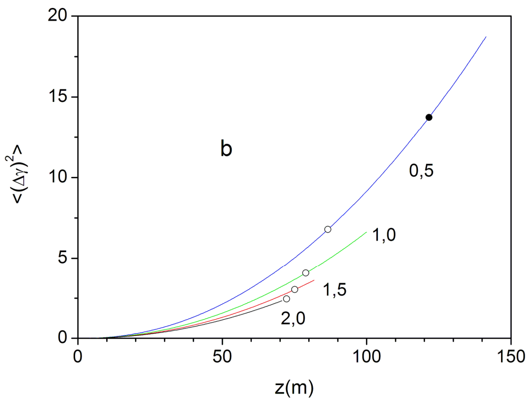


Fig. 2. Dependence of the normalized longitudinal momentum spread $\left\langle(\Delta\gamma)^2\right\rangle$ on the z coordinate in meters; I_b=1 kA (a), 4 kA (b). The numbers on the curves represent the radiation wavelength λ in angstroms

The dependences on the z coordinate were calculated up to the values of L_u. The white circles on the curves indicate the values z_{rel} of the momentum spread in coordinates in which the displacement of the electrons relative to the equilibrium trajectory, due to the radiation spread in the momentum, reaches half the wavelength of the radiation: $2z_{rel}\left\langle(\Delta p_z)^2\right\rangle^{1/2}_{z=z_{rel}} = \gamma_{z0}^2 p_{z0}\lambda$. The black circles on the curves correspond to the displacement of the electrons by the wavelength λ of the radiation.

It can be seen from the Fig. 2 that the spread in momenta increases with an increase in the electron energy. The distance z_{rel} also increases with an increase in the electron energy in proportional to $\gamma_0^{1/2}$ at $z>z_r$. Since the characteristic length of the undulator L_u is proportional to the energy of the electrons, the relative distance z_{rel}/L_u decreases with increasing the energy of the electrons.

CONCLUSIONS

The analysis shows that at the initial stage of the beam motion, when the distance traveled by it in the undulator is small compared to the distance z_c, the rms spread $<(\Delta p_z)^2>^{1/2}$ linearly depends on the coordinate z, for $z>z_r$. The evolution of electrons in momentum space at $z<<z_c$ corresponds to the pre-Brownian motion of electrons.

The distance z_c is equal in order of magnitude to the distance, during which time τ_c ($\tau_c=z_c/v_z$) the thermal dispersal of two electrons relative to each other reaches half the wavelength of the radiation: $v_{th}\,\tau_c = \sqrt{2}\lambda/\pi \approx \lambda/2$. The time τ_c does not depend on either the beam density or the force of the pair interaction of electrons and is the duration of the "synchronous" interaction of two electrons. During this time, the phase of the force acting on the test electron changes to π.

At distances much larger than z_c, the force of pair interaction changes rapidly. As a result, at large distances ($z>>z_r$) the diffusion coefficient does not depend on the coordinate, and the rms spread in the longitudinal momentum is proportional to $z^{1/2}$. In this case, the kinetic stage of electron diffusion in momentum space is realized, and their motion becomes completely random.

As follows from the above calculations, the radiative interaction of electrons can lead to an increase in the energy spread in the beams, which are used to obtain coherent electromagnetic radiation of the nanometer and a shorter wavelength range.

For example, for an electron beam and an undulator with the parameters given above (see Fig. 2), an increase in the energy spread as a result of radiation relaxation can prevent the production of coherent radiation with a wavelength less than 1.5 Å at I_b =1 kA and 1.0 Å at I_b = 4 kA, respectively.

REFERENCES

1. V.V. Ognivenko. Momentum spread in a relativistic electron beam in an undulator // *J. Exp. Theor. Phys.* 2012, v. 115, № 5, p. 938-946; *Zh. Eksp. Teor. Phys.* 2012, v. 142, № 5, p. 1067-1076.
2. V.V. Ognivenko. Dynamical derivation of momentum diffusion coefficients at collisions of relativistic charged particles // *J. Exp. Theor. Phys*. 2016, v. 122, № 1, p. 203-208; *Zh. Eksp. Teor. Phys*. 2016, v. 149, № 1, p. 230-236.

3. N.N. Bogolyubov. *Problems of Dynamical Theory in Statictical Physics.* Moscow: "Gostekhteorizdat", 1946; Interscience, New York, 1962.
4. K.-J. Kim. *Nucl. Instr. and Meth*. 1986, v. A 250, p. 396.
5. C. Pellegrini. *Particle accelerators*. 1990, v. 33, p. 159.
6. P. Emma, R. Akre, J. Arthur, et al. First Lasing and Operation of an Ångstrom-Wavelength Free-Electron Laser // *Nature Photonics.* 2010, v. 4, p. 641-647.
7. H. Tanaka, M. Yabashi, et al. A compact X-ray free-electron laser emitting in the sub-ångström region // *Nature Photonics.* 2012, v. 6, p. 540-544.
8. H. Weise, W. Decking. Commissioning and first lasing of the European XFEL // *Proc. 38th Free-Electron Laser Conf.* Santa Fe, NM, USA. 2017, p. 9-13.
9. L. Giannessi, E. Allaria, L. Badano, et al. // *Proc. 39th Free-Electron Laser Conf.* Germany, Hamburg. 2019, p. 742.
10. R. Ganter, G. Aeppli, J. Alex, et al. // *Proc. 39th Free-Electron Laser Conf.* Germany, Hamburg. 2019, p. 753.
11. V.V. Ognivenko // *J. Exp. Theor. Phys*. 2021 v. 132, N 5, p. 766-775.
12. A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev. *Integrals and Series. Elementary Functions.* Moscow: "Nauka", 1981.
13. G. Uhlenbeck and G. Ford. *Lectures in Statistical Mechanics.* Am. Math. Soc. Philadelphia, 1963.
14. V.P. Silin. *Introduction to Kinetic Theory of Gases.* Moscow: "Nauka", 1971.
15. A.I. Akhiezer and S.V. Peletminskii. *Methods of Statistical Physics.* Moscow: "Nauka", 1977; Pergamon, Oxford, 1981.
16. P. Sprangle, R.A. Smith // *Phys. Rev.* 1980, A21, p. 293.

Article received 03.06.2021

ДИФФУЗИЯ ПО ИМПУЛЬСУ РЕЛЯТИВИСТСКИХ ЭЛЕКТРОНОВ С ТЕПЛОВЫМ РАЗБРОСОМ, ПРОХОДЯЩИХ В ОНДУЛЯТОРЕ

В.В. Огнивенко

Исследована диффузия по продольному импульсу электронов, движущихся в пространственно-периодическое магнитное поле ондулятора, с учетом их начального энергетического разброса. Получены выражения для коэффициента диффузии и определены зависимости его как от расстояния, пройденного электронами в ондуляторе, так и от величины начального энергетического разброса электронов. Обсуждается возможность уменьшения длины волны в рентгеновских лазерах на свободных электронах.

ДИФУЗІЯ ПО ІМПУЛЬСУ РЕЛЯТИВІСТСЬКИХ ЕЛЕКТРОНІВ З ТЕПЛОВИМ РОЗКИДОМ, ЩО ПРОХОДЯТЬ В ОНДУЛЯТОРІ

В.В. Огнівенко

Досліджено дифузію за поздовжнього імпульсу електронів, що рухаються в просторово-періодичне магнітне поле ондулятора, з урахуванням їх початкового енергетичного розкиду. Отримано вираз для коефіцієнта дифузії й визначені залежності його як від відстані, вздовж ондулятора, так і від величини початкового енергетичного розкиду електронів. Обговорюється можливість зменшення довжини хвилі в рентгенівських лазерах на вільних електронах.