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General formal local cohomology modules

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Abstract. Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. In this paper, we define and study general formal local cohomology modules. We denote the i-th general formal local cohomology module M with respect to Φ by $\mathfrak{F}^i_\Phi(M)$ and we investigate some finiteness and Artinianness properties of general formal local cohomology modules.

Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R, Φ a system of ideals of R and M is an R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_\mathfrak{a}(M)$. There are some generaliztions of local cohomology theory. The following one is given in [2]. A system of ideals of R is a non-empty set Φ of ideals of R such that, whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, there exists $\mathfrak{c} \in \Phi$ with $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. For such a system, there is a Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(R) \to \mathcal{C}(R)$ (where $\mathcal{C}(R)$ denotes the category of R-modules and R-homomorphisms) such that for every R-module M,

$$\Gamma_\Phi(M) := \{x \in M : \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \text{ in } \Phi\}.$$

In [2], $\Gamma_\Phi(-)$ is called the "general local cohomology functor with respect to Φ". For each $i \geqslant 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $\mathrm{H}^i_\Phi(-)$.

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For more details about general local cohomology modules see [2], [3].

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geqslant 0$; $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$.

The formal local cohomology modules have been studied by several authors; see for example [1], [4], [6], [9] and [10]. The purpose of this paper is to make a generalization of formal local cohomology theory as above. There are some generalization of formal local cohomology theory (see [7] and [11]). Here, we give a new generalization in terms of a system of ideals.

Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. For each $i \geqslant 0$; we define i-th general formal local cohomology of M with respect to Φ by

$$\mathfrak{F}^i_{\Phi}(M) := \varprojlim_{\mathfrak{a} \in \Phi} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}M).$$

Clearly, for an ideal $\mathfrak{a}$ of R, if we put $\Phi := \{\mathfrak{a}^i | i \in \mathbb{N}\}$ then the above definition coincides with the original definition $\mathfrak{F}^i_{\mathfrak{a}}(M)$.

In this paper, we get some results on Artinianness, vanishing and other properties of general formal local cohomology modules. Among other things, we will prove that, for any finitely generated R-module M we have:

$$\begin{aligned} &\inf\{i \in \mathbb{N} : \mathfrak{F}^i_{\Phi}(M) \text{ is not representable}\} \\ &\qquad = \inf\{i \in \mathbb{N} : \mathfrak{F}^i_{\Phi}(M) \text{ is not Artinian}\} \end{aligned}$$

and

$$\begin{aligned} &\sup\{i \in \mathbb{N} : \mathfrak{F}^i_{\Phi}(M) \text{ is not representable}\} \\ &\qquad = \sup\{i \in \mathbb{N} : \mathfrak{F}^i_{\Phi}(M) \text{ is not Artinian}\}. \end{aligned}$$

Also, we study the structure of 0-th general formal local cohomology module and we will prove that for a complete local ring $(R, \mathfrak{m})$,

$$\mathrm{Ass}_R \, \mathfrak{F}^0_{\Phi}(M) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim R/(\mathfrak{a} + \mathfrak{p}) = 0 \text{ for all } \mathfrak{a} \in \Phi\}.$$

Recall that, $\mathrm{Assh}_R(M)$ denotes the set $\{\mathfrak{p} \in \mathrm{Ass}\, M : \dim R/\mathfrak{p} = \dim M\}$. We show that $\mathfrak{F}^{\dim M}_{\Phi}(M)$ is Artinian and there exists an ideal $\mathfrak{a}$ in Φ such that $\mathrm{Att}_R \, \mathfrak{F}^d_{\Phi}(M) = \mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$.

1. Results

Assume that $(R, \mathfrak{m})$ is a local ring and that M is a finitely generated R-module. We investigate a generalization of formal local cohomology theory in terms of a system of ideals. A system of ideals of R is a non-empty set Φ of ideals of R such that, whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, there exists $\mathfrak{c} \in \Phi$ with $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. We define the relation $\leqslant$ on Φ by: $\mathfrak{a} \leqslant \mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$. It is easy to see that Φ is a direct set by this relation. Now, let $\mathfrak{a}, \mathfrak{b} \in \Phi$ such that $\mathfrak{a} \leqslant \mathfrak{b}$, M be an R-module.Then for each integer $n \geqslant 0$, the R-homomorphism $M/\mathfrak{b}M \to M/\mathfrak{a}M$ induces the R-homomorphism $\psi_{\mathfrak{a}}^{\mathfrak{b}} : \mathrm{H}_{\mathfrak{m}}^{n}(M/\mathfrak{b}M) \to \mathrm{H}_{\mathfrak{m}}^{n}(M/\mathfrak{a}M)$. Thus $\{\mathrm{H}_{\mathfrak{m}}^{n}(M/\mathfrak{a}M), \psi\}$ forms an inverse system of R-modules and R-homomorphisms over Φ. Now we are ready to give the following definition.

Definition 1. Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. For each $i \geqslant 0$; $\mathfrak{F}_{\Phi}^{i}(M) := \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}M)$ is called the i-th general formal local cohomology of M with respect to Φ.

Theorem 1. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. For each $i \geqslant 0$; $\mathfrak{F}_{\Phi}^{i}(M) \simeq \varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}_{\mathfrak{a}}^{i}(M)$.*

Proof. Let $\mathfrak{a}, \mathfrak{b} \in \Phi$ such that $\mathfrak{a} \leqslant \mathfrak{b}$. If n is an integer then the natural homomorphism $M/\mathfrak{b}^n M \to M/\mathfrak{a}^n M$ induces the homomorphism $\mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}^n M) \to \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^n M)$ for any integer $i \geqslant 0$. On the other hand, if $n \leqslant m$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}^n M) & \longrightarrow & \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^n M) \\ \uparrow & & \uparrow \\ \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}^m M) & \longrightarrow & \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{a}^m M) \end{array}$$

From the above diagram we get a homomorphism

$$\lambda_{\mathfrak{a}}^{\mathfrak{b}} : \varprojlim_{n} \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}^n M) \to \varprojlim_{n} \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}^n M)$$

and so, we have

$$\lambda_{\mathfrak{a}}^{\mathfrak{b}} : \mathfrak{F}_{\mathfrak{b}}^{i}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i}(M).$$

This shows that $\{\mathfrak{F}_{\mathfrak{a}}^{i}(M), \lambda\}_{\mathfrak{a}\in\Phi}$ is an inverse system of R-modules and R-homomorphisms over the directed set Φ. Thus we may form $\varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}_{\mathfrak{a}}^{i}(-)$.

But, for each integer $k \in \mathbb{N}$ and any ideal $\mathfrak{a} \in \Phi$ there exists an ideal $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \mathfrak{a}^k$. Thus, by using a proof similar to the proof of [12, Lemma 3.8] for each integer k we have

$$\varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}M) \simeq \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)$$

and so

$$\begin{aligned}\varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}^i_{\mathfrak{a}}(M) &\simeq \varprojlim_{\mathfrak{a}\in\Phi} \varprojlim_{k} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M) \simeq \varprojlim_{k}\varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^k M)\\ &\simeq \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}M) \simeq \mathfrak{F}^i_{\Phi}(M).\end{aligned}$$ □

Let $(R,\mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let $\underline{x}$ denotes a system of elements of R such that $\mathfrak{m} = Rad(\underline{x}R)$. Let $\check{C}_{\underline{x}}$ denotes the Čech complex of R with respect to $\underline{x}$. For an R-module M and an ideal $\mathfrak{a}$, it is easy to see that there exists an inverse system of R-complexes $\{\check{C}_{\underline{x}} \otimes M/\mathfrak{a}M\}_{\mathfrak{a}\in\Phi}$. Hence, we may form the inverse limit $\varprojlim_{\mathfrak{a}\in\Phi}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}M)$. By a proof similar to the proof of [12, proposition 3.2] we obtain the next result.

Theorem 2. *With the previous notation, there is an isomorphism*

$$\mathfrak{F}^i_{\Phi}(M) \simeq \mathrm{H}^i(\varprojlim_{\mathfrak{a}\in\Phi}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}M))$$

for all $i \in \mathbb{Z}$.

Proof. It follows by a straightforward modification of the proof of [12, proposition 3.2]. □

Theorem 3. *Let* $(R,\mathfrak{m})$ *be a local ring,* Φ *a system of ideals of* R *and* M *a finitely generated* R*-module. Then* $\mathfrak{F}^i_{\Phi}(M) = 0$ *for all* $i > \dim(M)$.

Proof. Let $i > \dim(M)$. By [12, Theorem 4.5] $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $\mathfrak{a} \in \Phi$. Thus $\mathfrak{F}^i_{\Phi}(M) = \varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}^i_{\mathfrak{a}}(M) = 0$, as required. □

Let $f : R \to R'$ be a homomorphism of Noetherian commutative rings. Set $\Phi R' := \{\mathfrak{a}R' : \mathfrak{a} \in \Phi\}$. Then $\Phi R'$ is a system of ideals of R'. Now by using this notation we give the following result:

Theorem 4. *Let* $(R,\mathfrak{m})$ *be a local ring,* Φ *a system of ideals of* R *and* M *a finitely generated* R*-module. Then* $\mathfrak{F}^i_{\Phi}(M) \simeq \mathfrak{F}^i_{\Phi\widehat{R}}(\widehat{M})$ *for all* $i \in \mathbb{Z}$.

Proof. By [12, Proposition 3.3], $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}\widehat{R}}(\widehat{M})$. Thus $\varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}^i_{\mathfrak{a}\widehat{R}}(\widehat{M})$. Now Theorem 1 completes the proof. □

Recall that a dualizing complex $D^{\cdot}_R$ for a local ring $(R, \mathfrak{m})$ is a bounded complex of injective R-modules whose cohomology modules $\mathrm{H}^i(D^{\cdot}_R)$ are finitely generated R-modules for all $i \in \mathbb{Z}$. For more details see [13]. It is well known that R possesses a dualizing complex if and only if R is the factor ring of a Gorenstein ring. The next result is an expression of the general formal local cohomology in terms of a certain general local cohomology of the dualizing complex.

Theorem 5. *Let $(R, \mathfrak{m})$ be a local ring possessing a dualizing complex $D^{\cdot}_R$, Φ a system of ideals of R and M a finitely generated R-module. Then*

$$\mathfrak{F}^i_\Phi(M) \simeq \mathrm{Hom}_R(\mathrm{H}^{-i}_\Phi(\mathrm{Hom}_R(M, D^{\cdot}_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$.

Proof. By Local Duality Theorem there are the isomorphisms

$$\mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}M) \simeq \mathrm{Hom}_R(\mathrm{H}^{-i}(\mathrm{Hom}_R(M/\mathfrak{a}M, D^{\cdot}_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$ and $\mathfrak{a} \in \Phi$. Thus we have

$$\varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}M) \simeq \mathrm{Hom}_R(\mathrm{H}^{-i}(\varinjlim_{\mathfrak{a}\in\Phi} \mathrm{Hom}_R(M/\mathfrak{a}M, D^{\cdot}_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$. But $\varinjlim_{\mathfrak{a}\in\Phi} \mathrm{Hom}_R(M/\mathfrak{a}M, D^{\cdot}_R)) \simeq \Gamma_\Phi(\mathrm{Hom}_R(M, D^{\cdot}_R))$ and so

$$\varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}M) \simeq \mathrm{Hom}_R(\mathrm{H}^{-i}(\Gamma_\Phi(\mathrm{Hom}_R(M, D^{\cdot}_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$. Therefore

$$\mathfrak{F}^i_\Phi(M) \simeq \mathrm{Hom}_R(\mathrm{H}^{-i}_\Phi(\mathrm{Hom}_R(M, D^{\cdot}_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$, as required. □

Theorem 6. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and $0 \to A \to B \to C \to 0$ a short exact sequence of finitely generated R-modules. Then there is a long exact sequence*

$$\cdots \to \mathfrak{F}^i_\Phi(A) \to \mathfrak{F}^i_\Phi(B) \to \mathfrak{F}^i_\Phi(C) \to \mathfrak{F}^{i+1}_\Phi(A) \to \cdots .$$

Proof. It follows by an argument similar to the proof of [12, Theorem 3.11]. □

Theorem 7. *Let* $(R, \mathfrak{m})$ *be a local ring,* Φ *a system of ideals of* R *and* M *a finitely generated* R*-module. If* $w := \max\{\dim(M/\mathfrak{a}M) | \mathfrak{a} \in \Phi\}$ *is finite then* $\mathfrak{F}^w_\Phi(M) \neq 0$ *and* $\mathfrak{F}^i_\Phi(M) = 0$ *for all* $i > w$.

Proof. Let $i > w$. Since $i > \dim(M/\mathfrak{a}M)$ for all $\mathfrak{a} \in \Phi$, [12, Theorem 4.5] implies that $\mathfrak{F}^i_\mathfrak{a}(M) = 0$ for all $\mathfrak{a} \in \Phi$. Thus $\mathfrak{F}^i_\Phi(M) = \varprojlim_{\mathfrak{a}\in\Phi} \mathfrak{F}^i_\mathfrak{a}(M) = 0$. On the other hand, since w is finite there exists an ideal $\mathfrak{b} \in \Phi$ such that $\dim(M/\mathfrak{b}M) = w$. Now, put $\Psi = \{\mathfrak{c} \in \Phi \mid \mathfrak{c} \subseteq \mathfrak{b}\}$. Then Ψ is cofinal in Φ. Thus we may assume that $\mathfrak{a} \subseteq \mathfrak{b}$ for all $\mathfrak{a} \in \Phi$. Let $\mathfrak{c} \in \Phi$. It is easy to see that $\dim(\mathfrak{b}M/\mathfrak{c}M) \leqslant \dim M/\mathfrak{c}M \leqslant w$ and so the exact sequence $0 \to \mathfrak{b}M/\mathfrak{c}M \to M/\mathfrak{c}M \to M/\mathfrak{b}M \to 0$ induces $\mathrm{H}^w_\mathfrak{m}(M/\mathfrak{c}M) \to \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) \to 0$. Now for each $\mathfrak{d} \in \Phi$ with $\mathfrak{d} \leqslant \mathfrak{c}$ i.e $\mathfrak{c} \subseteq \mathfrak{d}$ we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}^w_\mathfrak{m}(M/\mathfrak{d}M) & \xrightarrow{f_\mathfrak{d}} & \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
\mathrm{H}^w_\mathfrak{m}(M/\mathfrak{c}M) & \xrightarrow{f_\mathfrak{c}} & \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) & \longrightarrow & 0
\end{array}$$

The family of R-modules $\{\ker f_\mathfrak{c}\}_{\mathfrak{c}\in\Phi}$, as a family of Artinian R-modules, satisfies the Mittag-Leffler condition. Hence the above diagram induces an exact sequence $\varprojlim_{\mathfrak{c}\in\Phi} \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{c}M) \to \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) \to 0$. By Theorem 1 we get $\mathfrak{F}^w_\Phi(M) \to \mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) \to 0$. By Grothendieck's non-vanishing Theorem $\mathrm{H}^w_\mathfrak{m}(M/\mathfrak{b}M) \neq 0$. Therefore $\mathfrak{F}^w_\Phi(M) \neq 0$, as required. □

Theorem 8. *Let* $(R, \mathfrak{m})$ *be a local ring,* Φ *a system of ideals of* R *and* M *a finitely generated* R*-module of dimension* d*. Then* $\mathfrak{F}^d_\Phi(M)$ *is homomorphic image of* $\mathrm{H}^d_\mathfrak{m}(M)$*, and so* $\mathfrak{F}^d_\Phi(M)$ *is Artinian.*

Proof. Let $\mathfrak{a} \in \Phi$. We have $\dim \mathfrak{a}M \leqslant \dim M$, so that, by the Grothendieck's Vanishing Theorem, the short exact sequence

$$0 \longrightarrow \mathfrak{a}M \longrightarrow M \longrightarrow M/\mathfrak{a}M \longrightarrow 0$$

induces an exact sequence

$$\mathrm{H}^d_\mathfrak{m}(M) \xrightarrow{\phi_\mathfrak{a}} \mathrm{H}^d_\mathfrak{m}(M/\mathfrak{a}M) \longrightarrow 0.$$

The family of R-modules $\{\ker \phi_\mathfrak{a}\}_{\mathfrak{a}\in\Phi}$, as a family of Artinian R-modules, satisfies the Mittag-Leffler condition. Therefore, the above exact sequence induces an exact sequence $\varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^d_\mathfrak{m}(M) \to \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^d_\mathfrak{m}(M/\mathfrak{a}M) \to 0$ and so we have the exact sequence $\mathrm{H}^d_{\mathfrak{m}}(M) \to \mathfrak{F}^d_{\Phi}(M) \to 0$, and the proof is complete. $\square$

In the next result, we investigate the 0-th general formal local cohomology module. Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. For a submodule N of M we denote the ultimate constant value of the increasing sequence

$$N \subseteq N :_M \mathfrak{a} \subseteq N :_M \mathfrak{a}^2 \subseteq \cdots \subseteq N :_M \mathfrak{a}^i \subseteq \cdots$$

by $N :_M \langle \mathfrak{a} \rangle$. Let $0 = \bigcap_{j=1}^n Q_j$ denotes a reduced primary decomposition of the zero submodule 0 in M and Q_j is a $\mathfrak{p}_j$-primary submodule of M, for all $j = 1, \cdots, n$. Put $T(\mathfrak{a}, M) := \{\mathfrak{p} \in \mathrm{Ass}_R M : \dim R/(\mathfrak{a} + \mathfrak{p}) > 0\}$ and $u_M(\mathfrak{a}) := \bigcap_{\mathfrak{p}_i \in T(\mathfrak{a},M)} Q_i$ also $T(\Phi, M) := \{\mathfrak{p} \in \mathrm{Ass}_R M :$ there exists $\mathfrak{a} \in \Phi$ such that $\dim R/(\mathfrak{a}+\mathfrak{p}) > 0\}$ and $u_M(\Phi) := \bigcap_{\mathfrak{p}_i \in T(\Phi,M)} Q_i$. With these notations we have:

Theorem 9. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Then*

i) $\bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = u_M(\Phi)$,

ii) $u_M(\Phi) = \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle)$,

iii) $\mathfrak{F}^0_{\Phi}(M) \simeq u_{\hat{M}}(\Phi \hat{R})$.

Proof. i) It is easy to see that

$$\bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{\mathfrak{p}_i \in T(\mathfrak{a},M)} Q_i = \bigcap_{\mathfrak{p}_i \in T(\Phi,M)} Q_i = u_M(\Phi).$$

ii) By [12, Lemma 4.1(a)], $u_M(\mathfrak{a}) = \bigcap_{n \geqslant 1} (\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle)$. Thus

$$u_M(\Phi) = \bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{n \geqslant 1} (\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle) \subseteq \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle).$$

Conversly, let $x \in \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M \langle \mathfrak{m} \rangle)$. Let $\mathfrak{a} \in \Phi$ be an ideal. Then there exists an integer u such that $x\mathfrak{m}^u \subseteq \mathfrak{a}M$. For any integer k, there exists an ideal $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \mathfrak{a}^k$. Since $x \in (\mathfrak{b}M :_M \langle \mathfrak{m} \rangle)$ there exists an integer t such that $x\mathfrak{m}^t \subseteq \mathfrak{b}M \subseteq \mathfrak{a}^k M$. Hence $x \in (\mathfrak{a}^k M :_M \langle \mathfrak{m} \rangle)$ and so $x \in \bigcap_{n \geqslant 1} (\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle)$ for each ideal $\mathfrak{a} \in \Phi$. Therefore $x \in \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{n \geqslant 1} (\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle) = u_M(\Phi)$.

iii) By Theorem 4 we may assume that $M = \hat{M}$ and $R = \hat{R}$. Let b be a proper ideal of R such that $\mathfrak{b} \in \Phi$. It is easy to see that $\bigcap_{\mathfrak{a} \in \Phi} \mathfrak{a}M \subseteq \bigcap_{n \geqslant 0} \mathfrak{b}^n M$. Thus Krull's intersection theorem implies that $\bigcap_{\mathfrak{a} \in \Phi} \mathfrak{a}M = 0$. Now the proof is a straightforward modification of the proof of [12, Lemma 4.1(c)]. $\square$

Corollary 1. *Let $(R, \mathfrak{m})$ be a complete local ring, Φ a system of ideals of R and M a finitely generated R-module. Then*

$$\operatorname{Ass}_R \mathfrak{F}^0_\Phi(M) = \{\mathfrak{p} \in \operatorname{Ass}_R M : \dim R/(\mathfrak{a}+\mathfrak{p}) = 0 \textit{ for all } \mathfrak{a} \in \Phi\}.$$

Proof. By [12, Lemma 2.7] $\operatorname{Ass}_R u_M(\Phi) = \operatorname{Ass}_R M \setminus T(\Phi, M)$. But

$$\operatorname{Ass}_R M \setminus T(\Phi, M) = \{\mathfrak{p} \in \operatorname{Ass}_R M : \dim R/(\mathfrak{a}+\mathfrak{p}) = 0 \text{ for all } \mathfrak{a} \in \Phi\}$$

and $\mathfrak{F}^0_\Phi(M) = u_M(\Phi)$ by Theorem 9(iii) and this finishes the proof. □

Corollary 2. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Then $\mathfrak{F}^0_\Phi(M) = 0$ if and only if $\operatorname{Ass}_{\hat{R}} \hat{M} = T(\Phi\hat{R}, \hat{M})$.*

Proof. By Theorem 3(iii) $\mathfrak{F}^0_\Phi(M) = 0$ if and only if $u_{\hat{M}}(\Phi\hat{R}) = 0$. But $\operatorname{Ass}_{\hat{R}} u_{\hat{M}}(\Phi\hat{R}) = \operatorname{Ass}_{\hat{R}} \hat{M} \setminus T(\Phi\hat{R}, \hat{M})$ by [12, Lemma 2.7]. Thus $u_{\hat{M}}(\Phi\hat{R}) = 0$ if and only if $\operatorname{Ass}_{\hat{R}} \hat{M} = T(\Phi\hat{R}, \hat{M})$ and the proof is complete. □

The next theorem gives a result for representable general formal local cohomology modules.

Theorem 10. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let i be an integer such that $\mathfrak{F}^i_\Phi(M)$ is nonzero and representable. Then there exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}_R \mathfrak{F}^i_\Phi(M)$.*

Proof. Let $\mathfrak{F}^i_\Phi(M) = S_1 + S_2 + ... + S_n$ be a minimal secondary representation of $\mathfrak{F}^i_\Phi(M)$ where S_j is non-zero and $\mathfrak{p}_j$-Secondary for $j = 1, 2, ..., n$.

Let $1 \leqslant j \leqslant n$. Since $S_j \neq 0$, there exists $0 \neq a = (a_i) \in S_j \subseteq \mathfrak{F}^i_\Phi(M) = \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}M)$.

Let a_k be the first nonzero component of a. Thus there exists an ideal $\mathfrak{a}_k \in \Phi$ such that $a_k \in \mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}_k M)$. We claim $\mathfrak{a}_k \subseteq \mathfrak{p}_j$. If not, then there exists $u \in \mathfrak{a}_k \setminus \mathfrak{p}_j$. Since $u \notin \mathfrak{p}_j$, we have $uS_j = S_j$. Thus $a \in S_j = uS_j \subseteq u\mathfrak{F}^i_\Phi(M)$ But $u\,\mathrm{H}^i_\mathfrak{m}(M/\mathfrak{a}_k M) = 0$ and so the k-th component of each element of $u\mathfrak{F}^i_\Phi(M)$ is zero. But $a \in u\mathfrak{F}^i_\Phi(M)$ and the k-th component of a is not zero. It follows that $\mathfrak{a}_k \subseteq \mathfrak{p}_j$ where $\mathfrak{a}_k \in \Phi$. Hence, we proved that for each integer $j \in \{1, \dots, n\}$ there exists an ideal $\mathfrak{b}_j \in \Phi$ such that $\mathfrak{b}_j \subseteq \mathfrak{p}_j$. Since Φ is a system of ideals there exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \mathfrak{b}_1\mathfrak{b}_2 \cdots \mathfrak{b}_n \subseteq \mathfrak{p}_j$ for all $j \in \{1, \dots, n\}$, this completes the proof. □

Corollary 3. *Let $(R,\mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let i be an integer such that $\mathfrak{F}^i_\Phi(M)$ is nonzero and representable. Then there exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}\mathfrak{F}^i_\Phi(M) = 0$.*

Proof. By [5, 7.2.11] $\bigcap_{\mathfrak{p} \in \operatorname{Att} \mathfrak{F}^i_\Phi(M)} \mathfrak{p} = \sqrt{(0 : \mathfrak{F}^i_\Phi(M))}$. Thus by Theorem 10 we conclude that there exists an ideal $\mathfrak{b}$ in Φ and an integer n such that, $\mathfrak{b}^n \mathfrak{F}^i_\Phi(M) = 0$. Since Φ is a system of ideals, there exists an ideal $\mathfrak{a}$ in Φ such that $\mathfrak{a} \subseteq \mathfrak{b}^n$. Therefore $\mathfrak{a}\mathfrak{F}^i_\Phi(M) = 0$, as desired. □

Let R be a ring, Φ a system of ideals of R and M an R-module. Recall that
$$\Gamma_\Phi(M) := \{x \in M : \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \text{ in } \Phi\}.$$
We say that M is Φ-torsion if $M = \Gamma_\Phi(M)$ and that M is Φ-torsion-free if $\Gamma_\Phi(M) = 0$. For a finitely generated R-module M, it is easy to see that M is Φ-torsion-free if and only if, for each $\mathfrak{a} \in \Phi$, $\mathfrak{a}$ contains a non-zero-divisor on M.

In order to state the next result we recall the concept of Matlis dual. Let M be an R-module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $\mathrm{D}(M) = \operatorname{Hom}_R(M, E(R/\mathfrak{m}))$ is called the Matlis dual of M.

Lemma 1. *Let $(R,\mathfrak{m})$ be a complete local ring, Φ a system of ideals of R and M a finitely generated R-module. Then*

(i) M is Φ-adically complete (i.e $M \simeq \varprojlim_{\mathfrak{a}\in\Phi}(M/\mathfrak{a}M)$),

ii) $\mathfrak{F}^0_\Phi(M)$ is finitely generated R-module.

Proof. i) Since M is finitely generated, $\mathrm{D}(M)$ is Artinian and so $\mathrm{D}(M)$ is $\mathfrak{m}$-torsion. For each $i \in \mathbb{N}$, there exists $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \mathfrak{m}^i$. Hence $\mathrm{D}(M)$ is Φ-torsion and we have
$$\mathrm{D}(M) = \bigcup_{\mathfrak{a}\in\Phi} (0 :_{\mathrm{D}(M)} \mathfrak{a}) \simeq \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{D}(M)).$$
Thus
$$M \simeq \mathrm{D}\,\mathrm{D}(M) \simeq \mathrm{D}(\varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{D}(M))) \simeq$$
$$\simeq \varprojlim_{\mathfrak{a}\in\Phi} R/\mathfrak{a} \otimes_R \mathrm{D}\,\mathrm{D}(M) \simeq \varprojlim_{\mathfrak{a}\in\Phi} M/\mathfrak{a}M.$$

ii) By definition $\mathfrak{F}^0_\Phi(M) = \varprojlim_{\mathfrak{a}\in\Phi} \mathrm{H}^0_\mathfrak{m}(M/\mathfrak{a}M)$. Since $\mathrm{H}^0_\mathfrak{m}(M/\mathfrak{a}M) \subseteq M/\mathfrak{a}M$ for all $\mathfrak{a} \in \Phi$, by (i) we get
$$\mathfrak{F}^0_\Phi(M) \subseteq \varprojlim_{\mathfrak{a}\in\Phi}(M/\mathfrak{a}M) \simeq M.$$

Since M is finitely generated we conclude that $\mathfrak{F}^0_\Phi(M)$ is finitely generated, as required. □

Lemma 2. *Let Φ be a system of ideals of R and $L \xrightarrow{f} M \xrightarrow{g} N$ be a exact sequence of R-modules and R-homomorphisms. Suppose that there exist two ideal $\mathfrak{a}$ and $\mathfrak{b}$ in Φ such that $\mathfrak{a}L = 0$ and $\mathfrak{b}N = 0$. Then there exists an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c}M = 0$.*

Proof. Since $\mathfrak{b}g(M) = 0$, we have $\mathfrak{b}M \subseteq \ker g = \operatorname{im} f$. But $\mathfrak{a}L = 0$, and so $\mathfrak{a}(\operatorname{im} f) = 0$. Thus $\mathfrak{a}\mathfrak{b}M = 0$. But, there exists an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. Therefore $\mathfrak{c}M = 0$ and the proof is complete. □

For the following proof we need the next Lemma.

Lemma 3. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let M be an Φ-torsion R-module. Then $\mathfrak{F}^i_\Phi(M) \cong \mathrm{H}^i_\mathfrak{m}(M)$. Therefore $\mathfrak{F}^i_\Phi(M)$ is Artinian for all $i \geqslant 0$.*

Proof. It is easy to see that, since M is finitely generated and Φ-torsion there exists an ideal $\mathfrak{a}$ in Φ such that $\mathfrak{a}M = 0$. We put $\Psi = \{\mathfrak{b} \in \Phi \mid \mathfrak{b} \subseteq \mathfrak{a}\}$. Then Ψ is cofinal in Φ. Thus we may assume that $\mathfrak{b} \subseteq \mathfrak{a}$ for all $\mathfrak{b} \in \Phi$ and so $\mathfrak{b}M = 0$ for all $\mathfrak{b} \in \Phi$. Hence

$$\mathfrak{F}^i_\Phi(M) \cong \varprojlim_{\mathfrak{b}\in\Phi} \mathrm{H}^i_\mathfrak{m}(M/\mathfrak{b}M) \cong \varprojlim_{\mathfrak{b}\in\Phi} \mathrm{H}^i_\mathfrak{m}(M) \cong \mathrm{H}^i_\mathfrak{m}(M)$$

for all $i \geqslant 0$, as desired. □

Theorem 11. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let $t \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$\mathfrak{F}^i_\Phi(M)$ is Artinian for all $i < t$,*
(ii) *$\mathfrak{F}^i_\Phi(M)$ is representable for all $i < t$,*
(iii) *there exists an ideal $\mathfrak{a}$ in Φ such that, $\mathfrak{a}\mathfrak{F}^i_\Phi(M) = 0$ for all $i < t$.*

Proof. (i) $\Rightarrow$(ii); Any Artinian R-module is representable.

(ii) $\Rightarrow$ (iii): By Corollary 3.

(iii) $\Rightarrow$ (i): We use induction on t. Since $\mathfrak{F}^i_\Phi(M) \simeq \mathfrak{F}^i_{\Phi\widehat{R}}(\widehat{M})$ by Theorem 4, we may assume that R is complete. Let $t = 1$. By Lemma 1(ii), $\mathfrak{F}^0_\Phi(M)$ is a finitely generated R-module. By assumption $\operatorname{Supp}_R(\mathfrak{F}^0_\Phi(M)) \subseteq V(\mathfrak{a})$ and so by Corollary 1 we conclude that $\operatorname{Supp}_R(\mathfrak{F}^0_\Phi(M)) \subseteq V(\mathfrak{m})$. Thus $\mathfrak{F}^0_\mathfrak{a}(M)$ is Artinian.

Now suppose, inductively, that $t > 0$ and $\mathfrak{F}^i_\Phi(M)$ is Artinian for all $i \leqslant t-2$. We show that $\mathfrak{F}^{t-1}_\mathfrak{a}(M)$ is Artinian. By Theorem 6, the short exact sequence

$$0 \longrightarrow \Gamma_\Phi(M) \longrightarrow M \longrightarrow M/\Gamma_\Phi(M) \longrightarrow 0$$

implies the long exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathfrak{F}^{i-1}_\Phi(\Gamma_\Phi(M)) &\longrightarrow \mathfrak{F}^{i-1}_\Phi(M) \\ &\longrightarrow \mathfrak{F}^{i-1}_\Phi(M/\Gamma_\Phi(M)) \longrightarrow \mathfrak{F}^{i}_\Phi(\Gamma_\Phi(M)) \longrightarrow \cdots .\end{aligned}$$

But $\mathfrak{F}^i_\Phi(\Gamma_\Phi(M))$ is Artinian for all i by Lemma 3. Thus by using the above long exact sequence it follows that $\mathfrak{F}^i_\Phi(M)$ is Artinian if and only if $\mathfrak{F}^i_\Phi(M/\Gamma_\Phi(M))$ is Artinian for all i. On the other hand, since Φ is a system of ideals, by Corollary 3 we can find an ideal $\mathfrak{b} \in \Phi$ such that $\mathfrak{b}\mathfrak{F}^i_\Phi(\Gamma_\Phi(M)) = 0$ for all $i \leqslant t$. By assumption and lemma 2 we conclude that there exists an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c}\mathfrak{F}^i_\Phi(M/\Gamma_\Phi(M)) = 0$ for all $i < t$. Therefore we can and do assume that M is an Φ-torsion-free R-module. Since $\mathfrak{a} \in \Phi$, it is easy to see that $\mathfrak{a}$ contains an element r which is a non-zerodivisor on M. The short exact sequence

$$0 \longrightarrow M \xrightarrow{r} M \longrightarrow M/rM \longrightarrow 0$$

induces a long exact sequence

$$0 \to \mathfrak{F}^0_\Phi(M) \xrightarrow{r} \mathfrak{F}^0_\Phi(M) \to \cdots \to \mathfrak{F}^i_\Phi(M) \xrightarrow{r} \mathfrak{F}^i_\Phi(M) \to \mathfrak{F}^i_\Phi(M/rM) \to \cdots .$$

By assumption and the above long exact sequence and lemma 2, it follows that there exists an ideal $\mathfrak{b} \in \Phi$ such that $\mathfrak{b}\mathfrak{F}^i_\Phi(M/rM) = 0$ for all $i < t-1$. Thus, by the inductive hypothesis, we conclude that $\mathfrak{F}^{t-2}_\Phi(M/rM)$ is Artinian. Since $r\mathfrak{F}^{t-1}_\Phi(M) \subseteq \mathfrak{a}\mathfrak{F}^{t-1}_\Phi(M) = 0$, the above long exact sequence implies that $\mathfrak{F}^{t-2}_\Phi(M/rM) \longrightarrow \mathfrak{F}^{t-1}_\Phi(M) \longrightarrow 0$ is exact. But $\mathfrak{F}^{t-2}_\Phi(M/rM)$ is Artinian and so $\mathfrak{F}^{t-1}_\mathfrak{a}(M)$ is Artinian, as required. □

Theorem 12. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module. Let $t \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$\mathfrak{F}^i_\Phi(M)$ is Artinian for all $i > t$,*
(ii) *$\mathfrak{F}^i_\Phi(M)$ is representable for all $i > t$,*
(iii) *there exists an ideal $\mathfrak{a}$ in Φ such that, $\mathfrak{a}\mathfrak{F}^i_\Phi(M) = 0$ for all $i > t$.*

Proof. (i) $\Rightarrow$(ii): It is clear.

(ii) $\Rightarrow$ (iii): By Corollary 3.

(iii) $\Rightarrow$ (i): The proof can be easily obtained by extending the proof of [4, Theorem 2.9] *mutatis mutandis* to this general case. $\square$

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. By Theorem 8 $\mathfrak{F}^d_\Phi(M)$ is Artinian. In the next result we determine the set $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M)$.

Theorem 13. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M a finitely generated R-module of dimension d. Then there exists an ideal $\mathfrak{a}$ in Φ such that $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M) = \mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$.*

Proof. Let $w := \max\{\dim(M/\mathfrak{a}M) : \mathfrak{a} \in \Phi\}$. If $w < d$ then $\mathfrak{F}^d_\Phi(M) = 0$ by Theorem 7 and so there is nothing to prove. Thus we suppose that $w = d$.

By Theorem 10 there exists an ideal $\mathfrak{a} \in \Phi$ such that $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M) \subseteq \mathrm{V}(\mathfrak{a})$. But by Theorem 8 and [5, 7.3.2] $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M) \subseteq \mathrm{Att}_R\,\mathrm{H}^d_\mathfrak{m}(M) = \mathrm{Assh}_R(M)$. Thus $\mathrm{Att}_R\,\mathfrak{F}^d_\mathfrak{a}(M) \subseteq \mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$.

Conversly, assume that $\mathfrak{a} \in \Phi$. We show that $\mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M)$. Let $\mathfrak{p} \in \mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$. By [8, 6.8], there exists a $\mathfrak{p}$-primary submodule N of M such that $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$ and $\mathfrak{p} = \sqrt{(0 : (M/N))}$. Thus $\dim M/N = \dim R/\mathfrak{p} = \dim M$. Since $\mathfrak{a} \subseteq \mathfrak{p}$ we have $\sqrt{\mathfrak{a}} \subseteq \sqrt{(0 : (M/N))}$.

Thus we can see that $\mathrm{Supp}_R((M/N)/\mathfrak{a}(M/N)) = \mathrm{Supp}_R(M/N)$ and $\dim((M/N)/\mathfrak{a}(M/N)) = \dim(M/N)$. Now by Theorem 7, $\mathfrak{F}^d_\Phi(M/N) \neq 0$. Hence

$$\phi \neq \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M/N) \subseteq \mathrm{Att}_R\,\mathrm{H}^d_\mathfrak{m}(M/N) \subseteq \mathrm{Ass}(M/N) = \{\mathfrak{p}\}$$

Therefore we have $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M/N) = \{\mathfrak{p}\}$. But the exact sequence

$$0 \to N \to M \to M/N \to 0$$

induces $\mathfrak{F}^d_\Phi(M) \to \mathfrak{F}^d_\Phi(M/N) \to 0$. Thus $\{\mathfrak{p}\} = \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M/N) \subseteq \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M)$. Therefore $\mathfrak{p} \in \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M)$. This completes the proof. $\square$

Corollary 4. *Let $(R, \mathfrak{m})$ be a local ring, Φ a system of ideals of R and M and N be two finitely generated R-modules of dimension d such that $\mathrm{Supp}_R M = \mathrm{Supp}_R N$. Then $\mathrm{Att}_R\,\mathfrak{F}^d_\Phi(M) = \mathrm{Att}_R\,\mathfrak{F}^d_\Phi(N)$.*

Proof. By Theorem 13 there exist two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in Φ such that $\operatorname{Att}_R \mathfrak{F}^d_\Phi(M) = \operatorname{Assh}_R M \cap \operatorname{V}(\mathfrak{a})$ and $\operatorname{Att}_R \mathfrak{F}^d_\Phi(N) = \operatorname{Assh}_R N \cap \operatorname{V}(\mathfrak{b})$. But by assumption we have $\operatorname{Assh}_R M = \operatorname{Assh}_R N$. On the other hand, by using the proof of Theorem 13,

$$\operatorname{Att}_R \mathfrak{F}^d_\Phi(M) = \operatorname{Assh}_R M \cap \operatorname{V}(\mathfrak{a}) = \operatorname{Assh}_R N \cap \operatorname{V}(\mathfrak{a}) \subseteq \operatorname{Att}_R \mathfrak{F}^d_\Phi(N).$$

Similarly $\operatorname{Att}_R \mathfrak{F}^d_\Phi(N) \subseteq \operatorname{Att}_R \mathfrak{F}^d_\Phi(M)$. This completes the proof. □

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