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RESEARCH ARTICLE

On the structure of the algebra of derivations of cyclic Leibniz algebras

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Abstract. We describe the algebra of derivation of finite-dimensional cyclic Leibniz algebra.

Introduction

Let L be an algebra over finite field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] \text{ for all } a,b,c \in L.$$

We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

Leibniz algebras were first in the paper of A. Bloh [1], but the term "Leibniz algebra" appears in the book of J.-L. Loday [2] and his article [3]. In [4] J. Loday and T. Pirashvili began the real study of properties of Leibniz algebras. The theory of Leibniz algebras was developed very intensively in many different directions. Some of the results of this theory were presented in the book [5]. Note that Lie algebras are a partial case of Leibniz algebras. Conversely, if L is a Leibniz algebra in which $[a,a] = 0$ for every element $a \in L$, then it is a Lie algebra. Thus, Lie algebras can be characterized as anticommutative Leibniz algebras.

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Like the Lie algebras, the structure of Leibniz algebras is strongly affected by their algebras of derivations.

Denote by $\mathrm{End}_F(L)$ the set of all linear transformations of L. Then L is an associative algebra by the operations $+$ and $\circ$. As usual, $\mathrm{End}_F(L)$ is a Lie algebra by the operations $+$ and $[\cdot,\cdot]$, where $[f,g] = f\circ g - g\circ f$ for all $f, g \in \mathrm{End}_F(L)$.

A linear transformation f of a Leibniz algebra L is called a *derivation*, if

$$f([a,b]) = [f(a),b] + [a,f(b)] \text{ for all } a,b \in L.$$

Let $\mathrm{Der}(L)$ be the subset of all derivations of L. It is possible to prove that $\mathrm{Der}(L)$ is a subalgebra of a Lie algebra $\mathrm{End}_F(L)$. $\mathrm{Der}(L)$ is called the *algebra of derivations* of a Leibniz algebra L.

For a Leibniz algebra, the following result shows the influence of the algebra of derivations on its structure: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $\mathrm{Der}(A)$ [6, Proposition 3.2].

Among the Leibniz algebras, it is natural to study the structure of their algebras of derivations for cyclic Leibniz algebras. The structure of cyclic Leibniz algebras was described in [7]. In what follows, we will demonstrate this structure.

Let L be a cyclic Leibniz algebra, $L = \langle a\rangle$, and we suppose that L has a finite dimension over a field F. Then there exists a positive integer n such that L has a basis $a_1,\dots,a_n$, where $a_1 = a$, $a_2 = [a_1,a_1]$, $\dots$, $a_n = [a_1,a_{n-1}]$, $[a_1,a_n] = \alpha_2a_2 + \ldots + \alpha_na_n$ [7]. Moreover, $[L,L] = \mathrm{Leib}(L) = Fa_2 + \ldots + Fa_n$ [7]. We fix these designations.

Here appear the following types of cyclic Leibniz algebras.

First case: $[a_1,a_n] = 0$. In this case, L is nilpotent, and we say that L is a *cyclic algebra of type* (I).

The structures of the algebras of derivations of these Leibniz algebras have been described in [7].

The next type of cyclic Leibniz algebras appears in the following way. In this case, $[a_1,a_n] = \alpha_2a_2 + \ldots + \alpha_na_n$ and $\alpha_2 \neq 0$. In some sense, it is a basic case. First, we recall some definitions.

The *left* (respectively *right*) *center* $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{x \in L|\ [x,y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{right}(L) = \{x \in L|\ [y,x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\mathrm{Leib}(L) \leqslant \zeta^{left}(L)$, so that $L/\zeta^{left}(L)$ is a Lie algebra. The right center is a subalgebra of L, and, in general, the left and right centers are different; they even may have different dimensions (see [6]).

Put $c = \alpha_2^{-1}(\alpha_2 a_1 + \ldots + \alpha_n a_{n-1} - a_n)$, then $[c, c] = 0$, moreover, Fc is a right center of $L, L = [L, L] \oplus Fc$ and $[c, b] = [a_1, b]$ for every element $b \in A$ [7]. In particular, $a_3 = [c, a_2]$, …, $a_n = [c, a_{n-1}]$, $[c, a_n] = \alpha_2 a_2 + \ldots + \alpha_n a_n$. In this case, we say that L is a *cyclic algebra of type* (II).

Put $A = [L, L]$ and define the mapping $\mathfrak{l}_c \colon L \to L$ by the rule $\mathfrak{l}_c(x) = [c, x]$ for every element $x \in L$. The restriction of a linear transformation $\mathfrak{l}_c$ on A has the following matrices in a basis $\{a_2, \ldots, a_n\}$:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \alpha_2 \\ 1 & 0 & 0 & \ldots & 0 & \alpha_3 \\ 0 & 1 & 0 & \ldots & 0 & \alpha_4 \\ 0 & 0 & 1 & \ldots & 0 & \alpha_5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \alpha_{n-1} \\ 0 & 0 & 0 & \ldots & 1 & \alpha_n \end{pmatrix}$$

These matrices are non-degenerate. Hence the restriction of $\mathfrak{l}_c$ on A is an F-automorphism of a linear space A. The first main result of this paper is following.

Theorem 1. *Let L be a cyclic Leibniz algebra of type (II) over a field F, and let D be the annihilator of a subspace Fc in algebra* $\mathrm{Der}(L)$. *Then the following assertions hold:*

(i) *D is an Abelian ideal having dimension* $\dim_F(L) - 1$; *the set* $\{\mathfrak{i}, \mathfrak{l}_c, \mathfrak{l}_c^2, \ldots, \mathfrak{l}_c^{n-2}\}$ *is a basis of D;*

(ii) *D has a codimension at most 1;*

(iii) *if $D \neq$* $\mathrm{Der}(L)$, *then* $\mathrm{char}(F)$ *divides* $\dim_F(L) - 1$.

Here $\mathfrak{i}$ is the mapping, defined by the rule: if $x = a + \sigma c$, $a \in A$, $\sigma \in F$, is an arbitrary element of L, then put $\mathfrak{i}(x) = a$.

Corollary $\mathbf{A_1}$. *Let L be a cyclic Leibniz algebra of type (II) over a field F. If F has a characteristic 0, then algebra* $\mathrm{Der}(L)$ *is Abelian and has a dimension* $\dim_F(L) - 1$.

Finally, consider the last type of finite-dimensional cyclic Leibniz algebra.

In this case: $[a_1, a_n] = \alpha_2 a_2 + \ldots + \alpha_n a_n$, but $\alpha_2 = 0$. Let t be the first index such that $\alpha_t \neq 0$. In other words, $[a_1, a_n] = \alpha_t a_t + \ldots + \alpha_n a_n$. By our condition, $t > 2$. Then

$$[a, a_n] = \alpha_t[a, a_{t-1}] + \ldots + \alpha_n[a, a_{n-1}] = [a, \alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1}],$$

which implies that $\alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1} - a_n \in \mathrm{Ann}_K^{right}(a_1)$. The fact that $\alpha_t \neq 0$ implies that $\alpha_t^{-1} \neq 0$, and then

$$\begin{aligned} d_{t-1} &= \alpha_t^{-1}(\alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1} - a_n) = \\ &= a_{t-1} + \beta_t a_t + \ldots + \beta_n a_n \in \mathrm{Ann}_K^{right}(a_1). \end{aligned}$$

Put

$$\begin{gathered} d_{t-2} = a_{t-2} + \beta_t a_{t-1} + \ldots + \beta_n a_{n-1}, \\ d_{t-3} = a_{t-3} + \beta_t a_{t-2} + \ldots + \beta_n a_{n-2}, \ldots, \\ d_1 = a_1 + \beta_t a_2 + \ldots + \beta_n a_{n-t+1}. \end{gathered}$$

Then

$$\begin{gathered} [d_1, d_1] = [a_1, d_1] = d_2, \\ [d_1, d_2] = [a_1, d_2] = d_3, \ldots, \\ [d_1, d_{t-2}] = [a_1, d_{t-2}] = d_{t-1}, \\ [d_1, d_{t-1}] = [a_1, d_{t-1}] = 0. \end{gathered}$$

It follows that the subspace $U = Fd_1 \oplus Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is a subalgebra, and, moreover, this subalgebra is nilpotent. Moreover, a subspace $[U, U] = Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is an ideal of L. Put further $d_t = a_t$, $d_{t+1} = a_{t+1}$, $\ldots$, $d_n = a_n$. The following matrix corresponds to this transaction:

$$\begin{pmatrix} 1 & \beta_t & \beta_{t+1} & \ldots & \beta_k & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \beta_t & \ldots & \beta_{k-1} & \beta_k & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & \beta_t & \ldots & \beta_{k-1} & \beta_k \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

This matrix is non-singular, which proves that the elements $\{d_1, \ldots, d_n\}$ present a new basis. We note that a subspace $V = Fd_t \oplus \ldots \oplus Fd_n$ is a subalgebra. Moreover, V is an ideal of D, because $[a_1, d_t] = d_{t+1}$, $\ldots$,

$[a_1, d_{n-1}] = d_n$, $[a_1, d_n] = \alpha_t d_t + \ldots + \alpha_n d_n$. Moreover, $[a_1, d_j] = [d_1, d_j]$ for all $j \geqslant t$ [7]. In this case, we say that L is a *cyclic algebra of type* (III).

Thus, $L = A \oplus Fd_1$, $A = V \oplus [U, U]$ is a direct sum of two ideals, $U = [U, U] \oplus Fd_1$ is a nilpotent cyclic subalgebra, i.e. is an algebra of type (I), and $V \oplus Fd_1$ is a cyclic subalgebra of type (II).

The second main result of this paper gives a description of the algebra of derivations of cyclic Leibniz algebras of type (III).

Theorem 2. *Let L be a cyclic Leibniz algebra of type (III) over a field F. Then* $\mathrm{Der}(L)$ *is a subdirect product of the algebras D_1 and D_2, where D_1 is the algebra of derivations of a cyclic nilpotent Leibniz algebra, D_2 is the algebra of derivations of a cyclic Leibniz algebra of type (II).*

We recall that a structure of the algebra of derivations of nilpotent cyclic Leibniz algebras was described in [8].

1. Algebra of derivation of a cyclic Leibniz algebra of type (II)

We show here some basic elementary properties of derivations, which have been proved in [9].

Lemma 1. *Let L be a Leibniz algebra over a field F, and let f be a derivation of L. Then $f(\zeta^{left}(L) \leqslant \zeta^{left}(L)$, $f(\zeta^{right}(L)) \leqslant \zeta^{right}(L)$ and $f(\zeta(L)) \leqslant \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivaion of L. Then $f(\zeta_\alpha(L)) \leqslant \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F, and let f be a derivaion of L. Then $f(\gamma_\alpha(L)) \leqslant \gamma_\alpha(L)$ for all ordinals α, in particular, $f(\gamma_\infty(L)) \leqslant \gamma_\infty(L)$.*

Corollary 2. *Let L be a cyclic Leibniz algebra of type (II) over a field F, $L = A \oplus S$, where $A = [L, L] = \mathrm{Leib}(L)$, $S = Fc = \zeta^{right}(L)$. If f is an derivaion of L, then $f(A) \leqslant A$, $f(S) \leqslant S$, in particular, $f(c) = \sigma c$ for some $\sigma \in F$.*

We start from the case of a non-nilpotent cyclic Leibniz algebra L, having dimension 2. In this case, $L = Fa_1 \oplus Fa_2$, where $[a_1, a_1] = a_2$, $[a_2, a_2] = [a_2, a_1] = 0$, $[a_1, a_2] = a_2$ (see, e.g., survey [10]). This algebra has an interesting property: every its subalgebra is an ideal. Note that the Leibniz algebras whose subalgebras are ideals were described in [11].

Let f be an arbitrary derivaion of L. We have $f(a_1) = \gamma a_1 + \alpha a_2$ for some elements $\alpha, \gamma \in F$. Then

$$\begin{gathered} f(a_2) = f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] = \\ = [\gamma a_1 + \alpha a_2, a_1] + [a_1, \gamma a_1 + \alpha a_2] = \\ = \gamma a_2 + \gamma a_2 + \alpha a_2 = (2\gamma + \alpha) a_2. \end{gathered}$$

Put $c = a_1 - a_2$, then $Fc = \zeta^{right}(L)$. We have

$$\begin{gathered} f(c) = f(a_1 - a_2) = f(a_1) - f(a_2) = \\ = \gamma a_1 + \alpha a_2 - (2\gamma + \alpha) a_2 = \gamma a_1 - 2\gamma a_2. \end{gathered}$$

On the other hand, **Lemma 1** shows that $f(c) \in Fc$. It is possible, only if $\gamma = 0$. In this case, $f(a_1) = \alpha a_2$ and $f(a_2) = \alpha a_2$. In this case, we can see that $\mathrm{Der}(L) \cong F$, in particular, $\mathrm{Der}(L)$ is Abelian and has a dimension 1.

Now, we suppose that $\dim_F(L) > 2$.

Lemma 3. *Let L be a cyclic Leibniz algebra of type (II) over a field F, and let D be the annihilator of a subspace Fc in algebra $\mathrm{Der}(L)$. Then D is an ideal of $\mathrm{Der}(L)$ and a factor-algebra $\mathrm{Der}(L)/D$ has dimension at most 1.*

Proof. Let f be an arbitrary derivaion of L. Since

$$f(Fc) = f(\zeta^{right}(L)) \leqslant \zeta^{right}(L)$$

by **Lemma 1**, we obtain that $f(c) = \alpha c$ for some element $\alpha \in F$.

Clearly, D is a subalgebra of $\mathrm{Der}(L)$. Let f be an arbitrary derivaion, and let g be an element of D. Then

$$\begin{gathered} [f, g](c) = (f \circ g)(c) - (g \circ f)(c) = f(g(c)) - g(f(c)) = \\ = f(0) - g(\alpha c) = -\alpha g(c) = 0, \\ [g, f](c) = -[f, g](c) = 0, \end{gathered}$$

so that D is an ideal of $\mathrm{Der}(L)$.

The factor-algebra of $\mathrm{Der}(L)/\mathrm{Ann}_{\mathrm{Der}(L)}(Fc)$ is isomorphic with some subalgebra of the algebra of linear transformations of a vector space Fc. It follows that this factor-algebra has dimension 0 or 1. □

Lemma 4. *Let L be a cyclic Leibniz algebra of type (II) over a field F. If L has a derivaion f such that $f(c) \neq 0$, then $\mathrm{char}(F)$ divides $\dim_F(L) - 1$.*

Proof. By Corollary 2 $f(c) = \sigma c$ for some non-zero element σ of a field F. Put $g = \sigma^{-1} f$. Then g is a derivation, and $g(c) = c$. Let x be an arbitrary element of L. Then $x = a + \lambda c$ for some element $\lambda \in F$. We have

$$[c, x] = [c, a + \lambda c] = [c, a] + [c, \lambda c] = [c, a]$$

and

$$g([c, x]) = g([c, a]) = [g(c), a] + [c, g(a)] =$$
$$= [c, a] + [c, g(a)] = [c, a + g(a)] = [c, (g + \mathrm{Id}_L)(a)]$$

(here, Id_L is an identity permutation of L).

Thus, we obtain $g(\mathfrak{l}_c(a)) = \mathfrak{l}_c((g + \mathrm{Id}_L)(a))$.

If h is a derivaion of L, then **Lemma 2** shows that $h(A) \leqslant A$. Define now the mapping $h^{\downarrow}\colon A \to A$ by the rule: $f^{\downarrow}(a) = f(a)$ for every $a \in A$. It is not hard to prove that $h^{\downarrow}$ is a linear transformation of a vector space A. Then we obtain

$$g^{\downarrow} \circ \mathfrak{l}_c^{\uparrow} = \mathfrak{l}_c^{\uparrow} \circ (g^{\uparrow} + \mathrm{Id}_A).$$

Denote, by G (respectively, M), the matrix of a linear mapping $g^{\uparrow}$ (respectively, $\mathfrak{l}_c^{\uparrow}$) in a basis $\{a_2, \ldots, a_n\}$. Then we obtain the matrix equality $GM = M(G + E)$. As we have seen above, the matrix M is not singular. Thus, we obtain $M^{-1}GM = G + E$.

Since $\mathrm{trace}(G) = \mathrm{trace}(M^{-1}GM)$, we obtain

$$\mathrm{trace}(G) = \mathrm{trace}(G + E) = \mathrm{trace}(G) + (n-1)1_F$$

(here, 1_F is an identity element of a field F). It follows that $(n-1)1_F = 0$. In this case, $\mathrm{char}(F)$ divides $n - 1$. □

Lemma 5. *Let L be a cyclic Leibniz algebra of type (II) over a field F, and let D be the annihilator of a subspace Fc in algebra* $\mathrm{Der}(L)$*. Then D is generated as a vector space by the derivations $\mathfrak{i}, \mathfrak{l}_c, \mathfrak{l}_c^2, \ldots, \mathfrak{l}_c^{n-2}$. Moreover, the set $\{\mathfrak{i}, \mathfrak{l}_c, \mathfrak{l}_c^2, \ldots, \mathfrak{l}_c^{n-2}\}$ is a basis of D, so that D is Abelian and has a dimension $n - 1$.*

Proof. We note that the mapping $\mathfrak{i}$ is a derivaion of L. Indeed, if $y = b + \tau c$, $b \in A$, $\tau \in F$, is another element of L, then put

$$\mathfrak{i}([x, y]) = \mathfrak{i}([a + \sigma c, b + \tau c]) = \mathfrak{i}(\sigma[c, b]) = \sigma[c, b],$$
$$[\mathfrak{i}(x), y] + [x, \mathfrak{i}(y)] = [a, b + \tau c] + [a + \sigma c, b] = \sigma[c, b].$$

Let f is an arbitrary derivaion of D. Then

$$(f \circ \mathfrak{l}_c)(x) = f(\mathfrak{l}_c(x)) = f(\mathfrak{l}_c(a + \sigma c)) =$$
$$= f([c, a + \sigma c]) = f([c, a]) = [f(c), a] + [c, f(a)] = [c, f(a)],$$

$$(\mathfrak{l}_c \circ f)(x) = \mathfrak{l}_c(f(x)) = \mathfrak{l}_c(f(a+\sigma c)) = \mathfrak{l}_c(f(a)) = [c, f(a)].$$

Since it is true for every element $x \in L$, we obtain that $f \circ \mathfrak{l}_c = \mathfrak{l}_c \circ f$. We have

$$a_3 = [c, a_2] = \mathfrak{l}_c(a_2),$$
$$a_4 = [c, a_3] = \mathfrak{l}_c(a_3) = \mathfrak{l}_c(\mathfrak{l}_c(a_2)) = \mathfrak{l}_c^2(a_2), \ldots,$$
$$a_n = [c, a_{n-1}] = \mathfrak{l}_c^{n-2}(a_2).$$

Note that

$$\mathfrak{l}_c^{n-1}(a_2) = \mathfrak{l}_c(\mathfrak{l}_c^{n-2}(a_2)) = \mathfrak{l}_c(a_n) = \alpha_2 + \alpha_3\mathfrak{l}_c(a_2) + \ldots + \alpha_n\mathfrak{l}_c^{n-2}(a_2),$$

so that we can define $\mathfrak{l}_c^k(a_2)$ (and, hence, $\mathfrak{l}_c^k(a)$ for arbitrary $a \in A$) for each positive integer k.

Let f be an arbitrary element of an ideal D. Let

$$f(a_2) = \beta_0 a_2 + \beta_1 a_3 + \ldots + \beta_{n-2}a_n$$

for some elements $\beta_0, \ldots, \beta_{n-2} \in F$. Then we obtain the presentation

$$f(a_2) = \beta_0\mathfrak{i}(a_2) + \beta_1\mathfrak{l}_c(a_2) + \beta_2\mathfrak{l}_c^2(a_2) + \ldots + \beta_{n-2}\mathfrak{l}_c^{n-2}(a_2) =$$
$$= (\beta_0\mathfrak{i} + \beta_1\mathfrak{l}_c + \beta_2\mathfrak{l}_c^2 + \ldots + \beta_{n-2}\mathfrak{l}_c^{n-2})(a_2).$$

Put $\mathfrak{d}_f = \beta_0\mathfrak{i} + \beta_1\mathfrak{l}_c + \beta_2\mathfrak{l}_c^2 + \ldots + \beta_{n-2}\mathfrak{l}_c^{n-2}$, then $f(a_2) = \mathfrak{d}_f(a_2)$.

If a is an arbitrary element of A, then $a = \sigma_0 a_2 + \sigma_1 a_3 + \ldots + \sigma_{n-2}a_n$ for some elements $\sigma_0, \ldots, \sigma_{n-2} \in F$. We have

$$f(a_3) = f(\mathfrak{l}_c(a_2)) = \mathfrak{l}_c(f(a_2)) = \mathfrak{l}_c(\mathfrak{d}_f(a_2)) = \mathfrak{d}_f(\mathfrak{l}_c(a_2)) = \mathfrak{d}_f(a_3).$$

Similarly, we obtain that $f(a_4) = \mathfrak{d}_f(a_4)$, $\ldots$, $f(a_n) = \mathfrak{d}_f(a_n)$. It follows that

$$f(a) = f(\sigma_0 a_2 + \sigma_1 a_3 + \sigma_2 a_4 + \ldots + \sigma_{n-2}a_n) =$$
$$= \sigma_0 f(a_2) + \sigma_1 f(a_3) + \sigma_2 f(a_4) + \ldots + \sigma_{n-2}f(a_n) =$$
$$= \sigma_0\mathfrak{d}_f(a_2) + \sigma_1\mathfrak{d}_f(a_3) + \sigma_2\mathfrak{d}_f(a_4) + \ldots + \sigma_{n-2}\mathfrak{d}_f(a_n) =$$
$$= \mathfrak{d}_f(\sigma_0 a_2 + \sigma_1 a_3 + \sigma_2 a_4 + \ldots + \sigma_{n-2}a_n) = \mathfrak{d}_f(a).$$

If $x = a + \sigma c$, $a \in A$, $\sigma \in F$, is an arbitrary element of L, then

$$f(x) = f(a + \sigma c) = f(a) + \sigma f(c) = f(a),$$

and it implies that $f(x) = \mathfrak{d}_f(x)$.

Since it is true for every element $x \in L$, we obtain that $f = \mathfrak{d}_f$.

We note that the mappings $\mathfrak{i}, \mathfrak{l}_c, \mathfrak{l}_c^2, \ldots, \mathfrak{l}_c^{n-2}$ are linearly independent. Indeed, suppose that $\lambda_0, \lambda_1, \ldots, \lambda_{n-2}$ are the elements of F such that

$$\lambda_0\mathfrak{i} + \lambda_1\mathfrak{l}_c + \lambda_2\mathfrak{l}_c^2 + \ldots + \lambda_{n-2}\mathfrak{l}_c^{n-2} = 0.$$

Then

$$(\lambda_0\mathfrak{i} + \lambda_1\mathfrak{l}_c + \lambda_2\mathfrak{l}_c^2 + \ldots + \lambda_{n-2}\mathfrak{l}_c^{n-2})(a_2) = 0.$$

On the other hand,

$$\begin{gathered}(\lambda_0\mathfrak{i} + \lambda_1\mathfrak{l}_c + \lambda_2\mathfrak{l}_c^2 + \ldots + \lambda_{n-2}\mathfrak{l}_c^{n-2})(a_2) = \\ = \lambda_0 a_2 + \lambda_1\mathfrak{l}_c(a_2) + \lambda_2\mathfrak{l}_c^2(a_2) + \ldots + \lambda_{n-2}\mathfrak{l}_c^{n-2}(a_2) = \\ = \lambda_0 a_2 + \lambda_1 a_3 + \lambda_2 a_4 + \ldots + \lambda_{n-2} a_n.\end{gathered}$$

The fact that $\{a_2, a_3, \ldots, a_n\}$ is a basis of A shows that

$$\lambda_0 = \lambda_1 = \ldots = \lambda_{n-2} = 0. \qquad \square$$

2. Proof of Theorem A

Assertion (i) follows from Lemmas 3 and 5. Assertion (ii) follows from Lemma 3. Assertion (iii) follows from Lemma 4.

The following natural question appears from Lemma 4.

Let L be a cyclic Leibniz algebra of type (ii). Is $f(c) = 0$ for an arbitrary derivaion f of L?

The following example gives a negative answer on this question.

Example 1. Let $L = Fc \oplus Fa_2 \oplus Fa_3 \oplus Fa_4$ be a cyclic Leibniz algebra of type (II), having dimension 4 over a field $\mathbb{F}_3$ of order 3. Let

$$[c, a_2] = a_3, \qquad [c, a_3] = a_4, \qquad [c, a_4] = a_2.$$

Consider a linear transformation f of L, defined by the rule

$$\begin{gathered} f(c) = c, \\ f(a_2) = 2a_3 + a_4, \\ f(a_3) = a_2 + a_3 + 2a_4, \\ f(a_4) = 2a_2 + a_3 + 2a_4. \end{gathered}$$

Let $x = \gamma c + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4$, $y = \lambda c + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4$ be the arbitrary elements of L. We have

$$\begin{gathered}[x, y] = [\gamma c + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4, \lambda c + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4] = \\ = [\gamma c, \lambda c + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4] = \\ = \gamma\lambda_2 a_3 + \gamma\lambda_3 a_4 + \gamma\lambda_4 a_2;\end{gathered}$$

$$\begin{gathered}f([x, y]) = f(\gamma\lambda_4 a_2 + \gamma\lambda_3 a_4 + \gamma\lambda_2 a_3) = \\ = \gamma\lambda_4 f(a_2) + \gamma\lambda_3 f(a_4) + \gamma\lambda_2 f(a_3) = \\ = \gamma\lambda_4(2a_3 + a_4) + \gamma\lambda_3(2a_2 + a_3 + 2a_4) + \gamma\lambda_2(a_2 + a_3 + 2a_4) = \\ = (\gamma\lambda_2 + 2\gamma\lambda_3)a_2 + (\gamma\lambda_2 + \gamma\lambda_3 + 2\gamma\lambda_4)a_3 + (2\gamma\lambda_2 + 2\gamma\lambda_3 + \gamma\lambda_4)a_4;\end{gathered}$$

$$\begin{gathered}f(x) = f(\gamma c + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4) = \\ = \gamma f(c) + \gamma_2 f(a_2) + \gamma_3 f(a_3) + \gamma_4 f(a_4) = \\ = \gamma c + \gamma_2(2a_3 + a_4) + \gamma_3(a_2 + a_3 + 2a_4) + \gamma_4(2a_2 + a_3 + 2a_4) = \\ = \gamma c + (2\gamma_4 + \gamma_3)a_2 + (2\gamma_2 + \gamma_3 + \gamma_4)a_3 + (\gamma_2 + 2\gamma_3 + 2\gamma_4)a_4;\end{gathered}$$

$$f(y) = \lambda c + (2\lambda_4 + \lambda_3)a_2 + (2\lambda_2 + \lambda_3 + \lambda_4)a_3 + (\lambda_2 + 2\lambda_3 + 2\lambda_4)a_4;$$

$$\begin{gathered}[f(x), y] = [\gamma c + (2\gamma_4 + \gamma_3)a_2 + (2\gamma_2 + \gamma_3 + \gamma_4)a_3 + (\gamma_2 + 2\gamma_3 + 2\gamma_4)a_4, \\ \lambda c + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4] = [\gamma c, \lambda c + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4] = \\ = \gamma\lambda_2 a_3 + \gamma\lambda_3 a_4 + \gamma\lambda_4 a_2;\end{gathered}$$

$$\begin{gathered}[x, f(y)] = [\gamma c + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4, \\ \lambda c + (2\lambda_4 + \lambda_3)a_2 + (2\lambda_2 + \lambda_3 + \lambda_4)a_3 + (\lambda_2 + 2\lambda_3 + 2\lambda_4)a_4] = \\ = [\gamma c, \lambda c + (2\lambda_4 + \lambda_3)a_2 + (2\lambda_2 + \lambda_3 + \lambda_4)a_3 + (\lambda_2 + 2\lambda_3 + 2\lambda_4)a_4] = \\ = (2\gamma\lambda_4 + \gamma\lambda_3)a_3 + (2\gamma\lambda_2 + \gamma\lambda_3 + \gamma\lambda_4)a_4 + (\gamma\lambda_2 + 2\gamma\lambda_3 + 2\gamma\lambda_4)a_2;\end{gathered}$$

$$\begin{gathered}[f(x), y] + [x, f(y)] = \gamma\lambda_2 a_3 + \gamma\lambda_3 a_4 + \gamma\lambda_4 a_2 + (2\gamma\lambda_4 + \gamma\lambda_3)a_3 + \\ + (2\gamma\lambda_2 + \gamma\lambda_3 + \gamma\lambda_4)a_4 + (\gamma\lambda_2 + 2\gamma\lambda_3 + 2\gamma\lambda_4)a_2 = \\ = (\gamma\lambda_2 + 2\gamma\lambda_4 + \gamma\lambda_3)a_3 + (\gamma\lambda_3 + 2\gamma\lambda_2 + \gamma\lambda_3 + \gamma\lambda_4)a_4 + \\ + (\gamma\lambda_4 + \gamma\lambda_2 + 2\gamma\lambda_3 + 2\gamma\lambda_4)a_2 = \\ = (\gamma\lambda_2 + 2\gamma\lambda_3)a_2 + (\gamma\lambda_2 + 2\gamma\lambda_4 + \gamma\lambda_3)a_3 + (2\gamma\lambda_3 + 2\gamma\lambda_2 + \gamma\lambda_4)a_4 = \\ = [f(x), f(y)].\end{gathered}$$

These equalities show that f is a derivaion of L.

3. Algebra of derivations of a cyclic Leibniz algebra of type (III). Proof of Theorem B

We have $L = A \oplus Fd_1$, $A = V \oplus [U, U]$, $U = Fd_1 \oplus Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is a nilpotent cyclic subalgebra, i.e. is an algebra of type (I). Moreover, a subspace $[U, U] = Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is an ideal of L. Furthermore, $V = Fd_t \oplus \ldots \oplus Fd_n$ is an ideal of L, and $[a_1, d_j] = [d_1, d_j]$ for all $j \geqslant t$. In other words, $V \oplus Fd_1$ is a cyclic subalgebra of type (II).

Since $L/V \cong U$ is a cyclic nilpotent Leibniz algebra,

$$\mathrm{Der}(L)/\operatorname{Ann}_{\mathrm{Der}(L)}(L/V) = D_1$$

is an algebra of derivations of a cyclic nilpotent Leibniz algebra. Since $L/[U, U] \cong V \oplus Fd_1$ is a cyclic Leibniz algebra of the second type, $\mathrm{Der}(L)/\operatorname{Ann}_{\mathrm{Der}(L)}(L/[U, U]) = D_2$ is the algebra of derivations of a cyclic Leibniz algebra of type (II).

Let $f \in \operatorname{Ann}_{\mathrm{Der}(L)}(L/V) \cap \operatorname{Ann}_{\mathrm{Der}(L)}(L/[U, U])$, and let x be an arbitrary element of L. Then $f(x) \in V$, and, on the other hand, $f(x) \in [U, U]$. It follows that

$$f(x) \in V \cap [U, U] = \langle 0 \rangle, \text{ so that } f(x) = x.$$

Thus $\operatorname{Ann}_{\mathrm{Der}(L)}(L/V) \cap \operatorname{Ann}_{\mathrm{Der}(L)}(L/[U, U]) = \langle 0 \rangle$, and Remak's theorem yields the embedding of algebra $\mathrm{Der}(L)$ into the direct product $D_1 \times D_2$.

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