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On the kernels of higher R-derivations of $R[x_1, \ldots, x_n]$*

S. Kour

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Abstract. Let R be an integral domain and $A = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables. In this article, we study the kernel of higher R-derivation D of A. It is shown that if R is a HCF ring and $\operatorname{tr.deg}_R(A^D) \leqslant 1$ then $A^D = R[f]$ for some $f \in A$.

Introduction

Derivations of polynomial rings and their kernels have been the objects of great interest for a long time because of their connections with several conjunctures and popular problems in Commutative Algebra and Algebraic Geometry. For a good collection of results on derivations of a polynomial ring containing a field of characteristic $p \geqslant 0$, one can refer to [3, 5–7].

Recently, the kernels of higher derivations have been studied by several mathematicians and many interesting and known results of derivations have been extended to higher derivations e.g. [5, 8]. Let R be a field. In [6], Nowicki and Nagata studied the structure of R-derivations of $R[x_1, x_2]$. They proved that the kernel of any non-trivial R-derivation of $R[x_1, x_2]$ will be of the form $R[f]$ for some $f \in R[x_1, x_2]$. The above result is generalised by Berson in [2] and by Kahoui in [3] to the polynomial ring $R[x_1, x_2]$ over a unique factorization domain R. More precisely, they proved that if R is a unique factorization domain and d is a non trivial R-derivation of

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$A = R[x_1, x_2]$, then there exists $f \in A$ such that $A^d = R[f]$. Nowicki in [7] observed that if R is a field and d is an R-derivation of $A = R[x_1, \ldots, x_n]$ with $\operatorname{tr.deg}_R(A^d) \leqslant 1$ then $A^d = R[f]$ for some $f \in A$.

In [8], Wada studied the kernels of higher R-derivations of $R[x_1, x_2]$ and $R(x_1, x_2)$, where R denotes a field. Later, Kojima and Wada in [5] extended the study of the kernels of higher R-derivations to the polynomial ring $R[x_1, x_2]$ over a HCF (Highest Common Factor) ring R. They proved that if R is a HCF ring, then the kernel of any non-trivial higher R-derivation of $R[x_1, x_2]$ will be of the from $R[h]$ for some $h \in R[x_1, x_2]$. The above result on the kernels of higher derivations is same as Theorem 2.8 of Nowicki and Nagata [6] and Corollary 3.2 of Kahoui [3] in the context of ordinary derivation. There are many other interesting results on derivations which have been extended to higher derivations.

Let R be a HCF ring and let $A = R[x_1, \ldots, x_n]$. In this article, we prove that if D is a non-trivial higher R-derivation on A with $\operatorname{tr.deg}_R(A^D) \leqslant 1$, then $A^d = R[f]$ for some $f \in A$. Kojima and Wada's Theorem 1.1 in [5] can be obtained as a corollary to our result. We would also like to mention that our result is obtained by using the arguments similar to Proposition 4.8 of [1] and Theorem 1.1 of [5].

1. Preliminaries

Let us first recall some definitions and basic facts about higher derivation. Let R be an integral domain and A be an R-algebra. Let F be an R-algebra containing A as R-subalgebra. Then a higher R-derivation on A into F is a set of R-linear homomorphisms $D = \{D_n\}_{n \geqslant 0}$ from A to F satisfying the following conditions:

1) D_0 is the identity map of A.
2) For any integer $n \geqslant 0$ and for any $a, b \in A$,

$$D_n(ab) = \sum_{i+j=n} D_i(a) D_j(b).$$

A higher R-derivation on A into F is called higher R-derivation of A (resp. a rational higher R-derivation of A) if $F = A$ (resp. $F = qt(A)$). If $D = \{D_n\}_{n \geqslant 0}$ is a higher R-derivation of A, then D_1 is a derivation of A. The kernel A^D of a higher R-derivation $D = \{D_n\}_{n \geqslant 0}$ is given by the set $\{a \in A \mid D_i(a) = 0 \; \forall \; i \geqslant 1\} = \cap_{i \geqslant 1} ker D_i$. It is easy to see that A^D is an R-subalgebra of A. We say that D is a non trivial if $A^D \neq A$.

Let $D = \{D_n\}_{n \geqslant 0}$ be a higher R-derivation on A. Then D defines an R-algebra homomorphism $\Phi_D : A \to A[|t|]$ given by $\Phi_D(a) = \sum_{i \geqslant 0} D_i(a) t^i$.

Φ_D is called the R-algebra homomorphism associated to D. Clearly, the kernel A^D can be seen as the set $\{a \in A \mid \Phi_D(a) = a\}$. The R-algebra homomorphism Φ_D is naturally extended to the $k = qt(R)$-algebra homomorphism $\Phi : qt(A) \to qt(A)[|t|]$ by defining

$$\Phi(\frac{a}{b}) = \frac{\Phi_D(a)}{\Phi_D(b)} \quad \forall\, a, b \in A,\ b \neq 0.$$

The k-algebra homomorphism Φ defines a higher k-derivation $\bar{D} = \{\bar{D}_n\}_{n\geqslant 0}$ on $qt(A)$ such that

$$\Phi(x) = \sum_{i\geqslant 0} \bar{D}_i(x)t^i \ \forall\ x \in qt(A).$$

Note that $\bar{D}_i|A = D_i\ \forall\ i \geqslant 0$. The k-derivation $\bar{D} = \{\bar{D}_n\}_{n\geqslant 0}$ is called the extension of D to $qt(A)$. From the definition, it is clear that $qt(A^D) \subseteq qt(A)^{\bar{D}}$.

Let D be a higher R derivation of A and let $A_k = k \otimes_R A$, where k denotes the quotient field $qt(R)$. Note that the R-algebra homomorphism $\Phi_D : A \to A[|t|]$ can be extended to a k-algebra homomorphism $\Phi_k : A_k \to A_k[|t|]$ defined as $\Phi_k(f/a) = \Phi_D(f)/a$ for $a \in A \setminus \{0\}$. Let D_k be the higher k-derivation given by Φ_k.

We first prove the following lemma which is similar to Lemma 2.4 of [5].

Lemma 1. *Let R be an integral domain and D be a non-trivial higher R-derivation of $A = R[x_1,\ldots,x_n]$. Then* $\operatorname{tr.deg}_R(A^D) \leqslant n-1$.

Proof. Let $k = qt(R)$ and let D_k be the higher k-derivation of $A_k = k\otimes_R A$ defined by Φ_k. Then $A_k^{D_k} \cap A = A^D$ and $qt(A^D) = qt(A_k^{D_k})$. Suppose, if possible, $\operatorname{tr.deg}_R(A^D) = n$. Then $\operatorname{tr.deg}_R(A^D) \leqslant \operatorname{tr.deg}_k(A_k^{D_k})$ implies that $\operatorname{tr.deg}_k(A_k^{D_k}) = n$. Now by Lemma 2.4 of [5], D_k is a trivial derivation. Therefore, D is a trivial derivation. □

2. Kernel of higher R-derivation

In this section, we discuss the kernel of higher R-derivation of the polynomial ring $R[x_1,\ldots,x_n]$ over a HCF ring R.

Theorem 1. *Let R be a HCF ring and let $A = R[x_1,\ldots,x_n]$ be the polynomial ring in n variables. Let D be a higher R-derivation of A with* $\operatorname{tr.deg}_R(A^D) \leqslant 1$. *Then $A^D = R[f]$ for some $f \in A$.*

Proof. We first consider the case where R is a field. In this case the argument goes along the lines of the proof of [7, Theorem 7.1.4], . Let $\operatorname{tr.deg}_R(A^D) = m$. Clearly, $m \leqslant 1$. If $m = 0$, then $A^D = R = R[1]$. If $m = 1$, then A^D is a Dedekind subring of A. Therefore, by Zaks' [9, Theorem 8], there exists $f \in A \setminus R$ such that $A^D = R[f]$.

Now, we consider the case where R is a HCF ring. Let $k = qt(R)$. Then the higher R-derivation D can be extended to a higher k-derivation D_k on $A_k = k \otimes_R A$, associated with the ring homomorphism $\Phi_k : A_k \to A_k[|t|]$. Note that $A_k^{D_k} \cap A = A^D$ and $qt(A^D) = qt(A_k^{D_k})$.

Since D_k is a higher k-derivation on A_k with $\operatorname{tr.deg}_k(A_k) \leqslant 1$, $A_k^{D_k} = k[f]$ for some $f \in A_k$. If $f \in k$, then $A_k^{D_k} = k$. Therefore, $A^D = A_k^{D_k} \cap A = k \cap A = R$. Suppose $f \in A_k \setminus k$. Clearly, $f \in A_k^{D_k}$. Since $k \subseteq A_k^{D_k}$, we may choose f in A^D. Furthermore, we may assume that the constant term of f is zero. Now the rest of the argument is similar to the proof of Proposition 4.8 of [1].

Observe that we may assume the greatest common divisor of the coefficients of f as polynomial in $x_1, x_2, \dots, x_n$ over R is one. If not, then let $r_1, \dots, r_m$ be the coefficients of f and let r be the greatest common divisor of $r_1, \dots, r_m$ (It exists as R is a HCF ring). Then, we can write $f = rf'$ and $f' \in A^D$.

Now recall that there exists a family $\{V_\alpha\}_{\alpha\in\Lambda}$ of valuation rings such that $R = \cap_\alpha V_\alpha$ and associated valuation maps are lattice homomorphisms. Observe that $(r_1, \dots, r_m)V_\alpha = V_\alpha$ for each $\alpha \in \Lambda$. Further, note that the valuation ring V_α can be extended to a valuation ring V'_α of $A_k = k[x_1, \dots, x_n]$ by defining the value of a polynomial to be the minimum of the values of its coefficients. Then x_i's are units in V'_α and the residues of the x_i modulo the maximal ideal of V'_α are algebraically independent over the residue field of V'_α. Since f has the constant term zero and $(r_1, \dots, r_m)V_\alpha = V_\alpha$, the residue of f modulo the maximal ideal of V'_α is transcendental over the residue field of V_α. Therefore, V'_α-value of any polynomial in $k[f]$ is the minimum of the values of its coefficients. Hence, $a_1 f + a_2 f^2 + \cdots + a_t f^t \in k[f] \cap A$, implies that for all $1 \leqslant i \leqslant t$, $a_i \in V_\alpha$. Since $R = \cap_\alpha V_\alpha$ we have $k[f] \cap A = R[f] = A^D$. □

Corollary 1. *[5, theorem 1.1] Let R be a HCF ring and let D be a non-trivial higher R-derivation of $A = R[x_1, x_2]$. Then there exists $f \in A$ such that $A^D = R[f]$.*

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Contact information

Surjeet Kour — Discipline of Mathematics, Indian Institute of Technology Delhi, India, 110016.
E-Mail(s): surjeetkour@maths.iitd.ac.in, koursurjeet@gmail.com
Web-page(s): sites.google.com/view/surjeetkour

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